Line Tension

1 Total Energy

We minimize the energy

$$E(x,y,C) := \int_\Gamma (c_0 + c_1 C) H^2 \mathrm{d}\Gamma + \sigma \int_\Gamma \left[\frac{\xi}{2} |\nabla_\parallel C|^2 + \frac{1}{4\xi}(C^2-1)^2 \right] \mathrm{d}\Gamma \tag{1}$$

where c_0, c_1, σ, ξ are constants. The mean curvature stiffness on upper and lower components of the membrane are $c_0 + c_1$ and $c_0 - c_1$, respectively; σ is the line tension constant; ξ represents the width of the phase field function.

For the axisymmetric case, the energy (1) is written as

$$E(x,y,C) := \int_0^\pi (c_0 + c_1 C) H^2 x \sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} + \sigma \int_0^\pi \left[\frac{\xi}{2} C'^2 + \frac{1}{4\xi}(C^2-1)^2 \right] x \sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} \tag{2}$$

where

$$C' = \frac{\dot{C}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

So the problem is converted into

$$\min_{x,y,C} E(x,y,C) \tag{3}$$

subject to

$$x\sqrt{\dot{x}^2 + \dot{y}^2} = \sin t$$
$$\int_0^\pi x^2 \dot{y}\ \mathrm{dt} = V$$
$$\int_0^\pi C x \sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} = 0$$

2 Euler-Lagrange Equations

Do the variation for (2), we can derive the Euler-Lagrange equations for the total energy.

2.1 Variation along Tangent Direction

First, we do the variation along the tangent direction, where the total energy $E(x, y, C)$ is supposed to be invariant. Similarly as we derive the variation along tangential direction for $\int H^2 x\sqrt{\dot{x}^2+\dot{y}^2}\mathrm{dt}$, we get

$$
\begin{aligned}
&\delta\int_0^\pi (c_0+c_1C)H^2x\sqrt{\dot{x}^2+\dot{y}^2}\mathrm{dt}\\
=&\int_0^\pi \delta(c_0+c_1C)H^2x\sqrt{\dot{x}^2+\dot{y}^2}+(c_0+c_1C)\delta\Big(H^2x\sqrt{\dot{x}^2+\dot{y}^2}\Big)\mathrm{dt}\\
=&\int_0^\pi c_1(C_x\delta x+C_y\delta y)H^2x\sqrt{\dot{x}^2+\dot{y}^2}+(c_0+c_1C)\delta\Big(H^2x\sqrt{\dot{x}^2+\dot{y}^2}\Big)\mathrm{dt}\\
=&\int_0^\pi c_1\dot{C}H^2xu+(c_0+c_1C)uH\big[\dot{k}_1x+\dot{x}(k_1-k_2)\big]+(c_0+c_1C)H^2(ux)^{\cdot}\mathrm{dt}\\
=&\int_0^\pi c_1\dot{C}H^2xu+(c_0+c_1C)uH\big[\dot{k}_1x+\dot{x}(k_1-k_2)-2x\dot{H}\big]-(c_0+c_1C)^{\cdot}H^2ux\mathrm{dt}\\
=&\int_0^\pi \dot{x}k_1-(xk_2)^{\cdot}\mathrm{dt}=0
\end{aligned}
$$

For the line tension part,

$$
\begin{aligned}
&\delta\int_0^\pi \Big[\frac{\xi}{2}C'^2+\frac{1}{4\xi}(C^2-1)^2\Big]x\sqrt{\dot{x}^2+\dot{y}^2}\ \mathrm{dt}\\
=&\int_0^\pi \Big[\xi C'\delta C'+\frac{1}{\xi}(C^2-1)C\delta C\Big]x\sqrt{\dot{x}^2+\dot{y}^2}+\Big[\frac{\xi}{2}C'^2+\frac{1}{4\xi}(C^2-1)^2\Big](ux)^{\cdot}\ \mathrm{dt}\\
=&\int_0^\pi \Big[\xi C'(C')^{\cdot}xu+\frac{1}{\xi}(C^2-1)C\dot{C}xu\Big]+\Big[\frac{\xi}{2}C'^2+\frac{1}{4\xi}(C^2-1)^2\Big](ux)^{\cdot}\ \mathrm{dt}=0
\end{aligned}
$$

where

$$\delta C=C_x\cos\phi u+C_y\sin\phi u=C'u$$

and

$$
\begin{aligned}
\delta C'=&\delta(C_x\cos\phi+C_y\sin\phi)\\
=&(C_{xx}\cos\phi+C_{xy}\sin\phi)u\cos\phi+C_x(-\sin\phi\phi')\\
&+(C_{xy}\cos\phi+C_{yy}\sin\phi)u\sin\phi+C_y(u\cos\phi\phi')\\
=&\frac{(C')^{\cdot}u}{\sqrt{\dot{x}^2+\dot{y}^2}}=C''u
\end{aligned}
$$

Since the tangent variations for the area and volume constraints are the same as the homogeneous energy case, we obtain

$$\boxed{\frac{\delta E}{\delta\mathbf{T}}=-x\dot{\mu}=0}$$

2.2 Variation along Normal Direction

We do the variation along the normal direction now. If we naturally extend $C(x, y)$ off the membrane such that

$$\frac{dC}{d\mathbf{n}} = 0$$

everywhere along the membrane. Then the variations of C and $\nabla_{\|}C$ along the normal direction are both 0, namely,

$$\frac{\mathrm{d}C}{\mathrm{d}\mathbf{n}} = 0, \quad \frac{\mathrm{d}\nabla_{\|}C}{\mathrm{d}\mathbf{n}} = 0.$$

Then

$$\begin{aligned}
&\delta \int_0^\pi (c_0 + c_1 C) H^2 x \sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} \\
&= \int_0^\pi \delta(c_0 + c_1 C) H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} + (c_0 + c_1 C)\delta\Big(H^2 x \sqrt{\dot{x}^2 + \dot{y}^2}\Big)\ \mathrm{dt} \\
&= \int_0^\pi (c_0 + c_1 C) H \left(\frac{\dot{u}x}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right)^{\cdot} + 2u(c_0 + c_1 C) H (H^2 - K) x \sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} \\
&= \int_0^\pi u \left(\frac{x\dot{\widetilde{H}}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right)^{\cdot} + 2u(c_0 + c_1 C) H (H^2 - K) x \sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt}
\end{aligned}$$

where

$$\widetilde{H} = (c_0 + c_1 C) H.$$

For the line tension part,

$$\begin{aligned}
&\delta \int_0^\pi \left[\frac{\xi}{2} C'^2 + \frac{1}{4\xi}(C^2 - 1)^2\right] x \sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} \\
&= \int_0^\pi \left[\xi C' \delta C' + \frac{1}{\xi}(C^2 - 1) C \delta C\right] x \sqrt{\dot{x}^2 + \dot{y}^2} + \left[\frac{\xi}{2} C'^2 + \frac{1}{4\xi}(C^2 - 1)^2\right](-2uHx\sqrt{\dot{x}^2 + \dot{y}^2})\ \mathrm{dt} \\
&= \int_0^\pi \xi C'^2 \phi' x \sqrt{\dot{x}^2 + \dot{y}^2} + \left[\frac{\xi}{2} C'^2 + \frac{1}{4\xi}(C^2 - 1)^2\right](-2uHx\sqrt{\dot{x}^2 + \dot{y}^2})\ \mathrm{dt}
\end{aligned}$$

where the first part of the integrand varnishes due to the fact that

$$\delta C = C_x(-u \sin\phi) + C_y(u \cos\phi) = 0$$

and

$$\begin{aligned}\delta C' =&\delta(C_x\cos\phi + C_y\sin\phi)\\ =&(C_{xx}(-\sin\phi) + C_{xy}\cos\phi)u\cos\phi + C_x(-\sin\phi u')\\ &+ (C_{xy}(-\sin\phi) + C_{yy}\cos\phi)u\sin\phi + C_y(\cos\phi u')\\ =&\frac{(C_x\cos\phi + C_y\sin\phi)\dot{\phi}}{\sqrt{\dot{x}^2+\dot{y}^2}} = C'\phi'\end{aligned}$$

By combining the normal variations for the area and volume constraints, and taking

$$Q := \frac{x^2}{\sin^2 t}\dot{\widetilde{H}}$$

the variation of total energy along the normal direction is

$$\boxed{\left[\dot{Q} + \cot tQ + 2\widetilde{H}(H^2-K)\right] - (2\mu H + 2p) + \sigma\xi C'^2\phi' - 2\sigma H\left[\frac{\xi}{2}C'^2 + \frac{1}{4\xi}(C^2-1)^2\right] - 2\lambda CH = 0}$$

2.3 Variation for C

One can continue with the variation with respect to C, and obtain

$$\boxed{c_1H^2x\sqrt{\dot{x}^2+\dot{y}^2} + \sigma\Big[-\xi(C'x)^{\cdot} + \frac{1}{\xi}(C^2-1)Cx\sqrt{\dot{x}^2+\dot{y}^2}\Big] + \lambda x\sqrt{\dot{x}^2+\dot{y}^2} = 0}$$

2.4 Euler-Lagrange Equations

We finally end up with Euler-Lagrange equations as follows:

$$\left[\dot{Q} + \cot tQ + 2\widetilde{H}(H^2-K)\right] - (2\mu H + 2p) + \sigma\xi C'^2\phi' - 2\sigma H\left[\frac{\xi}{2}C'^2 + \frac{1}{4\xi}(C^2-1)^2\right] - 2\lambda CH = 0$$

$$c_1H^2x + \sigma\Big[-\xi(C'x)' + \frac{1}{\xi}(C^2-1)Cx\Big] + \lambda x = 0$$

For the limiting behavior when t approaches boundaries, we have

$$\begin{aligned}&\dot{Q} = (\mu H + p) + \sigma H\left[\frac{1}{4\xi}(C^2-1)^2\right] + \lambda CH\\ &\dot{\mu} = 0\\ &\dot{D} = \frac{1}{2\xi^2}(C^2-1)C + \frac{c_1H^2+\lambda}{2\sigma\xi}\end{aligned}$$

Therefore, we end up with the following self-closed system:

$$
\begin{aligned}
\dot{Q} &= -\cot tQ - 2\widetilde{H}(H^2 - K) + 2(\mu H + p) - \sigma\xi C'^2\phi' + 2\sigma H\Big[\frac{\xi}{2}D^2 + \frac{1}{4\xi}(C^2-1)^2\Big] + 2\lambda CH \\
\dot{\widetilde{H}} &= \frac{\sin^2 t}{x^2}Q \\
\dot{\phi} &= \left(2H - \frac{\sin\phi}{x}\right)\frac{\sin t}{x} \\
\dot{x} &= \cos\phi\frac{\sin t}{x} \\
\dot{y} &= \sin\phi\frac{\sin t}{x} \\
\dot{V} &= \pi x\sin\phi\sin t \\
\dot{\mu} &= 0 \\
\dot{D} &= -\cot tD + \frac{c_1H^2+\lambda}{\sigma\xi} + \frac{1}{\xi^2}(C^2-1)C \\
\dot{C} &= \frac{\sin^2 t}{x^2}D \\
\dot{V}_c &= C\sin t \\
\dot{\lambda} &= 0
\end{aligned}
$$

3 Numerical Experiment

3.1 Homogeneous Case Where $c_1 = 0$

For the homogeneous case where $c_1 = 0$, we can make c_0 dimensionlessly to be 1, then the equations reduce into

$$
\begin{aligned}
&\big[\dot{Q} + \cot tQ + 2H(H^2-K)\big] - (2\mu H + 2p) + \sigma\xi C'^2\phi' - 2\sigma H\Big[\frac{\xi}{2}C'^2 + \frac{1}{4\xi}(C^2-1)^2\Big] - 2\lambda CH = 0 \\
&\sigma\Big[-\xi(C'x)' + \frac{1}{\xi}(C^2-1)Cx\Big] + \lambda x = 0
\end{aligned}
$$

To see if our diffuse interface model match up with the sharp interface model, let us first check if the condition

$$Q(0) = Q(\pi) = x(0) = 0 \Rightarrow x(\pi) = 0$$

still holds for the diffuse interface model.

The technique is similar as what we did for the sharp interface model. Multiplying both sides of the two equations by $\sin t \cos\phi$ and $\dot{C}\sin\phi$ yields

$$
\begin{aligned}
&(Q\sin t\cos\phi - H\sin^2\phi + xH^2\sin\phi - px^2)^{\cdot} \\
&\qquad - \mu(x\sin\phi)^{\cdot} - \lambda C(x\sin\phi)^{\cdot} - \sigma(x\sin\phi)^{\cdot}\Big[\frac{\xi}{2}C'^2 + \frac{1}{4\xi}(C^2-1)^2\Big] + \sigma\xi C'^2(\sin\phi)^{\cdot}x = 0 \\
&\sigma\Big[-\xi(C'x)^{\cdot}C'\sin\phi + \frac{1}{\xi}(C^2-1)C\dot{C}x\sin\phi\Big] + \lambda\dot{C}x\sin\phi = 0
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&C'^2(\sin\phi)^{\cdot}x + (C'x)^{\cdot}C'\sin\phi - \frac{1}{2}(x\sin\phi)^{\cdot}C'^2 \\
=&(C'x\sin\phi)^{\cdot}C' - \frac{1}{2}(x\sin\phi)^{\cdot}C'^2 \\
=&(C'^{\cdot})C'(x\sin\phi) + \frac{1}{2}(x\sin\phi)^{\cdot}C'^2 \\
=&\frac{1}{2}(C'^2x\sin\phi)^{\cdot}
\end{aligned}
$$

Substitute the second equality into the first one, we have the total integral

$$
\begin{aligned}
&(Q\sin t\cos\phi - H\sin^2\phi + xH^2\sin\phi - px^2)^{\cdot} - (\mu x\sin\phi)^{\cdot} - \lambda(x\sin\phi C)^{\cdot} - \\
&\qquad\qquad \left(x\sin\phi\Big[-\frac{\xi}{2}C'^2 + \frac{1}{4\xi}(C^2-1)^2\Big]\right)^{\cdot} = 0
\end{aligned}
$$

Integrating from 0 to π implies that $x(\pi) = 0$.

3.2 Nonhomogeneous Case Where $c_1 \neq 0$

For the nonhomogeneous case where $c_1 \neq 0$, we can make c_0 dimensionlessly to be 1, then the equations reduce into

$$
\begin{aligned}
&\big[\dot{Q} + \cot tQ + 2H(H^2-K)\big] - (2\mu H + 2p) + \sigma\xi C'^2\phi' - 2\sigma H\Big[\frac{\xi}{2}C'^2 + \frac{1}{4\xi}(C^2-1)^2\Big] - 2\lambda CH \\
&\qquad + \frac{c_1}{\sin t}\left(\frac{x^2(CH)^{\cdot}}{\sin t}\right)^{\cdot} + 2c_1CH(H^2-K) = 0 \\
&c_1H^2x + \sigma\Big[-\xi(C'x)' + \frac{1}{\xi}(C^2-1)Cx\Big] + \lambda x = 0
\end{aligned}
$$

Multiplying both sides of the two equations by $\sin t\cos\phi$ and $\sin\phi$ yields (we here only need consider the extra terms, and ignore the common factor c_1),

$$\begin{aligned}
&(\sin t\cos\phi P)^{\cdot} + P\sin t\sin\phi\dot{\phi} + 2CH(H^2-K)\sin t\cos\phi\\
=&(\sin t\cos\phi P)^{\cdot} + \frac{x^2}{\sin^2 t}(\dot{C}H)\sin t\sin\phi\dot{\phi} + C\left[\frac{x^2}{\sin^2 t}\dot{H}\sin t\sin\phi\dot{\phi} + 2CH(H^2-K)\sin t\cos\phi\right]\\
=&(\sin t\cos\phi P)^{\cdot} + \frac{x^2}{\sin^2 t}(\dot{C}H)\sin t\sin\phi\dot{\phi} + C(xH^2\sin\phi)^{\cdot} - C(H\sin^2\phi)^{\cdot}
\end{aligned}$$

where

$$P = x^2(CH)^{\cdot}/\sin^2 t$$

and

$$-xH^2\sin\phi\dot{C}.$$

Put them together, one has

$$\begin{aligned}
&(\sin t\cos\phi P)^{\cdot} + \frac{x^2}{\sin^2 t}(\dot{C}H)\sin t\sin\phi\dot{\phi} + C(xH^2\sin\phi)^{\cdot} - C(H\sin^2\phi)^{\cdot} - xH^2\sin\phi\dot{C}\\
=&(\sin t\cos\phi P)^{\cdot} + \frac{x^2}{\sin^2 t}(\dot{C}H)\sin t\sin\phi\Big(2H - \frac{\sin\phi}{x}\Big)\frac{\sin t}{x}\\
&\qquad\qquad + C(xH^2\sin\phi)^{\cdot} - C(H\sin^2\phi)^{\cdot} - xH^2\sin\phi\dot{C}\\
=&(\sin t\cos\phi P)^{\cdot} + 2xH^2\sin\phi\dot{C} - H\sin^2\phi\dot{C} + C(xH^2\sin\phi)^{\cdot} - C(H\sin^2\phi)^{\cdot} - xH^2\sin\phi\dot{C}\\
=&(\sin t\cos\phi P)^{\cdot} + xH^2\sin\phi\dot{C} - H\sin^2\phi\dot{C} + C(xH^2\sin\phi)^{\cdot} - C(H\sin^2\phi)^{\cdot}\\
=&(\sin t\cos\phi P + CxH^2\sin\phi - CH\sin^2\phi)^{\cdot}
\end{aligned}$$

Obviously, by integrating the total derivative one can derive that $x(\pi) = 0$.

4 Adhesion of Multi-component membrane

If we take the adhesion effect into the consideration, the axisymmetric membrane has the total energy written as

$$\begin{aligned}
E(x,y,C) := \int_0^\pi (c_0 + c_1 C)H^2 x\sqrt{\dot{x}^2+\dot{y}^2}\ \mathrm{dt} + \sigma\int_0^\pi \left[\frac{\xi}{2}C'^2 + \frac{1}{4\xi}(C^2-1)^2\right]x\sqrt{\dot{x}^2+\dot{y}^2}\ \mathrm{dt}\\
- w\int_0^\pi (c_0 + c_2 C)e^{-y^2/\delta^2}x\sqrt{\dot{x}^2+\dot{y}^2}\ \mathrm{dt}
\end{aligned}$$

subject to

$$\begin{aligned}
x\sqrt{\dot{x}^2+\dot{y}^2} &= \sin t\\
\int_0^\pi x^2\dot{y}\ \mathrm{dt} &= V\\
\int_0^\pi Cx\sqrt{\dot{x}^2+\dot{y}^2}\ \mathrm{dt} &= 0
\end{aligned}$$

4.1 Euler-Lagrange Equation

One can obtain Euler-Lagrange equations as follows:

$$\big[\dot{Q} + \cot tQ + 2\widetilde{H}(H^2 - K)\big] - (2\mu H + 2p) + \sigma\xi C'^2\phi' - 2\sigma H\Big[\frac{\xi}{2}C'^2 + \frac{1}{4\xi}(C^2-1)^2\Big]$$
$$- 2\lambda CH + 2w(1 + c_2C)e^{-y^2/\delta^2}\Big(\frac{y}{\delta^2}\cos\phi + H\Big) = 0$$

$$\dot{\mu} = 0$$

$$c_1H^2x + \sigma\Big[-\xi(C'x)' + \frac{1}{\xi}(C^2-1)Cx\Big] + \lambda x = 0$$

5 Coarsening

To view the coarsening process, or phase separation process, of the membrane with different components (red and blue) mixing up together, we apply a gradient flow approach, namely,

$$C_t = -\frac{\delta E}{\delta C}.$$

By discretizing the time derivative on the left hand, we get

$$\frac{C_{n+1} - C_n}{\Delta t} = -\frac{\delta E}{\delta C}(C_{n+1})$$

which is a implicit Euler method, or one can think of the solution C_{n+1} is a minimizer of the energy

$$E(x, y, C) + \int_0^\pi \frac{|C - C_n|^2}{2\Delta t} x\sqrt{\dot{x}^2 + \dot{y}^2}dt$$

5.1 Euler-Lagrange Equation

One can obtain Euler-Lagrange equations as follows:

$$\big[\dot{Q} + \cot tQ + 2\widetilde{H}(H^2 - K)\big] - (2\mu H + 2p) + \sigma\xi C'^2\phi' - 2\sigma H\Big[\frac{\xi}{2}\dot{C}^2 + \frac{1}{4\xi}(C^2-1)^2\Big]$$
$$- 2\lambda CH + 2w(1 + c_2C)e^{-y^2/\delta^2}\Big(\frac{y}{\delta^2}\cos\phi + H\Big) - \frac{|C - C_n|^2}{2\Delta t}H = 0$$

$$\dot{\mu} = 0$$

$$c_1H^2x + \sigma\Big[-\xi(C'x)' + \frac{1}{\xi}(C^2-1)Cx\Big] + \lambda x + \frac{C - C_n}{\Delta t}x = 0$$

6 Leonard-Jones Potential

Another way to eliminate the protrusion of the membrane shapes is to use some other adhesion potentials. One typical choice is Leonard-Jones potential

$$W(\mathbf{x}) = w(1 + c_2\eta) \cdot 4\zeta\left[\left(\frac{\beta}{\mathrm{d}(\mathbf{x})}\right)^{\alpha} - \left(\frac{\beta}{\mathrm{d}(\mathbf{x})}\right)^{\alpha/2}\right] \tag{4}$$

The key difference between the exponential potential and Leonard-Jones potential is that exponential potential is globally attractive, while there is a narrow region $\mathrm{d}(\mathbf{x}) \in (0, \beta)$ where Leonard-Jones potential is repulsive. Such a repulsive region can prevent the cell membrane to protrude into the substrate.

Total energy with Leonard-Jones potential is:

$$E(x, y, \eta) := \int_0^{\pi} (1 + c_1\eta)H^2 x\sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} + \sigma\int_0^{\pi}\left[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2 - 1)^2\right]x\sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt}$$
$$+ \int_0^{\pi} w(1 + c_2\eta) \cdot 4\zeta\left[\left(\frac{\beta}{x}\right)^{\alpha} - \left(\frac{\beta}{x}\right)^{\alpha/2}\right]\ \mathrm{dt}$$

With the Leonard-Jones potential, we have the following Euler-Lagrange equation:

$$\left[\dot{Q} + \cot tQ + 2\widetilde{H}(H^2 - K)\right] - (2\mu H + 2p) + \sigma\xi\eta'^2\phi' - 2\sigma H\left[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2 - 1)^2\right]$$
$$- 2\lambda\eta H - w(1 + c_2\eta)\left\{\frac{4\zeta\beta}{x^2}\left[\alpha\left(\frac{\beta}{x}\right)^{\alpha-1} - \frac{\alpha}{2}\left(\frac{\beta}{x}\right)^{\alpha/2-1}\right]\cos\phi + 8\zeta H\left[\left(\frac{\beta}{x}\right)^{\alpha} - \left(\frac{\beta}{x}\right)^{\alpha/2}\right]\right\} = 0$$

$$\dot{\mu} = 0$$

$$c_1H^2x + \sigma\left[-\xi(\eta' x)' + \frac{1}{\xi}(\eta^2 - 1)\eta x\right] + \lambda x + c_2 w \cdot 4\zeta\left[\left(\frac{\beta}{\mathrm{d}(\mathbf{x})}\right)^{\alpha} - \left(\frac{\beta}{\mathrm{d}(\mathbf{x})}\right)^{\alpha/2}\right] = 0$$

7 Double Obstacle Potential In Interfacial Energy

Since the double well potential

$$P(\eta) = \frac{1}{4\xi}(\eta^2 - 1)^2$$

can not fix η at ± 1 very well. Here we can consider another potential, which is so called double obstacle potential which is given by

$$P(\eta) = \alpha(1 - \eta^2) + (1 + \eta)\ln(1 + \eta) + (1 - \eta)\ln(1 - \eta) - 2\ln 2$$

where $P(\pm 1) = 0$ and $P(\eta)$ attains the minimum at η_1, η_2 which satisfy

$$P'(\eta) = \ln\frac{1 + \eta}{1 - \eta} - 2\alpha\eta = 0.$$

For the double obstacle potential, we first need to figure out the equilibrium solution of the equation

$$\xi\eta'' + \ln\frac{1+\eta}{1-\eta} - 2\alpha\eta = 0$$

The Euler-Lagrange equations for the double well potential combined with Leonard-Jones is derived from minimizing the following energy

$$\begin{aligned}
E(x,y,\eta) := &\int_0^\pi (1+c_1\eta)H^2 x\sqrt{\dot{x}^2+\dot{y}^2}\ \mathrm{dt} \\
&+\sigma\int_0^\pi \Big[\frac{\xi}{2}\eta'^2 + \alpha(1-\eta^2) + (1+\eta)\ln(1+\eta) + (1-\eta)\ln(1-\eta) - 2\ln 2\Big] x\sqrt{\dot{x}^2+\dot{y}^2}\ \mathrm{dt} \\
&\qquad + \int_0^\pi w(1+c_2\eta)\cdot 4\zeta\left[\left(\frac{\beta}{x}\right)^\alpha - \left(\frac{\beta}{x}\right)^{\alpha/2}\right]\ \mathrm{dt}
\end{aligned}$$

Actually we have

$$\begin{aligned}
&\big[\dot{Q} + \cot tQ + 2\widetilde{H}(H^2-K)\big] - (2\mu H + 2p + 2\lambda\eta H) + \sigma\xi\eta'^2\phi' \\
&-2\sigma H\Big[\frac{\xi}{2}\eta'^2 + \alpha(1-\eta^2) + (1+\eta)\ln(1+\eta) + (1-\eta)\ln(1-\eta) - 2\ln 2\Big] \\
&\quad -w(1+c_2\eta)\left\{\frac{4\zeta\beta}{x^2}\Big[\alpha\Big(\frac{\beta}{x}\Big)^{\alpha-1} - \frac{\alpha}{2}\Big(\frac{\beta}{x}\Big)^{\alpha/2-1}\Big]\cos\phi + 8\zeta H\Big[\Big(\frac{\beta}{x}\Big)^\alpha - \Big(\frac{\beta}{x}\Big)^{\alpha/2}\Big]\right\} = 0 \\
&\dot{\mu} = 0 \\
&c_1H^2x + \sigma\Big[-\xi(\eta' x)' + \Big(\ln\frac{1+\eta}{1-\eta} - 2\alpha\eta\Big)x\Big] + \lambda x + c_2 w\cdot 4\zeta\left[\left(\frac{\beta}{\mathrm{d}(\mathbf{x})}\right)^\alpha - \left(\frac{\beta}{\mathrm{d}(\mathbf{x})}\right)^{\alpha/2}\right] = 0
\end{aligned}$$

8 Anisotropic Energy

Consider the anisotropic energy

$$E(x,y,\eta) := \int_0^\pi \big(H + \alpha\eta(k_1 - k_2)\big)^2 x\sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt} + \sigma \int_0^\pi \Big[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2 - 1)^2\Big] x\sqrt{\dot{x}^2 + \dot{y}^2}\ \mathrm{dt}$$

The Euler-Lagrange equation is

$$\big[\dot{Q} + \cot tQ + 2H(H^2 - K)\big] - (2\mu H + 2p + 2\lambda\eta H) + \sigma\xi\eta'^2\phi' - 2\sigma H\Big[\frac{\xi}{2}\eta'^2 + \frac{1}{4\xi}(\eta^2 - 1)^2\Big]$$
$$+ 4\alpha\eta\Big[\dot{Q} + \cot tQ + \frac{(k_2\cos\phi)^{\cdot}}{\sin t} + \frac{1}{2}(k_1^2 + k_2^2)(k_1 - H)\Big] + 4\alpha^2\eta^2\big[\dot{Q} + \cot tQ + 2H(H^2 - K)\big] = 0$$

and

$$2\big(H + \alpha\eta(k_1 - k_2)\big)\cdot\alpha(k_1 - k_2)x + \sigma\Big[-\xi(C'x)' + \frac{1}{\xi}(C^2 - 1)Cx\Big] + \lambda x = 0$$