

Homework 05

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1(b).

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \text{ or } \begin{cases} 2u - v = 0 \\ -u + 2v - w = 2 \\ -v + 2w = 0 \end{cases}$$

Jacobi iteration can be obtained either from the matrix form:

$$\begin{aligned} \begin{bmatrix} u_{k+1} \\ v_{k+1} \\ w_{k+1} \end{bmatrix} &= D^{-1} \left(b + (L + U) \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{2} & \\ \frac{1}{2} & & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} +v_k \\ 2 + u_k + w_k \\ +u_k \end{bmatrix} = \begin{bmatrix} +\frac{1}{2}v_k \\ \frac{1}{2}(2 + u_k + w_k) \\ +\frac{1}{2}u_k \end{bmatrix} \end{aligned}$$

or directly from solving each equation for u, v, w respectively:

$$2u - v = 0 \Rightarrow u = \frac{1}{2}v \Rightarrow u_{k+1} = \frac{1}{2}v_k$$

$$-u + 2v - w = 2 \Rightarrow v = \frac{1}{2}(2 + u + w) \Rightarrow v_{k+1} = \frac{1}{2}(2 + u_k + w_k)$$

$$-v + 2w = 0 \Rightarrow w = \frac{1}{2}v \Rightarrow w_{k+1} = \frac{1}{2}v_k$$

The first two steps of computing Jacobi iteration are omitted.

Gauss-Seidel iteration can be obtained either from the matrix form:

$$\begin{aligned} \begin{bmatrix} u_{k+1} \\ v_{k+1} \\ w_{k+1} \end{bmatrix} &= D^{-1} \left(b + L \begin{bmatrix} u_{k+1} \\ v_{k+1} \\ w_{k+1} \end{bmatrix} + U \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & & \\ 1 & 0 & \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_{k+1} \\ v_{k+1} \\ w_{k+1} \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{2} & \\ \frac{1}{2} & & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} u_k \\ 2 + u_{k+1} + w_k \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}v_k \\ \frac{1}{2}(2 + u_{k+1} + w_k) \\ \frac{1}{2}u_{k+1} \end{bmatrix} \end{aligned}$$

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or directly from solving each equation for u , v , w respectively, and then add indices on each equation by using the latest informations.

$$2u - v = 0 \Rightarrow u = \frac{1}{2}v \Rightarrow u_{k+1} = \frac{1}{2}v_k$$

$$-u + 2v - w = 2 \Rightarrow v = \frac{1}{2}(2 + u + w) \Rightarrow v_{k+1} = \frac{1}{2}(2 + u_{k+1} + w_k)$$

$$-v + 2w = 0 \Rightarrow w = \frac{1}{2}v \Rightarrow w_{k+1} = \frac{1}{2}v_k$$

The first two steps of computing $\text{G}-\text{S}$ iteration are omitted.

z(b)

$$\begin{bmatrix} 1 & -8 & -2 \\ 1 & 1 & 5 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

Rearrange the equations by the following way:

$$\text{eqn } ① \rightarrow \text{eqn } ②$$

$$\text{eqn } ② \rightarrow \text{eqn } ③$$

$$\text{eqn } ③ \rightarrow \text{eqn } ①$$

then it becomes a strictly diagonally dominant system.

Or using more mathematical language: introduce a permutation matrix P

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and apply P from left on both sides of the system

$$P \cdot \begin{bmatrix} 1 & -8 & -2 \\ 1 & 1 & 5 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = P \cdot \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 1 \\ 1 & -8 & -2 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

the rest of work
is omitted.