Line Tension

1 Total Energy

We minimize the energy
\[ E(x, y, C) := \int_{\Gamma} (c_0 + c_1 C) H^2 d\Gamma + \sigma \int_{\Gamma} \left[ \frac{\xi}{2} |\nabla ||C|^2 + \frac{1}{4\xi} (C^2 - 1)^2 \right] d\Gamma \] (1)
where \( c_0, c_1, \sigma, \xi \) are constants. The mean curvature stiffness on upper and lower components of the membrane are \( c_0 + c_1 \) and \( c_0 - c_1 \), respectively; \( \sigma \) is the line tension constant; \( \xi \) represents the width of the phase field function.

For the axisymmetric case, the energy (1) is written as
\[ E(x, y, C) := \int_0^\pi (c_0 + c_1 C) H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} \, dt + \sigma \int_0^\pi \left[ \frac{\xi}{2} C''^2 + \frac{1}{4\xi} (C^2 - 1)^2 \right] x \sqrt{\dot{x}^2 + \dot{y}^2} \, dt \] (2)
where
\[ C'' = \frac{\dot{C}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \]
So the problem is converted into
\[ \min_{x, y, C} E(x, y, C) \] (3)
subject to
\[ x \sqrt{\dot{x}^2 + \dot{y}^2} = \sin t \]
\[ \int_0^\pi x^2 \dot{y} \, dt = V \]
\[ \int_0^\pi Cx \sqrt{\dot{x}^2 + \dot{y}^2} \, dt = 0 \]

2 Euler-Lagrange Equations

Do the variation for (2), we can derive the Euler-Lagrange equations for the total energy.
2.1 Variation along Tangent Direction

First, we do the variation along the tangent direction, where the total energy $E(x, y, C)$ is supposed to be invariant. Similarly as we derive the variation along tangential direction for $\int H^2x\sqrt{\dot{x}^2 + \dot{y}^2}dt$, we get

$$\delta \int_0^\pi (c_0 + c_1 C)H^2x\sqrt{\dot{x}^2 + \dot{y}^2}dt$$

$$= \int_0^\pi \delta(c_0 + c_1 C)H^2x\sqrt{\dot{x}^2 + \dot{y}^2} + (c_0 + c_1 C)\delta\left(H^2x\sqrt{\dot{x}^2 + \dot{y}^2}\right)dt$$

$$= \int_0^\pi c_1(C_x \delta x + C_y \delta y)H^2x\sqrt{\dot{x}^2 + \dot{y}^2} + (c_0 + c_1 C)\delta\left(H^2x\sqrt{\dot{x}^2 + \dot{y}^2}\right)dt$$

$$= \int_0^\pi c_1 \dot{C} H^2xu + (c_0 + c_1 C)uH[\dot{k}_1 x + \dot{x}(k_1 - k_2)] + (c_0 + c_1 C)H^2(ux) dt$$

$$= \int_0^\pi c_1 \dot{C} H^2xu + (c_0 + c_1 C)uH[\dot{k}_1 x + \dot{x}(k_1 - k_2) - 2x \dot{H}] - (c_0 + c_1 C) H^2ux dt$$

$$= \int_0^\pi \dot{x} k_1 - (xk_2) dt = 0$$

For the line tension part,

$$\delta \int_0^\pi \left[\frac{\xi}{2} C'^2 + \frac{1}{4\xi}(C^2 - 1)^2\right] x\sqrt{\dot{x}^2 + \dot{y}^2} dt$$

$$= \int_0^\pi \left[\xi C' \delta C' + \frac{1}{4\xi}(C^2 - 1)C \delta C\right] x\sqrt{\dot{x}^2 + \dot{y}^2} + \left[\frac{\xi}{2} C'^2 + \frac{1}{4\xi}(C^2 - 1)^2\right](ux) \cdot dt$$

$$= \int_0^\pi \left[\xi C'(C') xu + \frac{1}{\xi}(C^2 - 1)C \dot{C} xu\right] + \left[\frac{\xi}{2} C'^2 + \frac{1}{4\xi}(C^2 - 1)^2\right](ux) \cdot dt = 0$$

where

$$\delta C = C_x \cos \phi u + C_y \sin \phi u = C'u$$

and

$$\delta C'^2 = \delta(C_x \cos \phi + C_y \sin \phi)$$

$$= (C_{xx} \cos \phi + C_{xy} \sin \phi) u \cos \phi + C_x (-\sin \phi \phi')$$

$$+ (C_{xy} \cos \phi + C_{yy} \sin \phi) u \sin \phi + C_y (u \cos \phi \phi')$$

$$= \frac{(C')^2 u}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C''u$$

Since the tangent variations for the area and volume constraints are the same as the homogeneous energy case, we obtain

$$\frac{\delta E}{\delta T} = -x \mu = 0$$
2.2 Variation along Normal Direction

We do the variation along the normal direction now. If we naturally extend $C(x, y)$ off the membrane such that
\[ \frac{dC}{dn} = 0 \]
everywhere along the membrane. Then the variations of $C$ and $\nabla \parallel C$ along the normal direction are both 0, namely,
\[ \frac{dC}{dn} = 0, \quad \frac{d\nabla \parallel C}{dn} = 0. \]

Then
\[
\delta \int_0^\pi (c_0 + c_1 C) H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} \, dt
= \int_0^\pi \delta(c_0 + c_1 C) H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} + (c_0 + c_1 C) \delta \left( H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} \right) \, dt
= \int_0^\pi (c_0 + c_1 C) H \left( \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) \cdot + 2u(c_0 + c_1 C) H (H^2 - K) x \sqrt{\dot{x}^2 + \dot{y}^2} \, dt
= \int_0^\pi u \left( \frac{x \dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) \cdot + 2u(c_0 + c_1 C) H (H^2 - K) x \sqrt{\dot{x}^2 + \dot{y}^2} \, dt
\]

where
\[ \tilde{H} = (c_0 + c_1 C) H. \]

For the line tension part,
\[
\delta \int_0^\pi \left[ \frac{\xi}{2} C'^2 + \frac{1}{4\xi} (C^2 - 1)^2 \right] x \sqrt{\dot{x}^2 + \dot{y}^2} \, dt
= \int_0^\pi \left[ \xi C' \delta C' + \frac{1}{\xi} (C^2 - 1) C \delta C \right] x \sqrt{\dot{x}^2 + \dot{y}^2} + \left[ \frac{\xi}{2} C'^2 + \frac{1}{4\xi} (C^2 - 1)^2 \right] (-2uHx \sqrt{\dot{x}^2 + \dot{y}^2}) \, dt
= \int_0^\pi \xi C'^2 \phi' x \sqrt{\dot{x}^2 + \dot{y}^2} + \left[ \frac{\xi}{2} C'^2 + \frac{1}{4\xi} (C^2 - 1)^2 \right] (-2uHx \sqrt{\dot{x}^2 + \dot{y}^2}) \, dt
\]

where the first part of the integrand varnishes due to the fact that
\[ \delta C = C_x (-u \sin \phi) + C_y (u \cos \phi) = 0 \]
and

\[\delta C' = \delta(C_x \cos \phi + C_y \sin \phi)\]
\[= (C_{xx}(-\sin \phi) + C_{xy} \cos \phi)u \cos \phi + C_x(-\sin \phi u')\]
\[+ (C_{xy}(-\sin \phi) + C_{yy} \cos \phi)u \sin \phi + C_y(\cos \phi u')\]
\[= \frac{(C_x \cos \phi + C_y \sin \phi)\dot{\phi}}{\sqrt{x^2 + y^2}} = C'\dot{\phi}'\]

By combining the normal variations for the area and volume constraints, and taking

\[Q := \frac{x^2}{\sin^2 t} \dot{H}\]

the variation of total energy along the normal direction is

\[[\dot{Q} + \cot tQ + 2\tilde{H}(H^2 - K)] - (2\mu H + 2p) + \sigma \xi C'^{2}\phi' - 2\sigma H\left[\frac{\xi}{2}C'^{2} + \frac{1}{4\xi} (C^2 - 1)^2\right] - 2\lambda CH = 0\]

2.3 Variation for C

One can continue with the variation with respect to C, and obtain

\[c_1H^2x\sqrt{x^2 + y^2} + \sigma \left[ -\xi(C'x)' + \frac{1}{\xi} (C^2 - 1)Cx \sqrt{x^2 + y^2} \right] + \lambda x \sqrt{x^2 + y^2} = 0\]

2.4 Euler-Lagrange Equations

We finally end up with Euler-Lagrange equations as follows:

\[[\dot{Q} + \cot tQ + 2\tilde{H}(H^2 - K)] - (2\mu H + 2p) + \sigma \xi C'^{2}\phi' - 2\sigma H\left[\frac{\xi}{2}C'^{2} + \frac{1}{4\xi} (C^2 - 1)^2\right] - 2\lambda CH = 0\]
\[c_1H^2x + \sigma \left[ -\xi(C'x)' + \frac{1}{\xi} (C^2 - 1)Cx \right] + \lambda x = 0\]

For the limiting behavior when t approaches boundaries, we have

\[\dot{Q} = (\mu H + p) + \sigma H\left[\frac{1}{4\xi} (C^2 - 1)^2\right] + \lambda CH\]
\[\dot{\mu} = 0\]
\[\dot{D} = \frac{1}{2\xi^2} (C^2 - 1)C + \frac{c_1H^2 + \lambda}{2\sigma\xi}\]
Therefore, we end up with the following self-closed system:

\[
\begin{align*}
\dot{Q} &= -\cot tQ - 2\tilde{H}(H^2 - K) + 2(\mu H + p) - \sigma \xi C'^{\prime} \phi^\prime + 2\sigma H \left[ \frac{\xi}{2} D^2 + \frac{1}{4\xi} (C^2 - 1)^2 \right] + 2\lambda C H \\
\dot{\tilde{H}} &= \frac{\sin^2 t}{x^2} Q \\
\dot{\phi} &= \left( 2H - \frac{\sin \phi}{x} \right) \frac{\sin t}{x} \\
\dot{x} &= \cos \phi \frac{\sin t}{x} \\
\dot{y} &= \sin \phi \frac{\sin t}{x} \\
\dot{V} &= \pi x \sin \phi \sin t \\
\dot{\mu} &= 0 \\
\dot{D} &= -\cot tD + \frac{c_1 H^2 + \lambda}{\sigma \xi} + \frac{1}{\xi^2} (C^2 - 1) C \\
\dot{C}' &= \frac{\sin^2 t}{x^2} D \\
\dot{V}_c &= C \sin t \\
\dot{\lambda} &= 0
\end{align*}
\]

3 Numerical Experiment

3.1 Homogeneous Case Where \( c_1 = 0 \)

For the homogeneous case where \( c_1 = 0 \), we can make \( c_0 \) dimensionlessly to be 1, then the equations reduce into

\[
\begin{align*}
[\dot{Q} + \cot tQ + 2H(H^2 - K)] - (2\mu H + 2p) + \sigma \xi C'^{\prime} \phi^\prime - 2\sigma H \left[ \frac{\xi}{2} C'^{\prime} + \frac{1}{4\xi} (C^2 - 1)^2 \right] - 2\lambda C H &= 0 \\
\sigma \left[ -\xi (C'x)' + \frac{1}{\xi} (C^2 - 1) C x \right] + \lambda x &= 0
\end{align*}
\]

To see if our diffuse interface model match up with the sharp interface model, let us first check if the condition

\[ Q(0) = Q(\pi) = x(0) = 0 \Rightarrow x(\pi) = 0 \]

still holds for the diffuse interface model.
The technique is similar as what we did for the sharp interface model. Multiplying both sides of the two equations by \( \sin t \cos \phi \) and \( \dot{C} \sin \phi \) yields

\[
(Q \sin t \cos \phi - H \sin^2 \phi + xH^2 \sin \phi - px^2) - \mu(x \sin \phi) - \lambda C(x \sin \phi) - \sigma(x \sin \phi) \left[ \frac{\xi}{2} C'' \right] + \sigma \xi C'' \sin \phi \cdot x = 0
\]

\[
\sigma \left[ -\xi (C' x) C' \sin \phi + \frac{1}{\xi} (C^2 - 1) C \dot{C} \sin \phi \right] + \lambda C x \sin \phi = 0
\]

Notice that

\[
C'' \sin \phi \cdot x + (C' x) C' \sin \phi - \frac{1}{2} (x \sin \phi) C'' = 0
\]

Substitute the second equality into the first one, we have the total integral

\[
(Q \sin t \cos \phi - H \sin^2 \phi + xH^2 \sin \phi - px^2) - (\mu x \sin \phi) - \lambda (x \sin \phi) C' - \sigma(x \sin \phi) \left[ \frac{\xi}{2} C'' + \frac{1}{4\xi} (C^2 - 1)^2 \right] = 0
\]

Integrating from 0 to \( \pi \) implies that \( x(\pi) = 0 \).

### 3.2 Nonhomogeneous Case Where \( c_1 \neq 0 \)

For the nonhomogeneous case where \( c_1 \neq 0 \), we can make \( c_0 \) dimensionlessly to be 1, then the equations reduce into

\[
\left[ \dot{Q} + \cot tQ + 2H(H^2 - K) \right] - (2\mu H + 2p) + \sigma \xi C'' \phi' - 2\sigma H \left[ \frac{\xi}{2} C'' \right] + \lambda C \dot{H} + \frac{c_1}{\sin t} \left( \frac{x^2 (CH)}{\sin t} \right) = 0
\]

\[
c_1 H^2 x + \sigma \left[ -\xi (C' x) + \frac{1}{\xi} (C^2 - 1) C x \right] + \lambda x = 0
\]
Multiplying both sides of the two equations by \(\sin t \cos \phi\) and \(\sin \phi\) yields (we here only need consider the extra terms, and ignore the common factor \(c_1\)),

\[
(sint \cos \phi P)' + P \sin t \sin \phi \dot{\phi} + 2CH(H^2 - K) \sin t \cos \phi \\
= (sint \cos \phi P)' + \frac{x^2}{\sin^2 t} (\dot{CH}) \sin t \sin \phi \dot{\phi} + C \left[ \frac{x^2}{\sin^2 t} \dot{H} \sin t \sin \phi \dot{\phi} + 2CH(H^2 - K) \sin t \cos \phi \right] \\
= (sint \cos \phi P)' + \frac{x^2}{\sin^2 t} (\dot{CH}) \sin t \sin \phi \dot{\phi} + C(xH^2 \sin \phi)' - C(H \sin^2 \phi)' \\
\]

where

\[
P = \frac{x^2(CH)}{\sin^2 t}
\]

and

\[
-xH^2 \sin \phi \dot{C}.
\]

Put them together, one has

\[
(sint \cos \phi P)' + \frac{x^2}{\sin^2 t} (\dot{CH}) \sin t \sin \phi \dot{\phi} + C(xH^2 \sin \phi)' - C(H \sin^2 \phi)' - xH^2 \sin \phi \dot{C} \\
= (sint \cos \phi P)' + 2xH^2 \sin \phi \dot{C} - H \sin^2 \phi \dot{C} + C(xH^2 \sin \phi)' - C(H \sin^2 \phi)' - xH^2 \sin \phi \dot{C} \\
= (sint \cos \phi P)' + xH^2 \sin \phi \dot{C} - H \sin^2 \phi \dot{C} + C(xH^2 \sin \phi)' - C(H \sin^2 \phi)' \\
= (sint \cos \phi P + CXH^2 \sin \phi - CH \sin^2 \phi)'
\]

Obviously, by integrating the total derivative one can derive that \(x(\pi) = 0\).

### 4 Adhesion of Multi-component membrane

If we take the adhesion effect into the consideration, the axisymmetric membrane has the total energy written as

\[
E(x, y, C) := \int_0^\pi (c_0 + c_1C)H^2 x \sqrt{x^2 + y^2} \, dt + \sigma \int_0^\pi \left[ \frac{\xi}{2} \dot{C}^2 + \frac{1}{4\xi} (C^2 - 1)^2 \right] x \sqrt{x^2 + y^2} \, dt \\
- w \int_0^\pi (c_0 + c_2C)e^{-y^2/\delta^2} x \sqrt{x^2 + y^2} \, dt
\]

subject to

\[
x \sqrt{x^2 + y^2} = \sin t \\
\int_0^\pi x^2 y \, dt = V \\
\int_0^\pi Cx \sqrt{x^2 + y^2} \, dt = 0
\]
4.1 Euler-Lagrange Equation

One can obtain Euler-Lagrange equations as follows:

\[
\begin{align*}
\dot{Q} + \cot tQ + 2\tilde{H}(H^2 - K) - (2\mu H + 2p) + \sigma \xi C'^2 \phi' - 2\sigma H \left[ \frac{\xi}{2} C'^2 + \frac{1}{4\xi}(C^2 - 1)^2 \right] \\
- 2\lambda CH + 2w(1 + c_2 C)e^{-y^2/\delta^2} \left( \frac{y}{\delta^2} \cos \phi + H \right) = 0
\end{align*}
\]

\[
\dot{\mu} = 0
\]

\[
c_1 H^2 x + \sigma \left[ -\xi (C' x)' + \frac{1}{\xi} (C^2 - 1) C x \right] + \lambda x = 0
\]

5 Coarsening

To view the coarsening process, or phase separation process, of the membrane with different components (red and blue) mixing up together, we apply a gradient flow approach, namely,

\[
C_t = -\frac{\delta E}{\delta C}.
\]

By discretizing the time derivative on the left hand, we get

\[
\frac{C_{n+1} - C_n}{\Delta t} = -\frac{\delta E}{\delta C}(C_{n+1})
\]

which is a implicit Euler method, or one can think of the solution \(C_{n+1}\) is a minimizer of the energy

\[
E(x, y, C) + \int_0^\pi \frac{|C - C_n|^2}{2\Delta t} x \sqrt{x^2 + y^2} dt
\]

5.1 Euler-Lagrange Equation

One can obtain Euler-Lagrange equations as follows:

\[
\begin{align*}
\dot{Q} + \cot tQ + 2\tilde{H}(H^2 - K) - (2\mu H + 2p) + \sigma \xi C'^2 \phi' - 2\sigma H \left[ \frac{\xi}{2} C'^2 + \frac{1}{4\xi}(C^2 - 1)^2 \right] \\
- 2\lambda CH + 2w(1 + c_2 C)e^{-y^2/\delta^2} \left( \frac{y}{\delta^2} \cos \phi + H \right) - \frac{|C - C_n|^2}{2\Delta t} H = 0
\end{align*}
\]

\[
\dot{\mu} = 0
\]

\[
c_1 H^2 x + \sigma \left[ -\xi (C' x)' + \frac{1}{\xi} (C^2 - 1) C x \right] + \lambda x + \frac{C - C_n}{\Delta t} x = 0
\]
6 Leonard-Jones Potential

Another way to eliminate the protrusion of the membrane shapes is to use some other adhesion potentials. One typical choice is Leonard-Jones potential

\[ W(x) = w(1 + c_2 \eta) \cdot 4 \zeta \left[ \left( \frac{\beta}{d(x)} \right)^{\alpha} - \left( \frac{\beta}{d(x)} \right)^{\alpha/2} \right] \]  (4)

The key difference between the exponential potential and Leonard-Jones potential is that exponential potential is globally attractive, while there is a narrow region \(d(x) \in (0, \beta)\) where Leonard-Jones potential is repulsive. Such a repulsive region can prevent the cell membrane to protrude into the substrate.

Total energy with Leonard-Jones potential is:

\[
E(x, y, \eta) := \int_0^\pi (1 + c_1 \eta) H^2 x \sqrt{x^2 + y^2} \, dt + \sigma \int_0^\pi \left[ \frac{\kappa}{2} \eta^2 + \frac{1}{4\zeta} (\eta^2 - 1)^2 \right] x \sqrt{x^2 + y^2} \, dt \\
+ \int_0^\pi w(1 + c_2 \eta) \cdot 4 \zeta \left[ \left( \frac{\beta}{x} \right)^{\alpha} - \left( \frac{\beta}{x} \right)^{\alpha/2} \right] \, dt
\]

With the Leonard-Jones potential, we have the following Euler-Lagrange equation:

\[
\dot{\eta} + \cot \theta Q + 2 \tilde{H}(H^2 - K) - (2 \mu H + 2 \rho) + \sigma \xi \eta^2 \phi' - 2 \sigma H \left[ \frac{\xi}{2} \eta^2 + \frac{1}{4\zeta} (\eta^2 - 1)^2 \right] \\
- 2 \lambda \eta H - w(1 + c_2 \eta) \left\{ \frac{4 \zeta \beta}{x^2} \left[ \alpha \left( \frac{\beta}{x} \right)^{\alpha-1} - \frac{\alpha}{2} \left( \frac{\beta}{x} \right)^{\alpha/2-1} \right] \cos \phi + 8 \zeta H \left[ \left( \frac{\beta}{x} \right)^{\alpha} - \left( \frac{\beta}{x} \right)^{\alpha/2} \right] \right\} = 0
\]

\[
\dot{\mu} = 0
\]

\[
c_1 H^2 x + \sigma \left[ - \xi (\eta x)' + \frac{1}{\xi} (\eta^2 - 1) \eta x \right] + \lambda x + c_2 w \cdot 4 \zeta \left[ \left( \frac{\beta}{d(x)} \right)^{\alpha} - \left( \frac{\beta}{d(x)} \right)^{\alpha/2} \right] = 0
\]

7 Double Obstacle Potential In Interfacial Energy

Since the double well potential

\[
P(\eta) = \frac{1}{4\zeta} (\eta^2 - 1)^2
\]
can not fix \( \eta \) at \( \pm 1 \) very well. Here we can consider another potential, which is so called double obstacle potential which is given by

\[
P(\eta) = \alpha(1 - \eta^2) + (1 + \eta) \ln(1 + \eta) + (1 - \eta) \ln(1 - \eta) - 2 \ln 2
\]

where \( P(\pm 1) = 0 \) and \( P(\eta) \) attains the minimum at \( \eta_1, \eta_2 \) which satisfy

\[
P'(\eta) = \ln \frac{1 + \eta}{1 - \eta} - 2\alpha \eta = 0.
\]
For the double obstacle potential, we first need to figure out the equilibrium solution of the equation

$$\xi\eta'' + \ln \frac{1 + \eta}{1 - \eta} - 2\alpha\eta = 0$$

The Euler-Lagrange equations for the double well potential combined with Leonard-Jones is derived from minimizing the following energy

$$E(x, y, \eta) := \int_0^\pi (1 + c_1\eta)H^2 x\sqrt{x^2 + y^2} \, dt$$

$$+ \sigma \int_0^\pi \left[\xi\eta^2 + \alpha(1 - \eta^2) + (1 + \eta)\ln(1 + \eta) + (1 - \eta)\ln(1 - \eta) - 2\ln 2\right] x\sqrt{x^2 + y^2} \, dt$$

$$+ \int_0^\pi w(1 + c_2\eta) \cdot 4\zeta\left[\left(\frac{\beta}{x}\right)^\alpha - \left(\frac{\beta}{x}\right)^{\alpha/2}\right] \, dt$$

Actually we have

$$\dot{\dot{Q}} + \cot tQ + 2\tilde{H}(H^2 - K) - (2\mu + 2p + 2\lambda\eta H) + \sigma\xi\eta^2\phi'$$

$$- 2\sigma H\left[\frac{\xi}{2}\eta^2 + \alpha(1 - \eta^2) + (1 + \eta)\ln(1 + \eta) + (1 - \eta)\ln(1 - \eta) - 2\ln 2\right]$$

$$- w(1 + c_2\eta)\left\{4\zeta\beta x^2 \left[\alpha\left(\frac{\beta}{x}\right)^{\alpha-1} - \frac{\alpha}{2}\left(\frac{\beta}{x}\right)^{\alpha/2-1}\right] \cos \phi + 8\zeta H\left[\left(\frac{\beta}{x}\right)^\alpha - \left(\frac{\beta}{x}\right)^{\alpha/2}\right]\right\} = 0$$

$$\dot{\mu} = 0$$

$$c_1H^2 x + \sigma \left[ -\xi(\eta'x') + \left(\ln \frac{1 + \eta}{1 - \eta} - 2\alpha\eta\right)x \right] + \lambda x + c_2w \cdot 4\zeta\left(\left(\frac{\beta}{d(x)}\right)^\alpha - \left(\frac{\beta}{d(x)}\right)^{\alpha/2}\right) = 0$$
8 Anisotropic Energy

Consider the anisotropic energy

\[ E(x, y, \eta) := \int_0^\pi \left( H + \alpha \eta (k_1 - k_2) \right)^2 x \sqrt{\dot{x}^2 + \dot{y}^2} \, dt + \sigma \int_0^\pi \left[ \frac{\xi}{2} \eta'^2 + \frac{1}{4\xi} (\eta^2 - 1)^2 \right] x \sqrt{\dot{x}^2 + \dot{y}^2} \, dt \]

The Euler-Lagrange equation is

\[
\dot{Q} + \cot t Q + 2H(H^2 - K) - (2\mu H + 2p + 2\lambda H) + \sigma \xi \eta^2 \phi' - 2\sigma H \left[ \frac{\xi}{2} \eta'^2 + \frac{1}{4\xi} (\eta^2 - 1)^2 \right] \\
+ 4\alpha \eta \left[ \dot{Q} + \cot t Q + \frac{(k_2 \cos \phi)}{\sin t} + \frac{1}{2}(k_1^2 + k_2^2)(k_1 - H) \right] + 4\alpha^2 \eta^2 \left[ \dot{Q} + \cot t Q + 2H(H^2 - K) \right] = 0
\]

and

\[
2(H + \alpha \eta (k_1 - k_2)) \cdot \alpha (k_1 - k_2)x + \sigma \left[ -\xi (C'x)' + \frac{1}{\xi} (C^2 - 1)Cx \right] + \lambda x = 0
\]