

Quiz 06

1. Consider a linear system:

$$Ax = b : \begin{bmatrix} 1 & -2 \\ -3 & 3 \end{bmatrix} x = \begin{bmatrix} 5 \\ 5 \end{bmatrix},$$

find the optimal ω such that the corresponding relaxation method converges to the true solution of $Ax = b$ fastest.

Find Corresponding Jacobi Iteration Matrix :

$$B = D^{-1}(L+U) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2/3 \\ 0 & 2/3 & 0 \end{bmatrix}$$

then

$$|\lambda I - B| = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -2/3 \\ 0 & -2/3 & \lambda \end{vmatrix} = \lambda(\lambda^2 - \frac{4}{9}) = 0$$

$$\Rightarrow \lambda = 0, \lambda = \pm \frac{2}{3} \Rightarrow \rho(\lambda) = \frac{2}{3}$$

$$\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \rho(B)^2}} = \frac{2}{1 + \sqrt{1 - \frac{4}{9}}} = \frac{2}{1 + \frac{\sqrt{5}}{3}} = \frac{6}{3 + \sqrt{5}}$$

2. Find the minimizer for the following minimization problem

$$\min_{x \in \mathbb{R}^2} \|Ax - b\|_2$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

Normal Eqn reads

$$A^T A x = A^T b$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7/4 \\ 3/4 \end{bmatrix}$$