

Quiz 05

1. If $\|\delta I\| < 1$, show that $I + \delta I$ is invertible, and

$$\|(I + \delta I)^{-1}\| \leq \frac{1}{1 - \|\delta I\|}.$$

Invertibility of $I + \delta I$:

Assume $I + \delta I$ is NOT invertible, then $\exists \vec{x} \neq 0$ s.t.

$$(I + \delta I) \vec{x} = 0$$

which implies

$$\vec{x} = -\delta I \cdot \vec{x}$$

Taking norms on two sides:

$$\|\vec{x}\| = \|\delta I \cdot \vec{x}\| \leq \|\delta I\| \cdot \|\vec{x}\|$$

$$\Rightarrow \|\delta I\| \geq 1. \quad \text{Contradict with } \|\delta I\| < 1.$$

$$\begin{aligned} I &= (I + \delta I)(I + \delta I)^{-1} \\ &= (I + \delta I)^{-1} + \delta I \cdot (I + \delta I)^{-1} \end{aligned}$$

Taking norms from two sides:

$$\begin{aligned} 1 &= \|I\| = \|(I + \delta I)^{-1} + \delta I \cdot (I + \delta I)^{-1}\| \\ &\geq \|(I + \delta I)^{-1}\| - \|\delta I\| \|(I + \delta I)^{-1}\| \end{aligned}$$

then

$$\frac{1}{1 - \|\delta I\|} \geq \|(I + \delta I)^{-1}\|. \quad \text{Done.}$$

2. Given a linear system as follows

$$Ax = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix},$$

write out explicitly the Gauss-Seidel iteration applied to the above system. Is the Gauss-Seidel iteration convergent? Explain why.

$$3u + v + w = 6 \Rightarrow u = \frac{1}{3}(6 - v - w) \Rightarrow u^{(k+1)} = \frac{1}{3}(6 - v^{(k)} - w^{(k)})$$

$$u + 3v + w = 3 \Rightarrow v = \frac{1}{3}(3 - u - w) \Rightarrow v^{(k+1)} = \frac{1}{3}(3 - u^{(k+1)} - w^{(k)})$$

$$u + v + 3w = 5 \Rightarrow w = \frac{1}{3}(5 - u - v) \Rightarrow w^{(k+1)} = \frac{1}{3}(5 - u^{(k+1)} - v^{(k+1)})$$



the key here is to
use the newest values
of u, v, w to do
the update.

Gauss-Seidel iteration converges because

A is P.D.D.