Anisotropic Bending Energy

1 Total Energy

We minimize the energy

$$E(x,y,\eta) := \int_\Gamma \left[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\right]^2 \mathrm{d}\Gamma + \sigma \int_\Gamma \left[\frac{\xi}{2}|\nabla_\parallel \eta|^2 + \frac{1}{4\xi}(\eta^2 - 1)^2\right] \mathrm{d}\Gamma \tag{1}$$

where α, σ, ξ are constants. The parameter α control the anisotropy on the different phases of the vesicle membrane; σ is the line tension constant; ξ represents the width of the phase field function.

For the axisymmetric case, the energy (1) is written as

$$E(x,y,\eta) := \int_0^\pi [H + \alpha\eta \cdot \frac{k_1 - k_2}{2}]^2 x\sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} + \sigma \int_0^\pi \left[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2 - 1)^2\right] x\sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} \tag{2}$$

where

$$\eta' = \frac{\dot{\eta}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

So the problem is converted into

$$\min_{x,y,\eta} E(x,y,\eta) \tag{3}$$

subject to

$$x\sqrt{\dot{x}^2 + \dot{y}^2} = \sin t$$
$$\int_0^\pi x^2 \dot{y}\ \mathrm{dt} = V$$
$$\int_0^\pi \eta x\sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} = C$$

2 Euler-Lagrange Equations

Do the variation for (2), we can derive the Euler-Lagrange equations for the total energy.

2.1 Variation along Normal Direction

We do the variation along the normal direction now. If we naturally extend $\eta(x,y)$ off the membrane such that

$$\frac{\mathrm{d}\eta}{\mathrm{d}\mathbf{n}} = 0$$

everywhere along the membrane. Then the variations of η and $\nabla_{\|}\eta$ along the normal direction are both 0, namely,

$$\frac{\mathrm{d}\eta}{\mathrm{d}\mathbf{n}} = 0, \quad \frac{\mathrm{d}\nabla_{\|}\eta}{\mathrm{d}\mathbf{n}} = 0.$$

Then

$$\begin{aligned}
&\delta \int_0^\pi \Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]^2 x\sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} \\
&= \int_0^\pi 2\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]\delta\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big] x\sqrt{\dot{x}^2 + \dot{y}^2} + \Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]^2 \delta\Big(x\sqrt{\dot{x}^2 + \dot{y}^2}\Big)\ \mathrm{dt} \\
&= \int_0^\pi \Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]\delta\Big[(1+\alpha\eta)k_1 + (1-\alpha\eta)k_2\Big] x\sqrt{\dot{x}^2 + \dot{y}^2} \\
&\qquad\qquad - 2H\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]^2 \Big(x\sqrt{\dot{x}^2 + \dot{y}^2}\Big)\ \mathrm{dt} \\
&= \int_0^\pi \Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]\delta\Big[(1+\alpha\eta)(k_1 + k_2) - 2\alpha\eta k_2\Big] x\sqrt{\dot{x}^2 + \dot{y}^2} \\
&\qquad\qquad - 2H\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]^2 \Big(x\sqrt{\dot{x}^2 + \dot{y}^2}\Big)\ \mathrm{dt} \\
&= \int_0^\pi (1+\alpha\eta)\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]\delta(k_1 + k_2) x\sqrt{\dot{x}^2 + \dot{y}^2} \\
&\qquad\qquad - 2\alpha\eta\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]\delta(k_2) x\sqrt{\dot{x}^2 + \dot{y}^2} \\
&\qquad\qquad - 2H\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]^2 \Big(x\sqrt{\dot{x}^2 + \dot{y}^2}\Big)\ \mathrm{dt} \\
&= \int_0^\pi (1+\alpha\eta)\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]\Bigg[\Big(\frac{\dot{u}x}{\sqrt{\dot{x}^2 + \dot{y}^2}}\Big) + u(k_1^2 + k_2^2) x\sqrt{\dot{x}^2 + \dot{y}^2}\Bigg] \\
&\qquad\qquad - 2\alpha\eta\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big](\cos\phi\dot{u} + uk_2^2 x\sqrt{\dot{x}^2 + \dot{y}^2}) \\
&\qquad\qquad - 2H\Big[H + \alpha\eta \cdot \frac{k_1 - k_2}{2}\Big]^2 \Big(x\sqrt{\dot{x}^2 + \dot{y}^2}\Big)\ \mathrm{dt} \\
&= \int_0^\pi u\Big(\frac{x\dot{\widetilde{H}}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\Big)^{\cdot} + \Big[\widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}\cos\phi\Big]^{\cdot} u \\
&\qquad\qquad + u\Big[\widetilde{H}(k_1^2 + k_2^2) - \widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}k_2^2 - 2H\Big(\frac{\widetilde{H}}{1+\alpha\eta}\Big)^2\Big] x\sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt}
\end{aligned}$$

where

$$\widetilde{H} = (1+\alpha\eta)\Big[H + \alpha\eta\cdot\frac{k_1-k_2}{2}\Big].$$

For the line tension part,

$$\begin{aligned}
&\delta\int_0^\pi \Big[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2-1)^2\Big]x\sqrt{\dot{x}^2+\dot{y}^2}\ \mathrm{dt}\\
=&\int_0^\pi \Big[\xi\eta'\delta\eta' + \frac{1}{\xi}(\eta^2-1)\eta\delta\eta\Big]x\sqrt{\dot{x}^2+\dot{y}^2} + \Big[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2-1)^2\Big](-2uHx\sqrt{\dot{x}^2+\dot{y}^2})\ \mathrm{dt}\\
=&\int_0^\pi \xi\eta'^2\phi' x\sqrt{\dot{x}^2+\dot{y}^2} + \Big[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2-1)^2\Big](-2uHx\sqrt{\dot{x}^2+\dot{y}^2})\ \mathrm{dt}
\end{aligned}$$

where the first part of the integrand varnishes due to the fact that

$$\delta\eta = \eta_x(-u\sin\phi) + \eta_y(u\cos\phi) = 0$$

and

$$\begin{aligned}
\delta\eta' =&\delta(\eta_x\cos\phi + \eta_y\sin\phi)\\
=&(\eta_{xx}(-\sin\phi) + \eta_{xy}\cos\phi)u\cos\phi + \eta_x(-\sin\phi u')\\
&+ (\eta_{xy}(-\sin\phi) + \eta_{yy}\cos\phi)u\sin\phi + \eta_y(\cos\phi u')\\
=&\frac{(\eta_x\cos\phi + \eta_y\sin\phi)\dot{\phi}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \eta'\phi'
\end{aligned}$$

By combining the normal variations for the area and volume constraints, and taking

$$Q := \frac{x^2}{\sin^2 t}\dot{\widetilde{H}}$$

the variation of total energy along the normal direction is

$$\begin{aligned}
\dot{Q} + \cot tQ + \Big[\widetilde{H}(k_1^2+k_2^2) - \widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}k_2^2 - 2H\Big(\frac{\widetilde{H}}{1+\alpha\eta}\Big)^2\Big] + 2(\mu H + p + \lambda\eta H)\\
+\Big[\widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}\cos\phi\Big]^{\cdot}\Big/\sin t + \sigma\xi\eta'^2\phi' - 2\sigma H\Big[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2-1)^2\Big] = 0
\end{aligned}$$

2.2 Variation for η

One can continue with the variation with respect to η, and obtain

$$\alpha(k_1-k_2)\frac{\widetilde{H}}{1+\alpha\eta}x\sqrt{\dot{x}^2+\dot{y}^2} + \sigma\Big[-\xi(\eta' x)^{\cdot} + \frac{1}{\xi}(\eta^2-1)\eta x\sqrt{\dot{x}^2+\dot{y}^2}\Big] + \lambda x\sqrt{\dot{x}^2+\dot{y}^2} = 0$$

or equivalently

$$-\dot{D} - \cot tD + \frac{1}{\xi^2}(\eta^2-1)\eta + \frac{1}{\sigma\xi}\Big[\alpha(k_1-k_2)\frac{\widetilde{H}}{1+\alpha\eta} + \lambda\Big] = 0$$

2.3 Euler-Lagrange Equations

From the above derivation, the Euler-Lagrange equations are obtained:

$$\begin{aligned}
&\dot{Q} + \cot tQ + \left[\widetilde{H}(k_1^2+k_2^2) - \widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}k_2^2 - 2H\Big(\frac{\widetilde{H}}{1+\alpha\eta}\Big)^2\right] - 2(\mu H + p + \lambda\eta H) \\
&\qquad + \left[\widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}\cos\phi\right]^{\cdot}\Big/\sin t + \sigma\xi\eta'^2\phi' - 2\sigma H\left[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2-1)^2\right] = 0 \\
&-\dot{D} - \cot tD + \frac{1}{\xi^2}(\eta^2-1)\eta + \frac{1}{\sigma\xi}\left[\alpha(k_1-k_2)\frac{\widetilde{H}}{1+\alpha\eta} + \lambda\right] = 0
\end{aligned}$$

To implement the system, the limiting behaviors are analyzed as follows:

$$\begin{aligned}
\cot tQ &\to \dot{Q} \\
\widetilde{H} &\to (1+\alpha\eta)H \\
\left[\widetilde{H}(k_1^2+k_2^2) - \widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}k_2^2 - 2H\Big(\frac{\widetilde{H}}{1+\alpha\eta}\Big)^2\right] &\to (1+\alpha\eta)H\cdot 2H^2 - H\cdot 2\alpha\eta H^2 - 2H^3 = 0 \\
\sigma\xi\eta'^2\phi' &\to 0 \\
2\sigma H\left[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2-1)^2\right] &\to 2\sigma H\cdot\frac{1}{4\xi}(\eta^2-1)^2
\end{aligned}$$

notice that

$$\begin{aligned}
\left[\widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}\cos\phi\right]^{\cdot}\Big/\sin t =& \frac{\dot{\widetilde{H}}\frac{2\alpha\eta}{1+\alpha\eta}\cos\phi}{\sin t} + \frac{\widetilde{H}\frac{2\alpha\dot{\eta}}{1+\alpha\eta}\cos\phi}{\sin t} - \frac{\widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}\sin\phi\dot{\phi}}{\sin t} \\
=& \frac{Q}{\sin t}\frac{\sin^2 t}{x^2}\frac{2\alpha\eta}{1+\alpha\eta}\cos\phi + \frac{\widetilde{H}\frac{2\alpha\dot{\eta}}{1+\alpha\eta}\cos\phi}{\sin t} - \widetilde{H}\frac{2\alpha\eta}{1+\alpha\eta}\frac{\sin\phi}{\sin t}\dot{\phi} \\
\to& \dot{Q}\frac{2\alpha\eta}{1+\alpha\eta} + 0 - 2\alpha\eta H^3
\end{aligned}$$

Hence the limit of the Q-equation is

$$2\dot{Q} - 2(\mu H + p + \lambda\eta H) + \dot{Q}\frac{2\alpha\eta}{1+\alpha\eta} - 2\alpha\eta H^3 - 2\sigma H\cdot\frac{1}{4\xi}(\eta^2-1)^2 = 0$$

Similarly, the limit of the η-equation is

$$-2\dot{D} + \frac{1}{\xi^2}(\eta^2-1)\eta + \frac{\lambda}{\sigma\xi} = 0$$

Finally in the limit case, the Euler-Lagrange equations are

$$\begin{aligned}
\dot{Q} &= \frac{1}{1+\frac{\alpha\eta}{1+\alpha\eta}}\left[(\mu H + p + \lambda\eta H) + \alpha\eta H^3 + \sigma H\cdot\frac{1}{4\xi}(\eta^2-1)^2\right] \\
\dot{D} &= \frac{1}{2\xi^2}(\eta^2-1)\eta + \frac{\lambda}{2\sigma\xi}
\end{aligned}$$

```
function dydx = Aniso_ode2(x,y,p)

global xi sigma alpha

Q=y(1);  H_tilde=y(2);  phi=y(3);  r=y(4);  z=y(5);  V=y(6);  mu=y(7);
D=y(8); eta=y(9); V_eta=y(10); lamda=y(11);% E=y(12); LT=y(13);


          k2 = sin(phi)/r;
          H  = (H_tilde/(1+alpha*eta)+alpha*eta*k2)/(1+alpha*eta);
          k1 = 2*H-k2;
     eta_dot = D*(sin(x)/r)^2;
   eta_prime = D*(sin(x)/r);
     phi_dot = k1*sin(x)/r;
 H_tilde_dot = Q*(sin(x)/r)^2;
   Extra_dot = H_tilde_dot*(2*alpha*eta/(1+alpha*eta))*cos(phi)+...
                 H_tilde*(2*alpha*eta_dot/(1+alpha*eta)^2)*cos(phi)+...
                     H_tilde*(2*alpha*eta/(1+alpha*eta))*(-sin(phi))*phi_dot;


         Ptl = 1/(4*xi)*(eta^2-1)^2;
     Ptl_dot = 1/xi*(eta^2-1)*eta;


if (x<0.02*pi)
 dydx=[ 1/(1+alpha*eta/(1+alpha*eta))*( mu*H_tilde/(1+alpha*eta)+p+...
             lamda*eta*H_tilde/(1+alpha*eta)+sigma*H_tilde/(1+alpha*eta)*Ptl+...
               alpha*eta*(H_tilde/(1+alpha*eta))^3)
        Q
        H_tilde/(1+alpha*eta)
        cos(phi)
        sin(phi)
        pi*r*sin(x)*sin(phi)
        0
        Ptl_dot/(2*xi)+lamda/(2*sigma*xi)
        D
        eta*sin(x)
        0
        (H_tilde/(1+alpha*eta))^2*sin(x)+sigma*Ptl*sin(x)
        ];
elseif (x>=0.98*pi)
 dydx=[ 1/(1+alpha*eta/(1+alpha*eta))*( mu*H_tilde/(1+alpha*eta)+p+...
             lamda*eta*H_tilde/(1+alpha*eta)+sigma*H_tilde/(1+alpha*eta)*Ptl+...
               alpha*eta*(H_tilde/(1+alpha*eta))^3)
        Q
        H_tilde/(1+alpha*eta)
        cos(phi)
        sin(phi)
        pi*r*sin(x)*sin(phi)
        0
        Ptl_dot/(2*xi)+lamda/(2*sigma*xi)
        D
        eta*sin(x)
        0
        (H_tilde/(1+alpha*eta))^2*sin(x)+sigma*Ptl*sin(x)
        ];

else
 dydx=   [  - cot(x)*Q + 2*(mu*H+p+lamda*eta*H)-H_tilde*(k1^2+k2^2)+...
              H_tilde*(2*alpha*eta/(1+alpha*eta))*k2^2-...
                Extra_dot/sin(x)+2*H*(H_tilde/(1+alpha*eta))^2-...
                  sigma*xi*eta_prime^2*k1+2*sigma*H*(xi/2*eta_prime^2+Ptl)
            Q*(sin(x)/r)^2
            (2*H-sin(phi)/r)*sin(x)/r
            sin(x)*cos(phi)/r
            sin(x)*sin(phi)/r
            pi*r*sin(x)*sin(phi)
            0
            -cot(x)*D+ Ptl_dot/xi+(H_tilde/(1+alpha*eta)*alpha*(k1-k2)+lamda)/(sigma*xi)
            D*(sin(x)/r)^2
            eta*sin(x)
            0
            ( H_tilde/(1+alpha*eta) )^2*sin(x)+sigma*( xi/2*eta_prime^2+Ptl )*sin(x)
            ];

end
```