Incidence Hopf Algebras

William R. Schmitt

University of Memphis, Memphis, Tennessee 38152

Abstract: We present several results about incidence Hopf algebras of families of partially ordered sets, including a characterization of their algebra structure, a combinatorial technique for finding generating sets of primitive elements in the cocommutative case, and a determinant formula for the antipode which holds for a class including the Faà di Bruno Hopf algebra. We introduce a variety of examples of incidence Hopf algebras arising from families of graphs, matroids and distributive lattices, many of which generalize well-known Hopf algebras.

1 Introduction

For several years after the introduction of the notion of Hopf algebra, practically the only Hopf algebras that were dealt with were the ones obtained from groups and Lie algebras, and from some other special algebraic situations, such as the problems of algebraic number theory arising from the study of Witt vectors. Algebraic topology gradually contributed an increasingly complex variety of new Hopf algebras, which remain to this day a challenge to whoever were to nourish dreams of classification. Lastly, combinatorics began to pile up an impressive array of disparate constructions, which today, in retrospect, can be subsumed under the capacious umbrella of Hopf algebra.

It is the purpose of this paper to develop the theory underlying the Hopf algebras arising in combinatorics, giving special regard to new identities which can be obtained, and known identities which can be made obvious, by using the Hopf algebra formalism.

The first attempt at bringing Hopf algebra techniques to bear on combinatorics is the paper by Joni and Rota [13], which amounts essentially to a list of all combinatorial structures known to the authors which could be given – or which were naturally endowed with – a coalgebra or a bialgebra structure. Unfortunately, the authors did not contribute any structural considerations; indeed, they were unaware of the fact that almost all bialgebras considered in their paper are actually Hopf algebras. To be sure, the antipodes of these bialgebras are sometimes far from obvious, as the present author showed in his thesis [18].

The main thrust of the present paper is a classification of incidence Hopf algebras of partially ordered sets. In so doing, we were led to some results of independent interest, such as the logarithm formula for the determination of primitive elements (theorem 9.4, generalizing previous results of Ree [14] and Reutenauer [15]), a determinant formula for the antipode (theorem 8.1), and several special results relating to partially commutative Hopf algebras, which are natural generalizations of the partially commutative monoids introduced by Cartier and Foata [4]. In the wake of this classification, we study some Hopf algebras arising naturally from graphs and trees, which simplify some pioneering work of Grossmann and Larson [9].

We hope that the present work will be instrumental in including Hopf algebra techniques among the standard technical baggage of every combinatorialist, and in providing the algebraist a ground on which to test future conjectures on the structure of Hopf algebras.

2 Synopsis

Some familiarity with the elementary properties of Hopf algebras, which can be found in either Abe [1] or Sweedler [21], is assumed. However, the notion of incidence Hopf algebra is developed fully in the next three sections, as follows. Let K be a commutative ring with unit and let $\mathcal{P}$ be a set of finite intervals (partially ordered sets having unique minimal and maximal elements) which is closed under formation of subintervals. The *incidence coalgebra* of $\mathcal{P}$, over K, is the free K-module on the quotient of $\mathcal{P}$ by a suitable equivalence relation, with coproduct deriving from the natural splitting of an interval $[x, y]$ into all pairs of subintervals $([x, z], [z, y])$, for $x \leq z \leq y$. Whenever the set of intervals $\mathcal{P}$ is *hereditary*, that is, closed under direct product as well as under formation of subintervals, then direct product induces a bialgebra structure on the incidence coalgebra. If, in addition, the given equivalence relation on intervals meets a certain condition (satisfied by the isomorphism relation, in particular) then the incidence coalgebra is an irreducible

Hopf algebra, called the *incidence Hopf algebra* of $\mathcal{P}$. The antipode of an incidence Hopf algebra is always an involution, and an incidence coalgebra or Hopf algebra is filtered, according to the lengths of intervals in the family $\mathcal{P}$. In general, this filtration is different from the coradical filtration. If the intervals in $\mathcal{P}$ are graded, then their ranks determine a grading of the incidence Hopf algebra. For any incidence coalgebra C, an incidence Hopf algebra H can be constructed which contains C as a subcoalgebra and which is isomorphic to the free irreducible, commutative Hopf algebra over C.

In section six it is shown that, as algebras, incidence Hopf algebras are isomorphic to monoid algebras of free partially commutative monoids, i.e., free partially commutative algebras. In fact, any free partially commutative algebra can be given an incidence Hopf algebra structure. This characterizes the algebra structure of incidence Hopf algebras. On the other hand, it is shown in section seven that the dual algebra of an incidence Hopf algebra corresponds to a reduced incidence algebra in the sense of [7].

In section eight, a class of incidence Hopf algebras is introduced for which a determinant formula for the antipode holds. This class contains many well-known Hopf algebras, including the Faà di Bruno Hopf algebra. In the Faà di Bruno case, this antipode formula yields a determinant expression for the coefficients of the inverse of a formal power series under functional composition.

The structure of general irreducible cocommutative Hopf algebras is examined in section nine. When such a Hopf algebra H is *divisible* (a property generalizing being over a characteristic zero ring), there is a projection λ from H onto its Lie algebra of primitive elements. The map λ is useful for finding generating sets of primitive elements of H. These results are applied to cocommutative incidence Hopf algebras in section ten; in particular, divisibility of cocommutative incidence Hopf algebras is characterized combinatorially, and a combinatorial formula for the projection λ is obtained.

Basic examples of incidence Hopf algebras appear throughout the first eleven sections. The last four sections are dedicated to a detailed examination of various classes of incidence Hopf algebras arising from families of graphs, matroids and distributive lattices. These sections are intended to provide algebraists with concrete new examples of Hopf algebras, and to show combinatorialists the kinds of algebraic structure underlying many families of familiar objects.

3 Incidence Coalgebras

If P is a partially ordered set, or *poset*, for short, and $x \leq y$ in P, the *interval* $[x, y]$ is the set $\{z \in P : x \leq z \leq y\}$. P is *locally finite* if all of its intervals are finite. If the poset P is an interval, then the unique minimal and maximal elements of P are denoted by 0_P and 1_P, respectively. All posets considered in this paper will be locally finite intervals, unless specifically stated otherwise. In order to avoid possible set theoretic difficulties, we will assume that all posets and other combinatorial objects have underlying sets which are contained in some fixed universal set. Thus any family, or class, of structures we consider will actually be a set.

A family of posets $\mathcal{P}$ is *interval closed*, if it is non-empty and, for all $P \in \mathcal{P}$ and $x \leq y \in P$, the interval $[x, y]$ belongs to $\mathcal{P}$. An *order compatible relation* on an interval closed family $\mathcal{P}$ is an equivalence relation $\sim$ such that, whenever $P \sim Q$ in $\mathcal{P}$, there exists a bijection $\varphi : P \longrightarrow Q$ such that $[0_P, x] \sim [0_Q, \varphi(x)]$ and $[x, 1_P] \sim [\varphi(x), 1_Q]$, for all $x \in \mathcal{P}$. The map φ is called an *order compatible* bijection from P to Q, and in general depends on P and Q.

Poset isomorphism is an obvious example of an order compatible relation. In most examples considered here, the family $\mathcal{P}$ will consist of posets with additional structure, and $P \sim Q$ in $\mathcal{P}$ will mean that there exists an isomorphism from P to Q which preserves the additional structure. It is not the case, however, that posets which are related by an order compatible relation are necessarily isomorphic. For example, let $\mathcal{P}$ be the set of all intervals in either of the posets P_1 and P_2 whose Hasse diagrams are shown in figure 1. For $Q, R \in \mathcal{P}$, define $Q \sim R$ to mean that Q and R are isomorphic, or $Q = P_1$ and $R = P_2$. Then $\sim$ is order compatible and $P_1 \sim P_2$, but P_1 and P_2 are not isomorphic as posets.

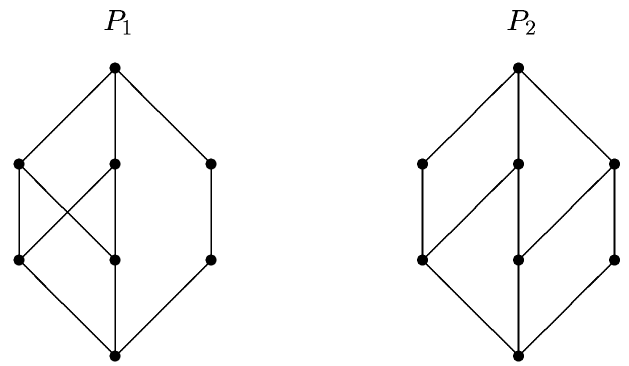

FIGURE 1

Suppose K is a commutative ring with 1, and $\sim$ is an order compatible relation on an interval closed family $\mathcal{P}$. The quotient set $\mathcal{P}/\sim$ is denoted by $\tilde{\mathcal{P}}$. The $\sim$-equivalence class, or *type*, of a poset $P \in \mathcal{P}$ is denoted by $[P]$. If $P = [x, y]$, then we write $[x, y]$ to denote either P or $[P]$; it should always be clear from the context which is meant. Let $C(\mathcal{P})$ denote free K-module generated by $\tilde{\mathcal{P}}$. Define linear maps $\Delta : C(\mathcal{P}) \longrightarrow C(\mathcal{P}) \otimes C(\mathcal{P})$ and $\epsilon : C(\mathcal{P}) \longrightarrow K$ by

$$\Delta[P] = \sum_{x \in P} [0_P, x] \otimes [x, 1_P]$$

and

$$\epsilon[P] = \begin{cases} 1 & \text{if } |P| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

for all $[P] \in \tilde{\mathcal{P}}$.

Theorem 3.1 *If $\sim$ is an order compatible relation on an interval closed family $\mathcal{P}$, then $C(\mathcal{P})$ is a K-coalgebra with comultiplication Δ and counit ϵ.*

Proof The fact that Δ is well-defined is a direct consequence of the fact that $\sim$ is order compatible. It follows immediately from the definitions that

$$(\Delta \otimes I) \circ \Delta[P] = (I \otimes \Delta) \circ \Delta[P] = \sum_{x \leq y \in P} [0_P, x] \otimes [x, y] \otimes [y, 1_P],$$

and

$$(\epsilon \otimes I) \circ \Delta[P] = (I \otimes \epsilon) \circ \Delta[P] = [P],$$

for all $[P] \in \tilde{\mathcal{P}}$. Therefore $C(\mathcal{P})$ is a coalgebra. $\square$

$C(\mathcal{P})$ is called the *incidence coalgebra* of the family $\mathcal{P}$ (*modulo the relation* $\sim$).

For all $[P], [Q], [R] \in \tilde{\mathcal{P}}$, the *section coefficient* $([P]; [Q], [R]) = (P; Q, R) \in K$ is the coefficient of $[Q] \otimes [R]$ in $\Delta[P]$, i.e., the number of elements $x \in P$ such that $[0_P, x] \sim Q$ and $[x, 1_P] \sim R$. Thus $\Delta[P]$ can be written as

$$\sum_{[Q],[R] \in \tilde{\mathcal{P}}} (P; Q, R)[Q] \otimes [R],$$

and so

$$\begin{aligned} (\Delta \otimes I) \circ \Delta[P] &= \sum_{[T],[S]} (P; T, S) \left(\sum_{[Q],[R]} (T; Q, R) Q \otimes R \right) \otimes S \\ &= \sum_{[Q],[R],[S]} \left(\sum_{[T]} (P; T, S)(T; Q, R) \right) Q \otimes R \otimes S, \end{aligned}$$

and

$$\begin{aligned} (I \otimes \Delta) \circ \Delta[P] &= \sum_{[Q],[U]} (P; Q, U) Q \otimes \left(\sum_{[R],[S]} (U; R, S) R \otimes S \right) \\ &= \sum_{[Q],[R],[S]} \left(\sum_{[U]} (P; Q, U)(U; R, S) \right) Q \otimes R \otimes S. \end{aligned}$$

Therefore the coassociativity of Δ is equivalent to the identity

$$\sum_{[U] \in \tilde{\mathcal{P}}} (P; Q, U)(U; R, S) = \sum_{[T] \in \tilde{\mathcal{P}}} (P; T, S)(T; Q, R),$$

for all $[P], [Q], [R], [S] \in \tilde{\mathcal{P}}$. The common value of the two sides of the above is denoted by $(P; Q, R, S)$. Similarly, one can define *multisection coefficients* $(P; Q_1, \ldots, Q_n)$, for any $n \geq 3$. The following example shows that section coefficients generalize binomial coefficients.

Example 3.1 (Binomial Coalgebra) Let $\mathcal{B}$ be the family of finite boolean algebras (i.e., posets which are isomorphic to lattices of subsets of finite sets, ordered by inclusion) and let $\sim$ be the isomorphism relation on $\mathcal{B}$. If V is a finite set and $U \subseteq W \subseteq V$, then the isomorphism class of the interval $[U, W]$ is uniquely determined by the cardinality of $W - U$. If $|W - U| = n$, let x_n denote this type. Then the incidence coalgebra $C(\mathcal{B})$ is the free module $K\{x_0, x_1, x_2, \ldots\}$, with coproduct Δ and counit ϵ given by

$$\Delta(x_n) = \sum_{k=0}^{n} \binom{n}{k} x_k \otimes x_{n-k},$$

and

$$\epsilon(x_n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \geq 0$. The coalgebra $C(\mathcal{B})$ is called the *binomial coalgebra*.

Recall that a *filtration* of a coalgebra C is a sequence of submodules of C, $C_0 \subseteq C_1 \subseteq \cdots$, such that $C = \cup_{n \geq 0} C_n$ and $\Delta(C_n) \subseteq \sum_{k=1}^{n} C_k \otimes C_{n-k}$, for all $n \geq 1$.

For any poset P, the *length* $l(P)$ of P is defined to be one less than the largest number of elements ocurring in any chain in P. Suppose $C(\mathcal{P})$ is an incidence coalgebra. If $P, Q \in \mathcal{P}$ and $P \sim Q$, then $l(P) = l(Q)$, so l is well-defined on the set of types $\tilde{\mathcal{P}}$. For all integers $n \geq 0$, let C_n be the submodule of $C(\mathcal{P})$ generated by those types $[P]$ in $\tilde{\mathcal{P}}$ such that $l([P]) \leq n$.

Proposition 3.2 *If $\mathcal{P}$ is an interval closed family of posets with order compatible relation $\sim$, then the sequence $C_0 \subseteq C_1 \subseteq \cdots$ is a filtration of the coalgebra $C(\mathcal{P})$.*

Proof If P is any poset and $x \in P$, then $l([0_P, x]) + l([x, 1_P]) \leq l(P)$. Therefore $\Delta(C_n) \subseteq \sum_{k=1}^{n} C_k \otimes C_{n-k}$. □

A chain $x_0 < x_1 < \cdots < x_n$ in a poset is *saturated* if, for $1 \leq i \leq n$, $x_{i-1} \leq y \leq x_i$ implies $y = x_{i-1}$ or $y = x_i$. A poset P is *graded* if the lengths of all saturated chains between 0_P and 1_P are the same, in which case the length of P is called the *rank* of P and denoted by $r(P)$. If an interval closed family $\mathcal{P}$ consists of graded posets, then the rank function is well-defined on types. For all $n \geq 0$, let $C(n) \subseteq C(\mathcal{P})$ be the submodule of $C(\mathcal{P})$ generated by types in $\tilde{\mathcal{P}}$ of rank n.

Proposition 3.3 *If $\mathcal{P}$ is an interval closed family of graded posets with order compatible relation $\sim$, then $C(\mathcal{P}) = \bigoplus_{n \geq 0} C(n)$ is a graded coalgebra.*

Proof If P is a graded poset and $x \in P$, then $r([0_P, x]) + r([x, 1_P]) = r(P)$. Hence $\Delta(C(n)) \subseteq \sum_{k=1}^{n} C(k) \otimes C(n-k)$, for all $n \geq 0$. Also, it is clear that $\epsilon(C(n)) = 0$, for $n \neq 0$. □

4 Incidence Hopf Algebras

The *direct product* of posets P_1 and P_2 is the cartesian product $P_1 \times P_2$, partially ordered by the relation $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_i \leq y_i$ in P_i, for $i = 1, 2$. We will always make the necessary identifications so that direct product is an associative (but not necessarily commutative) operation.

A *hereditary family* is an interval closed family of posets $\mathcal{P}$ which is also closed under formation of direct products. Suppose $\mathcal{P}$ is a hereditary family and $\mathcal{P}_\circ = \{R \in \mathcal{P} : R \neq P \times Q, \text{ for any} P, Q \in \mathcal{P}\}$ is the set of *indecomposable* elements of $\mathcal{P}$. It follows that $\mathcal{P}$ is a semigroup under direct product, which is generated by $\mathcal{P}_\circ$. Let $\sim$ be an order compatible relation on $\mathcal{P}$ which is also a semigroup congruence, i.e., whenever $P \sim Q$ in $\mathcal{P}$, then $P \times R \sim Q \times R$ and $R \times P \sim R \times Q$, for all $R \in \mathcal{P}$. Then the set of types $\tilde{\mathcal{P}} = \mathcal{P}/\sim$ is a semigroup, with product induced by direct product of posets. The congruence $\sim$ is *reduced* if, whenever $P, Q \in \mathcal{P}$ and $|Q| = 1$, then $P \times Q \sim Q \times P \sim P$. In this case, the set of types $\tilde{\mathcal{P}}$ is a monoid, with identity element 1 equal to the type of any one point interval.

An order compatible relation on a hereditary family $\mathcal{P}$ which is also a reduced congruence is called a *Hopf relation* on $\mathcal{P}$.

Suppose K is a commutative ring with 1, and $\sim$ is a Hopf relation on a hereditary family $\mathcal{P}$. The monoid structure of $\tilde{\mathcal{P}}$ induces a product on the incidence coalgebra $C(\mathcal{P})$, making $C(\mathcal{P})$ an algebra, isomorphic to the monoid algebra of $\tilde{\mathcal{P}}$ over K. Let $H(\mathcal{P})$ denote $C(\mathcal{P})$ together with this algebra structure.

Theorem 4.1 ([18]) *If $\mathcal{P}$ is a hereditary family and $\sim$ is a Hopf relation on $\mathcal{P}$, then $H(\mathcal{P})$ is a Hopf algebra over K. The antipode $S : H(\mathcal{P}) \longrightarrow H(\mathcal{P})$ is given by*

$$S[P] = \sum_{k \geq 0} \sum_{\substack{x_0 < \cdots < x_k \\ x_0 = 0_P \\ x_k = 1_P}} (-1)^k \prod_{i=1}^{k} [x_{i-1}, x_i], \tag{4.1}$$

for all $[P] \in \tilde{\mathcal{P}}$.

Proof The fact that Δ is an algebra map (and thus $H(\mathcal{P})$ is a bialgebra) follows directly from the observation that if P_1 and P_2 are posets and $(x_1, x_2) \leq (y_1, y_2)$ in $P_1 \times P_2$, then $[(x_1, x_2), (y_1, y_2)] = [x_1, y_1] \times [x_2, y_2]$.

It is easy to show that the operator $S' : H(\mathcal{P}) \longrightarrow H(\mathcal{P})$ defined recursively by $S'(1) = 1$ and

$$S'[P] = - \sum_{\substack{x \in P \\ x \neq 0_P}} [0_P, x] S'[x, 1_P]$$

or, dually,

$$S'[P] = - \sum_{\substack{x \in P \\ x \neq 1_P}} (S'[0_P, x])[x, 1_P]$$

for all $[P] \neq 1$ in $\mathcal{P}$, is an antipode for $H(\mathcal{P})$. The operator S defined by equation (4.1) satisfies these recursion relations, and is thus equal to the antipode S'. □

$H(\mathcal{P})$ is the *incidence hopf algebra* of the family $\mathcal{P}$ (*modulo the relation* $\sim$).

Because of the fact that Δ is an algebra map, the section coefficients satisfy the additional identity

$$(P \times Q; R, S) = \sum_{\substack{[R_1][R_2] = [R] \\ [S_1][S_2] = [S]}} (P; R_1, S_1)(Q; R_2, S_2),$$

for all $[P], [Q], [R], [S] \in \tilde{\mathcal{P}}$.

Example 4.1 (Binomial Hopf Algebra) Let $\mathcal{B}$ be the family of finite boolean algebras and let $\sim$ be isomorphism, as in example 3.1. If U and V are finite sets, then the direct product of the lattices of subsets of U and V is isomorphic to the lattice of subsets of the disjoint union $U + V$. Thus $\mathcal{B}$ is a hereditary family. Let x denote the isomorphism class of the lattice of subsets of a one element set. The incidence Hopf algebra $H(\mathcal{B})$ is isomorphic to the polynomial algebra $K[x]$, with coproduct Δ and counit ϵ given by

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k},$$

and

$$\epsilon(x^n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \geq 0$. The antipode of $H(\mathcal{B})$ is determined by $S(x) = -x$. $H(\mathcal{B})$ is called the *binomial Hopf algebra.* G.-C. Rota's umbral calculus, which provides a unified framework for studying polynomial sequences of binomial type (see, e.g., [16]), is essentially the study of the Hopf algebra $H(\mathcal{B})$.

Example 4.2 (Free Partially Commutative Hopf Algebras) Let A be any set, and let A^* denote the free monoid on A. Identify A in the usual manner with a subset of A^*. Suppose θ is a symmetric relation on A. The quotient of A^* by the congruence generated by the set of all pairs (xy, yx), for $(x, y) \in \theta$, is called the *free partially commutative monoid on* A, (*modulo the commutation relation* θ). Identify each element x of A with the boolean algebra of subsets of the the set $\{x\}$. Let B be the set of one-point intervals in these boolean algebras. The free semigroup $(A \cup B)^+$, consisting of all non-empty elements of $(A \cup B)^*$, is a hereditary family of posets. Let ρ be the symmetric relation on $(A \cup B)^+$ containing all pairs (xb, x) and (bx, x), for $x \in A$ and $b \in B$. The congruence $\sim$ on $(A \cup B)^+$ generated by $\theta \cup \rho$ is a Hopf relation, and the set of types $(A \cup B)^+/\sim$ is naturally identified with the free partially commutative monoid on A modulo θ. The incidence Hopf algebra of $(A \cup B)^+$, modulo $\sim$, is denoted by $K\langle A, \theta\rangle$, and is called the *Hopf algebra of partially commutative polynomials on A, modulo* θ. The coproduct of $K\langle A, \theta\rangle$ is determined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, for all $x \in A$.

For any word $w \in A^*$, let $[w] \in K\langle A, \theta\rangle$ be the partially commutative word represented by w, let $\tilde{w}$ be the reverse of w, and let $|w|$ denote the number of letters in w. Since the antipode S satisfies $S(x) = -x$ for all $x \in A$, and antipodes are always algebra anti-isomorphisms, it follows that

$$S[w] = (-1)^{|w|}[\tilde{w}], \tag{4.2}$$

for all partially commutative words $[w]$.

The Hopf algebras $K\langle A, \theta\rangle$ were introduced in [19], where identities involving partially commutative words were derived by comparing formulas (4.2) and (4.1) for the antipode.

In general, if a non-trivial Hopf algebra H is either commutative or cocommutative then the antipode of H has order two. Even though incidence Hopf algebras are generally neither commutative nor cocommutative, they still have this property.

Proposition 4.2 *The antipode S of an incidence Hopf algebra $H(\mathcal{P})$ satisfies $S \circ S = I$.*

Proof If $P \in \mathcal{P}$ has length zero, then clearly $S \circ S[P] = [P]$. If $l(P) \geq 1$, then by applying equation 4.1 twice, we have

$$\begin{aligned} S \circ S[P] &= \sum_{k \geq 0} \sum_{\substack{x_0 < \cdots < x_k \\ x_0 = 0_P \\ x_k = 1_P}} (-1)^k \sum_{U \subseteq \{1, \ldots, k-1\}} (-1)^{|U|+1} \prod_{i=1}^{k} [x_{i-1}, x_i] \\ &= \sum_{k \geq 0} \sum_{\substack{x_0 < \cdots < x_k \\ x_0 = 0_P \\ x_k = 1_P}} (-1)^{k+1} \delta_{1,k} \prod_{i=1}^{k} [x_{i-1}, x_i] \\ &= [P]. \end{aligned}$$

□

If $H(\mathcal{P})$ is an incidence Hopf algebra, let H_n be the submodule of $H(\mathcal{P})$ generated by those types in $\tilde{\mathcal{P}}$ having length less than or equal to n, for all $n \geq 0$. As we have seen before, $H_0 \subseteq H_1 \subseteq \cdots$ is a coalgebra filtration of $H(\mathcal{P})$. Furthermore, for any posets P and Q, $l(P \times Q) = l(P) + l(Q)$, and therefore $H_n H_k \subseteq H_{n+k}$, for all $n, k \geq 0$. Therefore we have the following proposition.

Proposition 4.3 *If $\mathcal{P}$ is a hereditary family of posets with order compatible relation $\sim$, then the sequence $H_0 \subseteq H_1 \subseteq \cdots$ is a filtration of the Hopf algebra $H(\mathcal{P})$.*

The filtration $H_0 \subseteq H_1 \subseteq \cdots$ is called the *length* filtration of $H(\mathcal{P})$. The length filtration differs from the coradical filtration of $H(\mathcal{P})$ because there are, in general, many primitive elements of $H(\mathcal{P})$ which are not contained in H_1 (see section 9).

If $\mathcal{P}$ consists of graded posets, let $H(n)$ be the submodule of $H(\mathcal{P})$ generated by types of rank n. Then $H(n)H(k) \subseteq H(n+k)$, for all $n, k \geq 0$. Therefore we have:

Proposition 4.4 *If $\mathcal{P}$ is a hereditary family of graded posets, then $H(\mathcal{P}) = \bigoplus_{n \geq 0} H(n)$ is a graded Hopf algebra.*

The grading $H(\mathcal{P}) = \bigoplus_{n \geq 0} H(n)$ is called the *rank* grading of $H(\mathcal{P})$.

Remarks:

Suppose $H(\mathcal{P})$ is an incidence Hopf algebra with length filtration $H_0 \subseteq H_1 \subseteq \cdots$. Because there is only one type of one-point interval in $\tilde{\mathcal{P}}$, it follows that $H_0 \cong K$, and therefore $H(\mathcal{P})$ is an irreducible Hopf algebra. A more general definition of incidence Hopf algebra results by replacing the condition that the order compatible congruence $\sim$ be reduced with the following, weaker, requirement: there exists $Q \in \mathcal{P}$ such that $P \sim P \times Q \sim Q \times P$, for all $P \in \mathcal{P}$, and for all $P \in \mathcal{P}$ with $|P| = 1$, there exists $R \in \mathcal{P}$ such that $P \times R \sim R \times P \sim Q$. In this case, the set of one point types is a group with identity element $[Q]$, and $H(\mathcal{P})$ is a Hopf algebra which is not necessarily irreducible.

One may define *incidence bialgebras* by weakening the condition that $\sim$ is reduced still further, requiring only that there exists $Q \in \mathcal{P}$ such that $P \sim P \times Q \sim Q \times P$, for all $P \in \mathcal{P}$. In this case, the set of one point types in $\tilde{\mathcal{P}}$ is a monoid with identity element $[Q]$, and $H(\mathcal{P})$ is a bialgebra, which, in general, does not have an antipode.

We will only consider reduced order compatible congruences (i.e. Hopf relations) in this paper, partly because one can say more about the structure of $H(\mathcal{P})$ in this case (theorem 6.4, for example), but also because this is a natural requirement from the point of view of combinatorics.

5 Free Commutative Incidence Hopf Algebras

Suppose $\mathcal{P}$ is an interval closed family of posets such that, whenever $P, Q \in \mathcal{P}$, the product $P \times Q$ does *not* belong to $\mathcal{P}$. Let $\mathcal{P}^*$ denote the family of all finite direct products of posets in $\mathcal{P}$. Then $\mathcal{P}^*$ is a hereditary family. Let $\sim$ be an order compatible relation on $\mathcal{P}$. We extend the definition of $\sim$ to all of $\mathcal{P}^*$ as follows: Given $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ in $\mathcal{P}$, let A be the set of all indices i such that $|P_i| \neq 1$, and let B be the set of all indices j such that $|Q_j| \neq 1$. Define $P_1 \times \cdots \times P_n \sim Q_1 \times \cdots \times Q_m$ in $\mathcal{P}^*$ whenever there exists a bijection $\sigma : A \longrightarrow B$ such that $P_i \sim Q_{\sigma(i)}$, for all $i \in A$. It follows that the extended relation $\sim$ is a Hopf relation on $\mathcal{P}^*$.

The incidence Hopf algebra $H(\mathcal{P}^*)$ is called the *free commutative incidence Hopf algebra* of $\mathcal{P}$ (*modulo* $\sim$). Let $\tilde{\mathcal{P}}_1 \subseteq \tilde{\mathcal{P}}$ denote the set of types of non-singleton posets in $\mathcal{P}$. As an algebra $H(\mathcal{P}^*)$ is isomorphic to the polynomial algebra $K[\tilde{\mathcal{P}}_1]$. $H(\mathcal{P}^*)$ is the free, irreducible, commutative Hopf algebra on the coalgebra $C(\mathcal{P})$, in the sense that any coalgebra map from $C(\mathcal{P})$ into an irreducible, commutative Hopf algebra factors uniquely through the inclusion $C(\mathcal{P}) \hookrightarrow H(\mathcal{P}^*)$.

One may similarly define free, and free partially commutative, incidence Hopf algebras of an interval closed family modulo order compatible relation.

Example 5.1 (Standard Incidence Hopf Algebra of a Poset) Suppose P is any poset, not necessarily having unique minimal and maximal elements. The set $I(P)$ of all intervals in P is obviously interval closed. Let $\sim$ be the isomorphism relation on $I(P)$. The free commutative incidence Hopf algebra $H(I(P)^*)$ is called the *standard incidence Hopf algebra* of P, and denoted by $H(P)$.

For example, suppose P is an interval of length one. Let x denote the isomorphism class of P. Then $H(P) \cong K[x]$, the binomial Hopf algebra (see example 4.1).

Example 5.2 (Linear Orders) Suppose $\mathcal{L}_\circ$ is the family of all finite linearly ordered sets, and $\sim$ is the isomorphism relation on $\mathcal{L}_\circ$. Let x_n denote the type of a linearly ordered set of length n, and let $\mathcal{L} = \mathcal{L}_\circ^*$ be the set of all finite direct products of elements of $\mathcal{L}_\circ$. The free commutative incidence Hopf algebra $H(\mathcal{L})$ is isomorphic to the polynomial algebra $K[x_1, x_2, \ldots]$, with coproduct given by

$$\Delta(x_n) = \sum_{k=0}^{n} x_k \otimes x_{n-k},$$

for all $n \geq 0$. According to equation 4.1, the antipode of $H(\mathcal{L})$ is given by

$$S(x_n) = \sum_{k \geq 1} (-1)^k \sum_{\substack{n_1 + \cdots + n_k = n \\ n_j \geq 1}} x_{n_1} \cdots x_{n_k}, \tag{5.1}$$

for all $n \geq 1$.

6 Hereditary Families of Posets

In this section, we show that the monoid of types of any hereditary family modulo a Hopf relation is free partially commutative. Thus any incidence Hopf algebra is the monoid algebra of a free partially commutative monoid.

If $\mathcal{P}$ is a hereditary family with Hopf relation $\sim$, the set of indecomposable elements of the monoid $\tilde{\mathcal{P}}$ is denoted by $\tilde{\mathcal{P}}_\circ$. Note that $1 \notin \tilde{\mathcal{P}}_\circ$, and also that $[P] \in \tilde{\mathcal{P}}_\circ$ implies $P \in \mathcal{P}_\circ$, but the converse is not necessarily true. The following lemma shows that factorization into indecomposables in $\tilde{\mathcal{P}}$ is unique, up to possible rearrangement of factors.

Lemma 6.1 *Suppose $P = P_1 \times \cdots \times P_n$ and $Q = Q_1 \times \cdots \times Q_m$, where $[P_i], [Q_j] \in \tilde{\mathcal{P}}_\circ$, for $1 \leq i \leq n$, $1 \leq j \leq m$. If $P \sim Q$, then $m = n$ and there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $P_i \sim Q_{\sigma(i)}$, for $1 \leq i \leq n$.*

Proof The result clearly holds if either m or n is equal to 1. Suppose $m, n > 1$ are given and the result holds for all smaller values of m and n. Let $\varphi : P \longrightarrow Q$ be an order compatible bijection. Switching the roles of P and Q if necessary, choose i such that $|P_i| = \max\{|P_j|, |Q_k| \;:\; 1 \leq j \leq n,\ 1 \leq k \leq m\}$. If $\varphi(0_{P_1}, \ldots, 1_{P_i}, \ldots, 0_{P_n}) = (x_1, \ldots, x_m) \in Q$, then

$$\begin{aligned} P_i &\sim [0_P, (0_{P_1}, \ldots, 1_{P_i}, \ldots, 0_{P_n})] \\ &\sim [0_Q, (x_1, \ldots, x_m)] \\ &= [0_{Q_1}, x_1] \times \cdots \times [0_{Q_m}, x_m], \end{aligned}$$

which contradicts the fact that $[P_i] \in \tilde{\mathcal{P}}_\circ$ unless $x_k = 0_{Q_k}$, for all but one value of k. Hence $\varphi(0_{P_1}, \ldots, 1_{P_i}, \ldots, 0_{P_n}) = (0_{Q_1}, \ldots, x_j, \ldots, 0_{Q_m})$ and thus $P_i \sim [0_{Q_j}, x_j] \subseteq Q_j$, for some j. So, by the maximality of $|P_i|$, it follows that $x_j = 1_{Q_j}$ and thus $P_i \sim Q_j$.

Therefore $[(0_{P_1}, \ldots, 1_{P_i}, \ldots, 0_{P_n}), 1_P] \sim [(0_{Q_1}, \ldots, 1_{Q_j}, \ldots, 0_{Q_m}), 1_Q]$, which implies that $P_1 \times \cdots \times \hat{P_i} \times \cdots \times P_n \sim Q_1 \times \cdots \times \hat{Q_j} \times \cdots \times Q_m$ (the "hat" designates a missing term). Hence, by the inductive hypothesis, $m = n$ and there exists a bijection $\rho : \{1, \ldots, \hat{\imath}, \ldots, n\} \longrightarrow \{1, \ldots, \hat{\jmath}, \ldots, n\}$ such that $P_k \sim Q_{\rho(k)}$ for all $k \neq i$. The map ρ extends in the obvious manner to a permutation of $\{1, \ldots, n\}$ having the desired property. □

From now on, let S_n denote the group of permutations of $\{1, \ldots, n\}$.

Lemma 6.2 *Suppose $u = a_1 a_2 \cdots a_n$ and $v = b_1 b_2 \cdots b_n$ are words of length $n > 2$ over some alphabet. For $1 \leq i \leq n$, let $u_i = a_1 \cdots \hat{a_i} \cdots a_n$ and $v_i = b_1 \cdots \hat{b_i} \cdots b_n$. If there exists a permutation $\sigma \in S_n$ such that $u_i = v_{\sigma(i)}$ for $1 \leq i \leq n$, then $u = v$.*

Proof Suppose $u \neq v$ and let $k = \min\{i : a_i \neq b_i\}$. Suppose $k > 1$. If $\sigma(k-1) < k$, then $u_{k-1} = v_{\sigma(k-1)}$ implies that $a_k = b_k$. Hence $\sigma(k-1) \geq k$, and therefore $u_{k-1} = v_{\sigma(k-1)}$ implies $a_k = b_{k-1}$. Similarly, $b_k = a_{k-1}$, and hence $a_k = b_k$, contrary to the definition of k. Therefore we must have $k = 1$. Since $n > 2$, there exists $i > 1$ such that $\sigma(i) > 1$. But then $u_i = v_{\sigma(i)}$ implies that $a_1 = b_1$, a contradiction; thus $u = v$. □

Lemma 6.3 *Suppose $[P], [Q] \in \tilde{\mathcal{P}}_\circ$, $P \times Q \not\sim Q \times P$ and $P_1 \times \cdots \times P_n \sim Q_1 \times \cdots \times Q_n$ for some $n \geq 1$, where $[P_i], [Q_i] \in \{[P], [Q]\}$, for $1 \leq i \leq n$. Then $P_i \sim Q_i$, for $1 \leq i \leq n$.*

Proof The result is trivial for n equal to 1 or 2; suppose it is true for all n less than some $k > 2$ and let $n = k$. Let $\varphi : P_1 \times \cdots \times P_n \longrightarrow Q_1 \times \cdots \times Q_n$ be an order compatible bijection. Then, by lemma 6.1, there is a permutation $\sigma \in S_n$ such that $\varphi(0_{P_1}, \ldots, 1_{P_i}, \ldots, 0_{P_n}) = (0_{Q_1}, \ldots, 1_{Q_{\sigma(i)}}, \ldots, 0_{Q_n})$, for $1 \leq i \leq n$. Hence

$$P_1 \times \cdots \times \hat{P_i} \times \cdots \times P_n \sim Q_1 \times \cdots \times \hat{Q}_{\sigma(i)} \times \cdots \times Q_n,$$

for $1 \leq i \leq n$. Therefore, by the inductive hypothesis,

$$([P_1], \ldots, \widehat{[P_i]}, \ldots, [P_n]) = ([Q_1], \ldots, \widehat{[Q_{\sigma(i)}]}, \ldots, [Q_n]),$$

for $1 \leq i \leq n$. Thus, by lemma 6.2, $P_i \sim Q_i$, for $1 \leq i \leq n$. □

Theorem 6.4 *If $\sim$ is a Hopf relation on a hereditary family $\mathcal{P}$, then $\tilde{\mathcal{P}}$ is a free partially commutative monoid on the set of indecomposable types $\tilde{\mathcal{P}}_\circ$.*

Proof Suppose $P \sim Q$, where $P = P_1 \times \cdots \times P_n$, $Q = Q_1 \times \cdots \times Q_n$, and $[P_i], [Q_i] \in \tilde{\mathcal{P}}_\circ$, for $1 \leq i \leq n$. For any permutation $\tau \in S_n$, let $T(\tau)$ be the set of increasing pairs whose order is reversed by τ, i.e.,

$$T(\tau) = \{(i, j) \ : \ 1 \leq i < j \leq n \text{ and } \tau(i) > \tau(j)\},$$

and let $t(\tau) = |T(\tau)|$. By lemma 6.1, there exists $\sigma \in S_n$ such that

$$P_i \sim Q_{\sigma(i)}, \text{ for } 1 \leq i \leq n. \tag{6.1}$$

Suppose such σ is chosen with $t(\sigma)$ as small as possible. Let $P_{i_1}, \ldots, P_{i_k}$ be the factors of P, and $Q_{j_1}, \ldots, Q_{j_r}$ be the factors of Q (in the same order as they occur in P and Q respectively) having the same type as either P_i or P_j. It follows from lemma 6.1 and the definition of order compatibility, that $k = r$ and

$$P_{i_1} \times \cdots \times P_{i_k} \sim Q_{j_1} \times \cdots \times Q_{j_k}.$$

If $P_{i_r} \sim Q_{j_r}$ for $1 \leq r \leq k$, then there exists another permutation $\sigma' \in S_n$ satisfying (6.1) with $t(\sigma') < t(\sigma)$. Therefore $P_{i_s} \not\sim Q_{j_s}$, for some s; and thus by lemma 6.3, $P_i \times P_j \sim P_j \times P_i$. Hence the permutation σ is a product of transpositions of subscripts corresponding to commuting types in $\tilde{\mathcal{P}}_\circ$. □

7 Duality

Suppose H is a K-coalgebra. Let H^* denote the K-module of all K-linear maps $\mathrm{Hom}(H, K)$ and, for any $f \in H^*$ and $x \in H$, let $\langle f, x \rangle$ denote the value which f takes on x. The transpose of the coproduct Δ is the map $\Delta^* : (H \otimes H)^* \longrightarrow H^*$ satisfying $\langle \Delta^*(h), x \rangle = \langle h, \Delta(x) \rangle$, for all $h \in (H \otimes H)^*$ and $x \in H$. The *convolution* product $\mu' : H^* \otimes H^* \longrightarrow H^*$ is defined as the composition of Δ^* with the natural map $H^* \otimes H^* \longrightarrow (H \otimes H)^*$, i.e., $fg = \mu'(f \otimes g) = (f \otimes g) \circ \Delta$, for all $f, g \in H^*$. Thus H^* is a K-algebra, having the counit ϵ of H as identity. H^* is called the *dual algebra* of H.

The subset $\mathrm{Alg}(H, K)$ of H^* consisting of all algebra maps from H to K is a group under convolution, called the group of multiplicative functions on H, and is denoted by $M(H)$. The inverse of any $f \in M(H)$ is given by the composition $f \circ S$, where S is the antipode of H.

If $H = H(\mathcal{P})$ is the incidence Hopf algebra of a hereditary family $\mathcal{P}$, modulo a Hopf relation $\sim$, then the dual $H(\mathcal{P})^*$ is called the *incidence algebra of* $\mathcal{P}$ (*reduced modulo* $\sim$). $H(\mathcal{P})^*$ can be identified with the set of all maps from $\tilde{\mathcal{P}}$ to K, and the group of multiplicative functions $M(H(\mathcal{P}))$ can be identified with the set of all maps from $\tilde{\mathcal{P}}_\circ$ to K. The convolution of f and g in $H(\mathcal{P})^*$, or in $M(H(\mathcal{P}))$, is given explicitly by

$$\langle fg, [P] \rangle = \sum_{x \in P} \langle f, [0_P, x] \rangle \langle g, [x, 1_P] \rangle,$$

for all $[P] \in \tilde{\mathcal{P}}$. Note that this operation corresponds to the usual product in an incidence algebra, as defined in [17] or [7].

Suppose $H = \bigoplus_{n \geq 0} H(n)$ is a graded Hopf algebra over K, which is a free, locally finite K-module, that is, each $H(n)$ is a free K-module of finite rank. Let H' denote the K-module $\bigoplus_{n \geq 0} H(n)^*$. H' is a graded algebra with product μ' given on homogenous components by the composition

$$\sum_{k=0}^{n} H(k)^* \otimes H(n-k)^* \hookrightarrow \sum_{k=0}^{n} (H(k) \otimes H(n-k))^* \xrightarrow{\Delta^*} H(n)^*,$$

where Δ^* is the transpose of the coproduct Δ of H. Because H_n has finite rank for all n, it follows that $H(r)^* \otimes H(k)^* \cong (H(r) \otimes H(k))^*$, for all $r, k \geq 0$. Thus H' is a graded Hopf algebra with coproduct Δ' given on homogenous components by the composition

$$H(n)^* \xrightarrow{\mu^*} \sum_{k=0}^{n} (H(k) \otimes H(n-k))^* \xrightarrow{\cong} \sum_{k=0}^{n} H(k)^* \otimes H(n-k)^*,$$

where μ^* is the transpose of the product μ of H. The Hopf algebra H' is called the *graded dual* of H.

8 Uniform Families

A *uniform family* is a hereditary family $\mathcal{P}$ of graded posets together with a Hopf relation $\sim$ such that the following conditions are satisfied:
(1) $\tilde{\mathcal{P}}$ is commutative.
(2) If $[P] \in \tilde{\mathcal{P}}_\circ$, $y \in P$ and $y < 1_P$, then $[y, 1_P] \in \tilde{\mathcal{P}}_\circ$.
(3) For all $n \geq 1$, there exists exactly one type in $\tilde{\mathcal{P}}_\circ$ having rank n.

Given a uniform family $\mathcal{P}$, let x_n denote the unique indecomposable type of rank n, for each $n \geq 1$ and let $x_0 = 1$. The incidence Hopf algebra $H(\mathcal{P})$ is thus isomorphic, as a graded algebra, to the the polynomial algebra $K[x_1, x_2, \ldots]$, where $\deg x_n = n$, for all $n \geq 0$.

For all $n, k \geq 0$, define the *rank polynomial* $\mathbf{W}_{n,k} = \mathbf{W}_{n,k}(x_1, x_2, \ldots)$ in $H(\mathcal{P})$ by letting $\mathbf{W}_{0,0} = 1$, and for $n \geq 1$, choosing $[x, y]$ of rank n, belonging to $\tilde{\mathcal{P}}_\circ$, and setting

$$\mathbf{W}_{n,k} = \sum_{\substack{z \in [x,y] \\ r[z,y]=k}} [x, z].$$

Note that $\mathbf{W}_{n,n} = 1$ and $\mathbf{W}_{n,0} = x_n$, for all $n \geq 0$, and $\mathbf{W}_{n,k} = 0$, whenever $n < k$. It follows that the coproduct of $H(\mathcal{P})$ is given by

$$\Delta(x_n) = \sum_{k \geq 0} \mathbf{W}_{n,k} \otimes x_k,$$

for all $n \geq 0$. As we will see in example 14.1, the polynomials $\mathbf{W}_{n,k}$ generalize the partial Bell polynomials $\mathbf{B}_{n,k}$ (see [5], p. 133).

For all $n \geq 1$, let M_n denote the $n \times n$ matrix whose entry in row i, column j is equal to $\mathbf{W}_{n-i+1,n-j}$, for $1 \leq i, j \leq n$, and let M_0 be the 1×1 identity matrix.

Theorem 8.1 *If $\mathcal{P}$ is a uniform family, then the antipode S of $H(\mathcal{P}) \simeq K[x_1, x_2, \ldots]$ is given by $S(x_n) = (-1)^n \det M_n$, for $n \geq 0$.*

Proof The formula is obviously true for $n = 0$. Suppose that $n \geq 0$. The cofactor of $\mathbf{W}_{n,k}$ in M_n is equal to $\det M_k$, for $0 \leq k \leq n-1$. Expanding by cofactors along the top row of M_n yields the recursive formula

$$\det M_n = \sum_{k=0}^{n-1} (-1)^{n-k+1} \mathbf{W}_{n,k} \det M_k.$$

Define an algebra map $S' : H(\mathcal{P}) \longrightarrow H(\mathcal{P})$ by $S'(x_n) = (-1)^n \det M_n$, for all $n \geq 0$. It follows that

$$(-1)^n S'(x_n) = \sum_{k=0}^{n-1} (-1)^{n-k+1} \mathbf{W}_{n,k} (-1)^k S'(x_k),$$

and thus

$$0 = \sum_{k=0}^{n} \mathbf{W}_{n,k} S'(x_k).$$

In other words, $\mu \circ (I \otimes S') \circ \Delta(x_n) = 0$, for all $n \geq 1$, where $\mu : H(\mathcal{P}) \otimes H(\mathcal{P}) \longrightarrow H(\mathcal{P})$ is multiplication in $H(\mathcal{P})$, and $I : H(\mathcal{P}) \longrightarrow H(\mathcal{P})$ is the identity map. Thus, by uniqueness, $S' = S$, the antipode of $H(\mathcal{P})$. □

Example 8.1 (Linear Orders) The family $\mathcal{L}$ of products of linear orders (see example 5.2) is uniform, with rank polynomials given by $\mathbf{W}_{n,k} = x_{n-k}$, for all $n, k \geq 0$, where $x_r = 0$, for $r < 0$. Thus by theorem 8.1, the antipode of $H(\mathcal{L})$ can be expressed as

$$S(x_n) = (-1)^n \det \left(x_{j-i+1} \right)_{1 \leq i,j \leq n}, \tag{8.1}$$

for all $n \geq 0$.

For all multiplicative functions $f \in M(H(\mathcal{L}))$, let $f(t)$ be the power series $\sum_{n \geq 0} f(x_n) t^n$. The correspondence $f \longrightarrow f(t)$ defines an isomorphism from $M(H(\mathcal{L}))$ onto the group (under multiplication) of power series with coefficients in K, having constant term equal to one. Equations 5.1 and 8.1 thus provide formulas for the coefficients of the multiplicative inverse of any such series.

9 Cocommutative Structure Theory

If H is any K-coalgebra and $n \geq 1$, let $H^{(n)}$ denote the n-fold tensor product $H \otimes \cdots \otimes H$, and let $\Delta_n : H \longrightarrow H^{(n+1)}$ be the n-fold coproduct. The *coradical* R of H is defined to be the direct sum of all simple coalgebras of H. For all $n \geq i \geq 1$, let $H_i^{(n)}$ denote the n-fold tensor product $H \otimes \cdots \otimes H \otimes R \otimes H \otimes \cdots \otimes H$, having R as the ith factor. Define a sequence of submodules of H by setting $H_0 = R$, and $H_n = \Delta_{n-1}^{-1}(H_1^{(n)} + H_2^{(n)} + \cdots + H_n^{(n)})$, for all $n \geq 2$. It is not difficult to see that each H_n is a subcoalgebra of H and the sequence $H_0 \subseteq H_1 \subseteq \cdots$ is a filtration of H. The filtration $H_0 \subseteq H_1 \subseteq \cdots$ is called the *coradical filtration* of H (see [21]).

Suppose H is an irreducible K-bialgebra. Let $\mu : H \otimes H \longrightarrow H$, $\Delta : H \longrightarrow H \otimes H$, $\eta : K \longrightarrow H$, and $\epsilon : H \longrightarrow K$ be the product, coproduct, unit and counit of H, respectively. The K-module $\mathrm{Hom}(H, H)$ of all K-linear maps from H to itself is an algebra with convolution product, given by $fg = \mu \circ (f \otimes g) \circ \Delta$, for all $f, g \in \mathrm{Hom}(H, H)$. The convolution identity $1 \in \mathrm{Hom}(H, H)$, given by $1 = \eta \circ \epsilon$, should not be confused with the identity map I, satisfying $I(x) = x$, for all $x \in H$.

Let $H_0 \subseteq H_1 \subseteq \cdots$ be the coradical filtration of H and, for each $n \geq 0$, let J_n denote the ideal in $\mathrm{Hom}(H, H)$ consisting of all maps which annihilate H_n. Since H is irreducible, its coradical H_0 is equal to $\eta(K) \cong K$, hence $f \in J_0$ if and only if $f \circ \eta$ is identically zero. If $f \in J_0$, then $f^n \in J_{n-1}$, for all $n \geq 1$.

The ideals $J_0 \supseteq J_1 \supseteq \cdots$ form a local base for a topology on $\mathrm{Hom}(H, H)$, which is thus a topological algebra. In particular, if $f \in J_0$, then the sequence of powers $(f^n)_{n \geq 1}$ converges to zero.

As an example, note that $1 - I$ belongs to J_0, hence $S = 1/I = 1/(1 - (1 - I)) = \sum_{k \geq 0} (1 - I)^k$ converges in $\mathrm{Hom}(H, H)$. Therefore H is a Hopf algebra with antipode S. In the case that $H = H(\mathcal{P})$ is an incidence Hopf algebra, and $[P] \in \tilde{\mathcal{P}}$, we thus have

$$S[P] \;=\; \sum_{k \geq 0} (1 - I)^k [P]$$

$$= \sum_{k\geq 0}[(1-I)\otimes\cdots\otimes(1-I)]\circ\Delta_{k-1}[P]$$

$$= \sum_{k\geq 0}\sum_{\substack{x_0<\cdots<x_k\\ x_0=0_P\\ x_k=1_P}}(-1)^k\prod_{i=1}^{k}[x_{i-1},x_i],$$

which is simply equation 4.1 for the antipode of an incidence Hopf algebra.

If H is any K-Hopf algebra, the submodule of primitive elements $P(H) = \{x \in H : \Delta x = x\otimes 1 + 1\otimes x\}$ is a K-Lie algebra, with $[x,y] = xy - yx$, for all $x, y \in P(H)$. On the other hand, given any Lie algebra L over a field of characteristic zero, the universal enveloping algebra $U(L)$ can be equipped with a unique Hopf algebra structure such that $L = P(U(L))$. The following structure theorem is well-known (see, *e.g.*, [21], p. 274).

Theorem 9.1 *If H is an irreducible, cocommutative Hopf algebra over a field K of characteristic zero, then H is isomorphic to the universal enveloping algebra $U(P(H))$.*

Define a Hopf algebra H over a characteristic zero ring K to be *divisible* if, for all $n \geq 1$ and $x \in H$, $(I-1)^n(x)$ is divisible by n in H. Note that if K is a field of characteristic zero, then any Hopf algebra over K is divisible.

We now construct a linear projection $\lambda : H \longrightarrow P(H)$, in the case that H is irreducible, cocommutative and divisible over K. Among other things, the map λ will allow us to explicitly exhibit commutative, cocommutative, divisible incidence Hopf algebras as polynomial Hopf algebras with primitive indeterminants (i.e., as symmetric algebras, with the usual Hopf algebra structure).

If H is irreducible, then the tensor product $H\otimes H$ is also irreducible, and the K-module $\mathrm{Hom}(H\otimes H, H\otimes H)$ is a topological algebra in the same manner as $\mathrm{Hom}(H,H)$. If H is also divisible, and $f \in \mathrm{Hom}(H,H)$ is an algebra map or coalgebra map, then $1-f \in J_0$ and the series $\log(f) = -\sum_{k\geq 1}(1-f)^k/k$ and $\log(f\otimes f) = -\sum_{k\geq 1}(1\otimes 1 - f\otimes f)^k/k$ converge in $\mathrm{Hom}(H,H)$ and $\mathrm{Hom}(H\otimes H, H\otimes H)$, respectively.

Lemma 9.2 *Suppose H is irreducible, divisible and cocommutative, and $f \in Hom(H,H)$ is a coalgebra map. Then $\Delta\circ\log(f) = \log(f\otimes f)\circ\Delta$.*

Proof For all $k \geq 1$, let $\mu_k : H^{(k+1)} \longrightarrow H$ denote the k-fold product on H, and let $T_k : H^{(2k)} \longrightarrow H^{(2k)}$ be the "twist" map, given by

$$T(x_1\otimes y_1\otimes\cdots\otimes x_k\otimes y_k) = x_1\otimes\cdots\otimes x_k\otimes y_1\otimes\cdots\otimes y_k,$$

for all $x_i, y_i \in H$.

If we write $f_1 = 1-f$ and $f_2 = 1\otimes 1 - f\otimes f$, then $\Delta\circ f_1 = f_2\circ\Delta$, because f is a coalgebra map. Therefore, for all $k \geq 1$,

$$\begin{aligned}
\Delta\circ f_1^{k+1} &= \Delta\circ\mu_k\circ(f_1\otimes\cdots\otimes f_1)\circ\Delta_k \\
&= (\mu_k\otimes\mu_k)\circ T_{k+1}\circ(\Delta\otimes\cdots\otimes\Delta)\circ(f_1\otimes\cdots\otimes f_1)\circ\Delta_k \\
&= (\mu_k\otimes\mu_k)\circ T_{k+1}\circ(f_2\otimes\cdots\otimes f_2)\circ(\Delta\otimes\cdots\otimes\Delta)\circ\Delta_k \\
&= \mu_k^{(2)}\circ(f_2\otimes\cdots\otimes f_2)\circ\Delta_{2k+1}
\end{aligned}$$

where $\mu_k^{(2)}$ denotes the k-fold product on $H^{(2)}$. By cocommutativity, $\Delta_{2k+1} = \Delta_k^{(2)} \circ \Delta$, where $\Delta_k^{(2)}$ is the k-fold coproduct on $H^{(2)}$. Hence $\Delta \circ f_1^{k+1} = f_2^{k+1} \circ \Delta$, and thus the result follows from the series definition of $\log(f)$. □

Lemma 9.3 *Suppose $H_0 \subseteq H_1 \subseteq \cdots$ is a filtration of an irreducible Hopf algebra H (e.g., the coradical filtration; or the length filtration, if H is an incidence Hopf algebra). Let $H^+ = \ker \epsilon$, and $H_n^+ = H_n \cap H^+$, for all $n \geq 0$. Then for all $x \in H_n^+$, $\Delta(x) = x \otimes 1 + 1 \otimes x + y$, where $y \in H_{n-1}^+ \otimes H_{n-1}^+$.*

Proof Let $y = \Delta(x) - x \otimes 1 - 1 \otimes x$. Since $x \in H^+$, we have $(I \otimes \epsilon)(y) = (I \otimes \epsilon) \circ \Delta(x) - x - 0 = x - x = 0$. Therefore $y \in H \otimes H^+$. Similarly, $y \in H^+ \otimes H$, and thus $y \in H^+ \otimes H^+$. Since $H_0 \subseteq H_1 \subseteq \cdots$ is a filtration, $\Delta(x) \in \sum_{i=0}^n H_i \otimes H_{n-i}$. Also, $x \otimes 1 + 1 \otimes x \in H_n \otimes H_0 + H_0 \otimes H_n$, and so

$$\begin{aligned} y &\in (H^+ \otimes H^+) \cap \sum_{i=0}^n H_i \otimes H_{n-i} \\ &= \sum_{i=0}^n H_i^+ \otimes H_{n-i}^+. \end{aligned}$$

But $H_0^+ = \{0\}$, hence

$$y \in \sum_{i=1}^n H_i^+ \otimes H_{n-i}^+ \subseteq H_{n-1}^+ \otimes H_{n-1}^+.$$

□

Theorem 9.4 *If H is a divisible, cocommutative, irreducible Hopf algebra, and $f \in Hom(H, H)$ is a coalgebra map, then $\log(f) : H \longrightarrow P(H)$.*

Proof Let $H_0 \subseteq H_1 \subseteq \cdots$ be a filtration of H. Since $\eta(K) = H_0$ and the composition $\epsilon \circ \eta$ is the identity on K, it follows that $H = H_0 \oplus H^+$. Given any $x \in H$ write $x = a + b$, where $a \in H_0$ and $b \in H^+$. Then $\log(f)(x) = \log(f)(a) + \log(f)(b) = \log(f)(b)$. Hence it suffices to show that $\log(f)(x) \in P(H)$, for all $x \in H^+$. By lemma 9.2, $\Delta \circ \log(f)(x) = \log(f \otimes f) \circ \Delta(x)$, for all $x \in H$. But

$$\begin{aligned} \log(f \otimes f) &= \log[(f \otimes 1)(1 \otimes f)] \\ &= \log(f \otimes 1) + \log(1 \otimes f) \\ &= \log(f) \otimes 1 + 1 \otimes \log(f), \end{aligned}$$

and thus it follows from lemma 9.3 that $\log(f)(x) \in P(H)$, for all $x \in H^+$. □

In [14], R. Ree proved a theorem which is equivalent to the above in the case K is a field of characteristic zero and H is a tensor algebra, i.e., the universal enveloping algebra of the free Lie algebra $P(H)$. His approach is quite different from the one taken here and, in particular, does not use the language of Hopf algebras.

Suppose $\sigma \in S_n$ is a given permutation. In the sequence $\sigma(1), \ldots, \sigma(n)$ insert a vertical bar, or "spacer", between $\sigma(i)$ and $\sigma(i+1)$ whenever $\sigma(i) > \sigma(i+1)$. This partitions the sequence into strictly increasing subsequences called *ascending runs*, e.g.,

$$\sigma(1), \ldots, \sigma(i_1) \mid \sigma(i_1 + 1), \ldots, \sigma(i_2) \mid \ \ldots \ \mid \sigma(i_{d_\sigma} + 1), \ldots, \sigma(n). \tag{9.1}$$

Let a_σ denote the number of ascending runs of σ.

Theorem 9.5 *Suppose H is a divisible, cocommutative, irreducible Hopf algebra, and $f \in Hom(H,H)$ is an algebra map. Then for $x_1, \ldots, x_n \in P(H)$,*

$$\log(f)(x_1 \cdots x_n) = \sum_{\sigma \in S_n} \frac{(-1)^{a_\sigma - 1}}{a_\sigma} \binom{n}{a_\sigma}^{-1} f(x_{\sigma(1)} \cdots x_{\sigma(n)}). \tag{9.2}$$

Proof Fix $k \geq 1$. For $1 \leq i \leq k$ and $1 \leq j \leq n$, set $x_j^{(i)} = 1 \otimes \cdots \otimes x_j \otimes \cdots \otimes 1$, where x_j is the ith factor and all other factors are equal to one. Therefore $\Delta_{k-1}(x_j) = \sum_{i=1}^k x_j^{(i)}$ and thus

$$\Delta_{k-1}(x_1 \cdots x_n) = \sum_{i_1, \ldots, i_n = 1}^{k} x_1^{(i_1)} \cdots x_n^{(i_n)}. \tag{9.3}$$

Let L_n denote the set of all linearly ordered partitions of $\{1, \ldots, n\}$ (an element of L_n is a partition of $\{1, \ldots, n\}$ together with a linear order on its set of blocks), and let $L_{n,k} = \{\pi \in L_n : |\pi| = k\}$. If $\pi = (B_1, \ldots, B_k) \in L_{n,k}$, then each B_i inherits a linear ordering from $\{1, \ldots, n\}$, and the concatenation of these ordered sets gives a new ordering $\pi'(1), \ldots, \pi'(n)$ of the set $\{1, \ldots, n\}$. It follows from equation 9.3 and the fact that f is an algebra map that

$$(f-1)^k(x_1 \cdots x_n) = \sum_{\pi \in L_{n,k}} f(x_{\pi'(1)} \cdots x_{\pi'(n)}). \tag{9.4}$$

For each $\sigma \in S_n$, let k_σ denote the number of elements π of $L_{n,k}$ such that $\pi' = \sigma$, so that

$$(f-1)^k(x_1 \cdots x_n) = \sum_{\sigma \in S_n} k_\sigma f(x_{\sigma(1)} \cdots x_{\sigma(n)}).$$

Given $\pi = (B_1, \ldots, B_k) \in L_{n,k}$, note that each ascending run of the permutation π' is a concatenation of consecutive B_i's. Therefore, given $\sigma \in S_n$, if $k < a_\sigma$, then $k_\sigma = 0$. And if $k \geq a_\sigma$, then each $\pi \in L_{n,k}$ with $\pi' = \sigma$ corresponds to a subdivision of the partition (9.1) into a total of k pieces, which can be constructed by inserting vertical bars into $k - a_\sigma$ of the available $n - a_\sigma$ spaces. Therefore

$$k_\sigma = \binom{n - a_\sigma}{k - a_\sigma},$$

for all $\sigma \in S_n$ and all $k \geq 1$. It is not difficult to show (using induction) that

$$\sum_{k \geq 1} \frac{(-1)^k}{k} \binom{n-a}{k-a} = \frac{(-1)^{a-1}}{a} \binom{n}{a}^{-1},$$

for all $n, k, a \geq 1$. Therefore equation 9.2 follows. □

C. Reutenauer [15] proved a special case of the above theorem under the restrictions that H is a tensor algebra, K a field of characteristic zero, and f the identity map.

Corollary 9.6 *If H is a divisible, cocommutative, irreducible Hopf algebra, $f \in Hom(H,H)$ is an algebra map, and $x_1, \ldots x_n \in P(H)$ commute, then*

$$\log(f)(x_1 \cdots x_n) = \begin{cases} f(x_1) & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\log(f)(x^n) = \delta_{n,1} \cdot f(x)$, for all $x \in P(H)$.

Proof Since the x_i commute, it follows from equation 9.4 in the proof of theorem 9.5 that

$$\log(f)(x_1 \cdots x_n) = f(x_1 \cdots x_n) \sum_{\pi \in L_n} \frac{(-1)^{|\pi|-1}}{|\pi|}.$$

Now,

$$\begin{aligned} \sum_{\pi \in L_n} \frac{(-1)^{|\pi|-1}}{|\pi|} &= \sum_{\gamma \in \Pi_n} (-1)^{|\gamma|-1}(|\gamma|-1)! \\ &= \sum_{\gamma \in \Pi_n} \mu_\pi(\gamma, 1_{\Pi_n}), \end{aligned}$$

where Π_n denotes the lattice of all partitions of the set $\{1, \ldots, n\}$, and μ_π is the partition Möbius function, which is defined by the condition

$$\sum_{\gamma \in \Pi_n} \mu_\pi(\gamma, 1_{\Pi_n}) = \delta_{n,1},$$

for all $n \geq 1$ (see [17]). Hence the result follows. □

In the characteristic zero case, corollary 9.6 follows immediately from theorem 9.5 and the Specht-Wever theorem (see [12]).

Whenever H is irreducible and divisible over K, we write λ for $\log(I)$, where $I : H \longrightarrow H$ is the identity map. If H is also cocommutative, then according to theorem 9.4 and corollary 9.6, λ is a surjection from H onto $P(H)$, whose restriction to $P(H)$ is the identity, that is, λ is a projection from H onto $P(H)$.

Example 9.1 (Binomial Hopf Algebra) The binomial Hopf algebra $K[x]$ (see example 4.1) is divisible, for any K. The projection $\lambda : K[x] \longrightarrow P(K[x])$ satisfies

$$\lambda(x^n) = \begin{cases} x & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 9.2 (Free Partially Commutative Hopf Algebras) If K is a field of characteristic zero, then by theorem 9.5, the projection λ from the free partially commutative Hopf algebra $K\langle A, \theta\rangle$ (see example 4.2) onto $P(K\langle A, \theta\rangle)$ is given by

$$\lambda[w] = \sum_{\sigma \in S_n} \frac{(-1)^{a_\sigma - 1}}{a_\sigma} \binom{n}{a_\sigma}^{-1} [x_{\sigma(1)} \cdots x_{\sigma(n)}],$$

whenever $w = x_1 \cdots x_n$, where $x_i \in A$, for $1 \leq i \leq n$.

One immediate consequence of theorem 9.1 is that an irreducible, cocommutative Hopf algebra over a field of characteristic zero is generated by primitive elements. The following proposition generalizes this fact to divisible Hopf algebras and, furthermore, gives a technique for finding generating sets of primitive elements.

Proposition 9.7 *Suppose H is a divisible, cocommutative, irreducible Hopf algebra, and $H_0 \subseteq H_1 \subseteq \cdots$ is a filtration of H. If S is a subset of H such that the subalgebra of H generated over K by $S_n = S \cap H_n$ contains H_n, for all $n \geq 0$, then $\lambda(S)$ generates H. Hence, in particular, H is generated by primitive elements.*

Proof Since $H_0 \subseteq H_1 \subseteq \cdots$ is a filtration, we have

$$\Delta_{k-1}(H_n) \subseteq \sum_{i_1+\cdots+i_k=n} H_{i_1} \otimes \cdots \otimes H_{i_k}, \tag{9.5}$$

for all $k \geq 2$. Let $H^+ = \ker \epsilon$ and $H_n^+ = H^+ \cap H_n$, for all $n \geq 0$. If $x \in H_n^+$, then by (9.5),

$$\lambda(x) = -\sum_{k\geq 1}(1-I)^k(x)/k = x + p(x_1, \dots, x_r), \tag{9.6}$$

where p is a (non-commutative) polynomial with coefficients in K, and $x_1, \dots, x_r \in H_{n-1}^+$. Let $\langle\lambda(S)\rangle$ be the subalgebra of H generated by $\lambda(S)$. The proof now follows by induction: If $x \in S_1$, then x can be written uniquely as $a+b$, where $a \in H_0$ and $b \in P(H)$, because $H = H_0 \oplus H^+$ and $H_1^+ \subseteq P(H)$. Hence $\lambda(x) = b$ and so $x = a + \lambda(x)$. Thus S_1, and therefore H_1, is contained in $\langle\lambda(S)\rangle$. Now suppose $H_{n-1} \subseteq \langle\lambda(S)\rangle$. If $x \in S_n$, then it follows from equation 9.6 that $x \in \langle\lambda(S)\rangle$. Therefore $H_n \subseteq \langle\lambda(S)\rangle$. □

10 Cocommutative Incidence Hopf Algebras

Suppose $\mathcal{P}$ is a hereditary family of posets. A Hopf relation $\sim$ on $\mathcal{P}$ is *self-dual* if, for all $P \in \mathcal{P}$, there exists a bijection $\varphi = \varphi_P$ from P to itself such that $[0_P, x] \sim [\varphi(x), 1_P]$, for all $x \in P$. The incidence Hopf algebra $H(\mathcal{P})$ is cocommutative if and only if the relation $\sim$ is self-dual. In particular, whenever $\mathcal{P}$ consists of self-dual posets, then isomorphism is a self dual relation.

The monoid of types $\tilde{\mathcal{P}}$ is *divisible* if, for all $k \geq 1$, and $[P], [Q] \in \tilde{\mathcal{P}}$, the number of length k chains $0_P = x_0 < \cdots < x_k = 1_P$ in P such that $\prod_{i=1}^k [x_{i-1}, x_i] = [Q]$ is a multiple of k. For example, if $\mathcal{P}$ consists of boolean algebras and $\sim$ is isomorphism, then $\tilde{\mathcal{P}}$ is divisible. Clearly, $H(\mathcal{P})$ is divisible if and only if $\tilde{\mathcal{P}}$ is divisible.

Theorem 10.1 *If $H = H(\mathcal{P})$ is a divisible, cocommutative incidence Hopf algebra, then the projection $\lambda : H \longrightarrow P(H)$ is determined by*

$$\lambda[P] = \sum_{k\geq 1} \sum_{\substack{x_0<\cdots<x_k \\ x_0=0_P \\ x_k=1_P}} \frac{(-1)^{k-1}}{k} \prod_{i=1}^k [x_{i-1}, x_i], \tag{10.1}$$

for all $[P] \in \tilde{\mathcal{P}}$.

Proof Immediate from the power series expansion of $\log(I)$. □

Note that formula (10.1) for $\lambda = \log(I)$ also holds when H is non-cocommutative, but in that case, the image of λ is not necessarily contained in $P(H)$.

If $H = H(\mathcal{P})$ is commutative, then by theorem 6.4, as an algebra, H is isomorphic to the polynomial algebra $K[\tilde{\mathcal{P}}_\circ]$. If H is also divisible and cocommutative, then it follows from proposition 9.7 that the set $\lambda(\tilde{\mathcal{P}}_\circ)$ generates H. The following theorem shows that $H(\mathcal{P})$ is isomorphic to the symmetric algebra on the free module $K\{\lambda(\tilde{\mathcal{P}}_\circ)\}$, equipped with the usual Hopf algebra structure (see [21], p. 59).

Theorem 10.2 *If $H = H(\mathcal{P})$ is a divisible, commutative and cocommutative incidence Hopf algebra, then H is isomorphic to the polynomial Hopf algebra $K[\lambda(\tilde{\mathcal{P}}_\circ)]$, having primitive indeterminates.*

Proof By proposition 9.7, the set of primitive elements $\lambda(\tilde{\mathcal{P}}_\circ)$ generates H. It remains to show independence. Suppose $r \geq 1$ is minimal such that there exist $[P_1], \ldots, [P_r] \in \tilde{\mathcal{P}}_\circ$ and a non-zero polynomial p with coefficients in K satisfying $p(\lambda[P_1], \ldots, \lambda[P_r]) = 0$. Assume that $l(P_1) \geq l(P_i)$, for $1 \leq i \leq r$; and arrange p according to descending powers of $\lambda[P_1]$, i.e.,

$$\begin{aligned} p(\lambda[P_1], \ldots, \lambda[P_r]) &= p_0(\lambda[P_2], \ldots, \lambda[P_r])(\lambda[P_1])^n + p_1(\lambda[P_2], \ldots, \lambda[P_r])(\lambda[P_1])^{n-1} \\ &\quad + \cdots + p_n(\lambda[P_2], \ldots, \lambda[P_r]) \\ &= 0, \end{aligned}$$

where $p_0, \ldots, p_n$ are polynomials with coefficients in K, for some $n \geq 1$. By equation 10.1, $\lambda[P_1] = [P_1] + g([Q_1], \ldots, [Q_k])$, for some k, where g is a polynomial with coefficients in K and $l(Q_i) < l(P_1)$, for $1 \leq i \leq k$. Hence the coefficient of $[P_1]^n$ in $p(\lambda[P_1], \ldots, \lambda[P_r])$ is equal to $p_0(\lambda[P_2], \ldots, \lambda[P_r])$, which must be zero. Thus by the minimality of r, p_0 is identically zero. Similarly, $p_1, \ldots, p_n$ are identically zero, and therefore so is p. Hence the set $\lambda(\tilde{\mathcal{P}}_\circ)$ is algebraically independent over K. □

Example 10.1 (Linear Orders) Suppose $H(\mathcal{L})$ is the free commutative incidence Hopf algebra of the family of linear orders (see examples 5.2 and 8.1). If K is a field of characteristic zero, the projection $\lambda : H(\mathcal{L}) \longrightarrow P(H(\mathcal{L}))$ is given, according to equation 10.1, by

$$\lambda(x_n) = \sum_{k \geq 1} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i \geq 1}} \frac{(-1)^{k-1}}{k} x_{n_1} \cdots x_{n_k},$$

for all $n \geq 0$. Thus $H(\mathcal{L})$ is isomorphic to the polynomial Hopf algebra $K[\lambda(x_1), \lambda(x_2), \ldots]$, where the $\lambda(x_i)$ are primitive.

Suppose $H(\mathcal{P})$ is a divisible, commutative and cocommutative incidence Hopf algebra. Because $I = \exp(\lambda)$ in $\mathrm{Hom}(H, H)$, we have the following polynomial expression for the basis element $[P] \in \tilde{\mathcal{P}}$ in terms of primitive elements:

$$[P] = \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{x_0 < \cdots < x_k \\ x_0 = 0_P \\ x_k = 1_P}} \prod_{i=1}^{k} \lambda[x_{i-1}, x_i]. \tag{10.2}$$

11 Commutative Incidence Hopf Algebras

A Hopf relation $\sim$ on a hereditary family of graded posets $\mathcal{P}$ is *locally finite* if there are finitely many types $[P] \in \tilde{\mathcal{P}}$ of rank n, for all $n \geq 0$. Suppose such $\mathcal{P}$ and $\sim$ are given, and $\bigoplus_{n \geq 0} H(\mathcal{P})(n)$ is the rank grading of the incidence Hopf algebra $H(\mathcal{P})$. For all $[P] \in \tilde{\mathcal{P}}$, define P' in the graded dual $H(\mathcal{P})'$ by

$$\langle P', [Q] \rangle = \begin{cases} 1 & \text{if } [P] = [Q] \\ 0 & \text{otherwise,} \end{cases}$$

for all $[Q] \in \tilde{\mathcal{P}}$. The set $\mathcal{P}' = \{P' : [P] \in \tilde{\mathcal{P}}\}$ is thus a basis for $H(\mathcal{P})'$. The product in $H(\mathcal{P})'$ is given by

$$P'Q' = \sum_{[R]\in\tilde{\mathcal{P}}} (R; P, Q)R',$$

and the coproduct Δ' is given by

$$\Delta'(P') = \sum_{\substack{[Q],[P]\in\tilde{\mathcal{P}} \\ [Q][R]=[P]}} Q' \otimes R',$$

for all $P', Q' \in \mathcal{P}'$.

Now suppose that K is a field of characteristic zero and the monoid of types $\tilde{\mathcal{P}}$ is commutative. Write $L(\mathcal{P})$ for the Lie algebra of primitive elements of $H(\mathcal{P})'$. $L(\mathcal{P})$ has basis $\mathcal{P}'_\circ = \{P' : [P] \in \tilde{\mathcal{P}}_\circ\}$, and bracket determined by

$$\begin{aligned}[P', Q'] &= P'Q' - Q'P' \\ &= \sum_{[R]\in\tilde{\mathcal{P}}_\circ} [(R; P, Q) - (R; Q, P)]R',\end{aligned}$$

for all $P', Q' \in \mathcal{P}'_\circ$. In particular, it follows from the above that if $\tilde{\mathcal{P}}_\circ$ is finite, then $L(\mathcal{P})$ is finite dimensional and nilpotent.

According to theorem 9.1, $H(\mathcal{P})'$ is isomorphic to the universal enveloping algebra of $L(\mathcal{P})$. It follows from the theorem of Poincaré-Birkhoff-Witt that if $\mathcal{P}'_\circ$ is totally ordered, then the set of monomials of the form $P'_1 \cdots P'_k$, where $k \geq 0$ and $P'_1 \leq \cdots \leq P'_k \in \mathcal{P}'_\circ$, is a basis for $H(\mathcal{P})'$.

12 Cocommutative Graph Hopf Algebras

Suppose G is a graph with vertex set $V(G)$ and edge set $E(G)$. If $U \subseteq V(G)$, then the *induced subgraph* $G|U$ is the graph having vertex set U and edge set consisting of all edges of G which have both end-vertices contained in U. The *sum* $G + H$ is the disjoint union of graphs G and H. The isomorphism class, or *type*, of a graph G is denoted by $[G]$. All graphs considered in this paper will be finite.

Suppose $\mathcal{G}$ is a family of graphs which is closed under formation of induced subgraphs and sums. Let $\tilde{\mathcal{G}}$ denote the set of isomorphism types of graphs in $\mathcal{G}$ and let $\tilde{\mathcal{G}}_\circ$ denote the set of isomorphism types of connected graphs in $\mathcal{G}$. A product is defined on $\tilde{\mathcal{G}}$ by $[G][H] = [G + H]$, for all $G, H \in \mathcal{G}$, and thus $\tilde{\mathcal{G}}$ can be identified with the free commutative monoid on $\tilde{\mathcal{G}}_\circ$.

For $G \in \mathcal{G}$, let $B(G)$ denote the lattice of subsets of $V(G)$, ordered by inclusion. Let $\mathcal{P}(\mathcal{G})$ be the hereditary family consisting of all finite products of intervals from the posets $B(G)$, for $G \in \mathcal{G}$. Suppose $G_i, G'_j \in \mathcal{G}$, $U_i \subseteq W_i \subseteq V(G_i)$ and $U'_j \subseteq W'_j \subseteq V(G'_j)$, for $1 \leq i \leq n$, $1 \leq j \leq m$. A Hopf relation $\sim$ is defined on $\mathcal{P}(\mathcal{G})$ by setting $[U_1, W_1] \times \cdots \times [U_n, W_n] \sim [U'_1, W'_1] \times \cdots \times [U'_m, W'_m]$ in $\mathcal{P}(\mathcal{G})$ if and only if $\sum_{i=1}^n G_i|(W_i - U_i)$ and $\sum_{j=1}^m G'_j|(W'_j - U'_j)$ are isomorphic graphs.

We identify the monoid of types $\widetilde{\mathcal{P}(\mathcal{G})}$ with the monoid of isomorphism types $\tilde{\mathcal{G}}$ *via* the correspondence $[U, W] \longleftrightarrow [G|(W - U)]$, whenever $U \subseteq W \subseteq V(G)$, for $G \in \mathcal{G}$. Under this identification, the set of indecomposable types corresponds to the set of types of connected graphs $\tilde{\mathcal{G}}_\circ$.

The incidence Hopf algebra $H(\mathcal{G}) = H(\mathcal{P}(\mathcal{G}))$ is therefore isomorphic to the polynomial algebra $K[\tilde{\mathcal{G}}_\circ]$, with cocommutative coproduct given by

$$\Delta[G] = \sum_{U \subseteq V(G)} [G|U] \otimes [G|(V(G) - U)],$$

for all $G \in \mathcal{G}$.

It follows from equation 4.1 that the antipode of $H(\mathcal{G})$ is given by

$$S[G] = \sum_{\pi} (-1)^{|\pi|} |\pi|! \prod_{B \in \pi} [G|B], \tag{12.1}$$

where the sum is over all partitions π of $V(G)$.

The monoid of types $\widetilde{\mathcal{P}(\mathcal{G})}$ is divisible, and therefore the map $\lambda : H(\mathcal{G}) \longrightarrow P(H(\mathcal{G}))$ is defined with no need for characteristic zero K. We obtain from equation 10.1 that

$$\lambda[G] = \sum_{\pi} (-1)^{|\pi|-1} (|\pi| - 1)! \prod_{B \in \pi} [G|B], \tag{12.2}$$

where, as above, the sum is over all partitions π of $V(G)$.

According to theorem 10.2, $H(\mathcal{G})$ is isomorphic to the polynomial Hopf algebra $K[\lambda(\tilde{\mathcal{G}}_\circ)]$. For any graph G, the poset $B(G)$ is graded of rank $|V(G)|$. Therefore, by proposition 4.4, $H(\mathcal{G})$ is a graded Hopf algebra, where for all $G \in \mathcal{G}$, $\deg[G] = |V(G)|$, and by equation 12.2, $\lambda[G]$ is homogeneous of degree $|V(G)|$ as well. Associate to the family $\mathcal{G}$ the *type sequence* $t(\mathcal{G}) = (t_1, t_2, \ldots)$ where, for all $k \geq 1$, t_k is the number of isomorphism types of connected graphs on k vertices in $\mathcal{G}$. We thus have the following proposition.

Proposition 12.1 *The Hopf algebra $H(\mathcal{G})$ is determined, up to graded Hopf algebra isomorphism, by the type sequence $t(\mathcal{G})$.*

Remark:

The construction of $H(\mathcal{G})$ makes sense when $\mathcal{G}$ is any family of structures on finite sets for which one has suitable notions of restriction to a subset and disjoint union. For example, $\mathcal{G}$ may consist of matroids, labelled graphs, or hypergraphs. These more general Hopf algebras are considered in [20], where the language of categories is used in order to give a precise definition of such a familiy of structures.

Example 12.1 (Independent Graphs) Let $\mathcal{I}$ be the set of all (finite) graphs with no edges, and let x denote the type of the single vertex graph. Then $H(\mathcal{I})$ is isomorphic to the binomial Hopf algebra $K[x]$ (see example 4.1).

Example 12.2 (Complete Graphs) Let $\mathcal{K}$ consist of all disjoint unions of complete graphs, and let x_n denote the type of the complete graph on n vertices. Then $H(\mathcal{K})$ is isomorphic to the polynomial algebra $K[x_1, x_2, \ldots]$, with coproduct given by

$$\Delta(x_n) = \sum_{k=0}^{n} \binom{n}{k} x_k \otimes x_{n-k},$$

for all $n \geq 0$. According to equation 12.1, the antipode of $H(\mathcal{K})$ can be written as

$$S(x_n) = \sum_{k=0}^{n} (-1)^k k! \mathbf{B}_{n,k}(x_1, x_2, \ldots), \tag{12.3}$$

where the $\mathbf{B}_{n,k}(x_1, x_2, \ldots)$ are the partial Bell polynomials.

The family $\mathcal{P}(\mathcal{K})$ is uniform, with x_n the unique connected type of rank n, for all $n \geq 0$ (as usual, $x_0 = 1$). The rank polynomials $\mathbf{W}_{n,k}$ are given by

$$\mathbf{W}_{n,k} = \binom{n}{k} x_{n-k},$$

for all $n, k \geq 0$, where $x_r = 0$, for $r < 0$. Therefore, by theorem 8.1, the antipode of $H(\mathcal{K})$ satisfies

$$S(x_n) = (-1)^n \det \left(\binom{n-i+1}{n-j} x_{j-i+1} \right)_{1 \leq i,j \leq n}, \tag{12.4}$$

for all $n \geq 0$.

By equation 12.2, the projection $\lambda : H(\mathcal{K}) \longrightarrow P(H(\mathcal{K}))$ is given by

$$\lambda(x_n) = \sum_{k \geq 1} (-1)^{k-1} (k-1)! \mathbf{B}_{n,k}(x_1, x_2, \ldots),$$

for all $n \geq 0$. The polynomials $\lambda(x_n)$ are known as the *logarithmic polynomials* (see [5]) which are usually defined by setting $\lambda(x_n)$ equal to the coefficient of $t^n/n!$ in the series $\log(\sum_{k \geq 0} x_k t^k/k!)$, for all $n \geq 0$.

Suppose K has characteristic zero. For a multiplicative function $f \in M(H(\mathcal{K}))$, let $f(t)$ be the divided powers series $\sum_{n \geq 0} f(x_n) t^n/n!$. The correspondence $f \longrightarrow f(t)$ defines an isomorphism from $M(H(\mathcal{K}))$ onto the group (under multiplication) of divided powers series with coefficients in K, having constant term equal to one. Equations 12.3 and 12.4 thus provide formulas for the coefficients of the multiplicative inverse of any such series.

13 Non-Commutative Graph Hopf Algebras

Suppose G_1 and G_2 are simple graphs (i.e., having no loops or multiple edges) with linearly ordered vertex sets. If $U \subseteq V(G_1)$, then U inherits a linear ordering from $V(G_1)$. Also, the disjoint union of $V(G_1)$ and $V(G_2)$ is linearly ordered by concatenation. Therefore the restriction $G_1|U$ and the disjoint union of G_1 and G_2 also have linearly ordered vertex sets.

Suppose $\mathcal{G}$ is a family of simple graphs with linearly ordered vertex sets, which is closed under formation of disjoint unions and induced subgraphs. Define $G_1, G_2 \in \mathcal{G}$ to be *isomorphic* if there exists an order-preserving graph isomorphism $f : V(G_1) \longrightarrow V(G_2)$. Let $\langle G \rangle$ denote the isomorphism class, or type, of $G \in \mathcal{G}$. The set $\tilde{\mathcal{G}}$ of all types of graphs in $\mathcal{G}$ is a monoid, with product induced by disjoint union. A type $\langle G \rangle$ factors in $\tilde{\mathcal{G}}$ if and only if G can be written as a disjoint union of graphs G_1 and G_2, where each vertex set $V(G_i)$ consists of an interval in the ordering of $V(G)$. Let $\tilde{\mathcal{G}}_\circ$ denote the set of indecomposable types in $\tilde{\mathcal{G}}$. Then $\tilde{\mathcal{G}}$ is isomorphic to the free (non-commutative) monoid on $\tilde{\mathcal{G}}_\circ$.

Let $H_l(\mathcal{G})$ denote the monoid algebra of $\tilde{\mathcal{G}}$ over K. A coproduct and counit are defined on $H_l(\mathcal{G})$ by

$$\Delta\langle G \rangle = \sum_{U \subseteq V(G)} \langle G|U \rangle \otimes \langle G|(V(G) - U) \rangle,$$

and

$$\epsilon\langle G\rangle = \begin{cases} 1 & \text{if } V(G) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

for all $G \in \mathcal{G}$. $H_l(\mathcal{G})$ is a cocommutative, non-commutative and non-divisible incidence Hopf algebra, arising from products of boolean algebras in the much the same manner as the commutative graph Hopf algebra $H(\mathcal{G})$ defined earlier.

As an example, let $\mathcal{G}$ be the family of all graphs with linearly ordered vertex sets. The set of types on n vertices in $\tilde{\mathcal{G}}$ is in natural one-to-one correspondence with the set of all simple graphs with vertex set $\{1, \ldots, n\}$. Therefore, the homogenous component $H_l(\mathcal{G})(n)$ in the rank grading of $H_l(\mathcal{G})$ has dimension $2^{\binom{n}{2}}$.

14 Hereditary Families of Graphs

If G is a simple graph (finite, as usual) and S is a subset of the edge set $E(G)$, let $G|S$ denote the subgraph of G consisting of all edges in S and all vertices which are incident to edges in S, and let $G \cdot S$ denote the contraction of G to S, i.e., the graph obtained from G by succesively removing edges in $E(G) - S$ from G, where the end-vertices of each edge are identified upon its removal. The resulting graph is independent of the particular order in which the elements of $E(G) - S$ are removed. Also, note that loops and parallel edges may be created by this process of identifying vertices; so in order to obtain a simple graph $G \cdot S$, all such loops are deleted and parallel edges are replaced by single edges. A subset $S \subseteq E(G)$ is *closed* if no loops are created while forming the contraction $G \cdot (E(G) - S)$. Equivalently, S is closed if and only if there exists a partition $\pi = \pi(S)$ of the vertex set $V(G)$, where the induced subgraphs $G|B$ are connected for all blocks $B \in \pi$, such that S is equal to the union of the edge sets $E(G|B)$, $B \in \pi$.

The *lattice of contractions* of G is the lattice $\mathcal{L}(G)$ of all closed subsets of $E(G)$, ordered by inclusion. Through the correspondence $S \longrightarrow \pi(S)$, $\mathcal{L}(G)$ can be identified with the lattice of all partitions of $V(G)$ whose blocks induce connected subraphs, ordered by refinement. Graphs having isomorphic lattices of contractions were characterized by Whitney in [22]. In particular, he showed that if G is simple and 3-connected, then the isomorphism class of G is uniquely determined by the isomorphism class of $\mathcal{L}(G)$.

A *hereditary family* of graphs is a class of simple graphs which is closed under formation of direct sums, restrictions to closed edge subsets, and contractions. If $\mathcal{G}$ is a hereditary family of graphs, G and H in $\mathcal{G}$ are *weakly isomorphic* if they are isomorphic after deleting all isolated vertices. Let $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}_\circ$ denote the sets of weak isomorphism types of graphs and connected graphs in $\mathcal{G}$, respectively. Direct sum induces a product on $\tilde{\mathcal{G}}$, which is thus isomorphic to the free commutative monoid on $\tilde{\mathcal{G}}_\circ$.

Given a hereditary family of graphs $\mathcal{G}$, let $H(\mathcal{G})$ denote the monoid algebra of $\tilde{\mathcal{G}}$ over K. A coproduct Δ and counit ϵ are defined on $H(\mathcal{G})$ by

$$\Delta[G] = \sum_{S \in \mathcal{L}(G)} [G|S] \otimes [G \cdot (E(G) - S)],$$

and

$$\epsilon[G] = \begin{cases} 1 & \text{if } E(G) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

for all $[G] \in \tilde{\mathcal{G}}$. Δ is coassociative because the operations of restriction and contraction commute with one another, i.e.,

$$(G|(S_1 + S_2)) \cdot S_1 = (G \cdot (S_1 + S_3))|S_1,$$

whenever $S_1 + S_2 + S_3 = E(G)$. $H(\mathcal{G})$ is an commutative incidence Hopf algebra, which is non-cocommutative, in general, whose underlying hereditary family of posets is the family of lattices of contractions of graphs in $\mathcal{G}$.

One can define hereditary classes of matroids analagously, and construct corresponding incidence Hopf algebras exactly as for graphs. However, one does not obtain a strict generalization of the graph Hopf algebras $H(\mathcal{G})$ in this manner, due to the fact that the isomorphism relation for graphs which are not 3-connected is finer than the isomorphism relation for corresponding graphic matroids.

Example 14.1 (The Faà di Bruno Hopf Algebra) Let $\mathcal{K}$ be the hereditary family consisting of all disjoint unions of complete graphs, and let x_n denote the type of the complete graph on $n+1$ vertices. The lattice of contractions of a graph of type x_n is the full partition lattice of an $n+1$-element set. Therefore $H(\mathcal{K})$ is isomorphic to the polynomial algebra $K[x_1, x_2, \ldots]$, with coproduct and counit given by

$$\Delta(x_n) = \sum_{k=0}^{n} \mathbf{B}_{n+1,k+1}(1, x_1, x_2 \ldots) \otimes x_k,$$

and

$$\epsilon(x_n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

for all $n \geq 0$, where $x_0 = 1$, and the $\mathbf{B}_{n,k}$ are the partial Bell polynomials. The Hopf algebra $\mathcal{F} = H(\mathcal{K})$, called the *Faà di Bruno Hopf Algebra* by Joni and Rota [13], was first studied by P. Doubilet, in [6].

Suppose K has characteristic zero. For $f \in M(\mathcal{F})$, let $f(t)$ be the divided powers series $\sum_{n\geq 1} f(x_{n-1})t^n/n!$. The correspondence $f \longrightarrow f(t)$ defines an anti-isomorphism from $M(\mathcal{F})$ onto the group (under composition) of divided powers series with coefficients in K, having zero constant term and coefficient of t equal to one. One form of the Lagrange inversion formula (see [5]) states that if $f(t) = \sum_{n\geq 1} f_{n-1}t^n/n!$ and $g(t) = \sum_{n\geq 1} g_{n-1}t^n/n!$ are such series, which are inverse to one another under functional composition, then the coefficients of $g(t)$ are given by

$$g_n = \sum_{k\geq 1} (-1)^k \mathbf{B}_{n+k,k}(0, f_1, f_2, \ldots).$$

In other words, the antipode S of $\mathcal{F}$ satisfies

$$S(x_n) = \sum_{k\geq 1} (-1)^k \mathbf{B}_{n+k,k}(0, x_1, x_2, \ldots), \tag{14.1}$$

for all $n \geq 1$. A combinatorial proof of equation 14.1, using the general equation (4.1) for the antipode of an incidence Hopf algebra, was given in [11].

The hereditary family of posets corresponding to $\mathcal{K}$ is the family of all intervals in partition lattices of finite sets, which is uniform. For all $n \geq 0$, x_n corresponds to the type of the lattice of partitions of an $(n+1)$-element set, and is the unique connected type of rank n. The rank polynomial $\mathbf{W}_{n,k}$ of $\mathcal{F}$ is equal to the partial Bell polynomial $\mathbf{B}_{n+1,k+1}(1, x_1, x_2, \ldots)$, for all $n, k \geq 0$, where $\mathbf{B}_{n,k} = 0$, for $n < k$. Therefore, by theorem 8.1, the antipode of $H(\mathcal{K})$ satisfies

$$S(x_n) = (-1)^n \det \left(\mathbf{B}_{n-i+2,n-j+1}(1, x_1, x_2, \ldots) \right)_{1 \leq i,j \leq n,} \tag{14.2}$$

for all $n \geq 1$. Equation 14.2 thus provides a determinant formula for the coefficients of the inverse of a formal power series under functional composition.

Let $L(\mathcal{K}) = P(\mathcal{F}')$ be the Lie algebra of primitive elements of the graded dual hopf algebra $\mathcal{F}'$. The set $\{x'_n : n \geq 1\}$ is a basis for $L(\mathcal{K})$. The product of x'_n and x'_m in $\mathcal{F}'$ is given by

$$x'_n x'_m = \binom{n+m+1}{m} x'_{n+m} + (1 + \delta_{n,m})(x_n x_m)'.$$

Hence, the bracket $[x'_n, x'_m] = x'_n x'_m - x'_m x'_n$ satisfies

$$\begin{aligned} [x'_n, x'_m] &= \left[\binom{n+m+1}{m} - \binom{n+m+1}{n} \right] x'_{n+m} \\ &= (m-n) \frac{(n+m+1)!}{(n+1)!(m+1)!} x'_{n+m}, \end{aligned}$$

for all $n, m \geq 1$. Putting $y'_n = (n+1)! x'_n$, for all $n \geq 1$, we have

$$[y'_n, y'_m] = (m-n) y'_{n+m},$$

for all $n, m \geq 1$. If K has characteristic zero, then $\{y'_1, y'_2, \ldots\}$ is a basis for $L(\mathcal{K})$, and since $\mathcal{F}'$ is cocommutative and connected, it is isomorphic to the universal enveloping algebra of $L(\mathcal{K})$, by theorem 9.1.

Example 14.2 (Paths) Let $\mathcal{N}$ be the family of all graphs which are disjoint unions of paths, and let b_n denote the type of a path of length n, for all $n \geq 0$. The family $\mathcal{N}$ is hereditary, and $H(\mathcal{N}) \cong K[b_1, b_2, \ldots]$ with coproduct and counit given by

$$\Delta(b_n) = \sum_{k \geq 1} \sum_{\substack{n_1 + \cdots + n_k = n+1 \\ n_i \geq 1}} \left(\prod_{i=1}^{k} b_{n_i - 1} \right) \otimes b_{k-1},$$

and

$$\epsilon(b_n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

for all $n \geq 0$.

If G is a path, then every subset of $E(G)$ is closed, so the lattice of contractions of G is a boolean algebra. Thus H($\mathcal{N}$) is the incidence Hopf algebra of a hereditary family of boolean algebras. This family is uniform, and the type b_n of the lattice of contractions of a path of length n is the unique indecomposable type of rank n. The rank polynomials of $H(\mathcal{N})$ are given by

$$\mathbf{W}_{n,k-1} = \sum_{\substack{n_1 + \cdots + n_k = n+1 \\ n_i \geq 1}} \prod_{i=1}^{k} b_{n_i - 1},$$

for all $n \geq 0$.

For $f \in M(H(\mathcal{N}))$, let $f(t)$ be the power series $\sum_{n\geq 1} f(b_{n-1})t^n$. The correspondence $f \longrightarrow f(t)$ defines an anti-isomorphism from $M(H(\mathcal{N}))$ onto the group (under composition) of power series with coefficients in K, having zero constant term and coefficient of t equal to one. Theorem 8.1 thus gives a determinant formula for the inverse of such series.

The algebra map $\alpha : \mathcal{F} \longrightarrow H(\mathcal{N})$ defined by $\alpha(x_n) = (n+1)!b_n$, for all $n \geq 0$, is a Hopf algebra map, which is an isomorphism when K has characteristic zero. The transpose $\alpha' : H(\mathcal{N})' \longrightarrow \mathcal{F}'$ satisfies $\alpha'(b'_n) = (n+1)!x'_n = y'_n$, for all $n \geq 0$. Thus the bracket in $L(\mathcal{N}) = P(H(\mathcal{N})')$ is given by $[b'_n, b'_m] = (m-n)b'_{n+m}$, for all $n, m \geq 1$, which is also not difficult to compute directly.

If K is the Lazard Ring $\pi_*(MU)$, then $H(\mathcal{N})$ is isomorphic as a Hopf algebra to the dual $MU_*(MU)$ of the Novikov algebra of operations on MU-cohomology (see [2]).

15 Minor-Closed Families of Matroids

Suppose $M = M(S)$ is a matroid on a (finite) set S. If $U \subseteq S$, then $M|U$ and $M \cdot U$ denote the restriction and contraction of M to U, respectively. A matroid N is a *minor* of M if it is obtained from M by any combination of restrictions and contractions. As in the case of graphs, these operations commute, so any chain $T \subseteq U \subseteq S$ determines a unique minor $M(T, U)$ of M, where $M(T,U) = (M|U) \cdot T = (M \cdot (S - (U - T)))|T$.

If $M = M(S)$ and $N = N(T)$ are matroids, then $M + N$ denotes the direct sum of M and N, whose underlying set is the disjoint union $S + T$. A matroid is *connected* if and only if it cannot be expressed as a direct sum of two non-trivial matroids.

Suppose $\mathcal{M}$ is a family of matroids which is closed under formation of minors and direct sums. Let $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_\circ$ denote the sets of isomorphism types of matroids and connected matroids in $\mathcal{M}$, respectively. Direct sum induces a product on $\tilde{\mathcal{M}}$, which is thus isomorphic to the free commutative monoid on $\tilde{\mathcal{M}}_\circ$. Let $H(\mathcal{M})$ denote the monoid algebra of $\tilde{\mathcal{M}}$ over K. A coproduct Δ and counit ϵ are defined on $H(\mathcal{M})$ by

$$\Delta[M] = \sum_{U \subseteq S} [M|U] \otimes [M \cdot (S - U)],$$

and

$$\epsilon[M] = \begin{cases} 1 & \text{if } S = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

for all $[M] = [M(S)] \in \tilde{\mathcal{M}}$. $H(\mathcal{M})$ is a commutative, usually non-cocommutative, incidence Hopf algebra, whose underlying hereditary family consists of boolean algebras of subsets of point sets of matroids in $\mathcal{M}$.

16 Distributive Lattices

All posets considered in this section are finite and do not necessarily have unique maximal and minimal elements.

An *order ideal* in a poset P is a subset L of P such that whenever $y \in L$ and $x \leq y$ in P then $x \in L$. A subposet Q of P is *convex* if, whenever $x \leq y$ in Q and $z \in P$ satisfies $x \leq z \leq y$, then $z \in Q$; that is, if and only if $Q = L - I$, for some order ideals $I \subseteq L$ in P.

Let $J(P)$ be the distributive lattice of all order ideals of the poset P, ordered by inclusion (see [3] for more about distributive lattices). The isomorphism class of P is uniquely determined by that of $J(P)$, and $J(P+Q)$ is naturally isomorphic to $J(P) \times J(Q)$, for all posets P and Q (where $P+Q$ denotes the disjoint union of P and Q). If P is a poset and $I \leq L$ in $J(P)$, then the interval $[I, L]$ in $J(P)$ is naturally isomorphic to the lattice of order ideals $J(L-I)$.

Suppose $\mathcal{F}$ is a family of posets which is closed under formation of disjoint unions and convex subposets. The set of distributive lattices $\mathcal{P} = \{J(P) : P \in \mathcal{F}\}$ is a hereditary family. Let $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}_\circ$ denote the sets of isomorphism classes of posets and connected posets, respectively, in $\mathcal{F}$. Disjoint union induces a product on $\tilde{\mathcal{F}}$, which is thus isomorphic to the free commutative monoid on $\tilde{\mathcal{F}}_\circ$. Letting $\sim$ be the isomorphism relation on $\mathcal{P}$, we can identify the monoid of types $\tilde{\mathcal{P}}$ with $\tilde{\mathcal{F}}$, and the set of connected types $\tilde{\mathcal{P}}_\circ$ with $\tilde{\mathcal{F}}_\circ$. The incidence Hopf algebra $H(\mathcal{F}) = H(\mathcal{P})$ is thus isomorphic to the polynomial algebra $K[\tilde{\mathcal{F}}_\circ]$, with coproduct and counit given by

$$\Delta[P] = \sum_{I \in J(P)} [I] \otimes [P - I]$$

and

$$\epsilon[P] = \begin{cases} 1 & \text{if } P = \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

for all $[P] \in \tilde{\mathcal{F}}$.

Example 16.1 (Rooted Forests) A *tree* is a (finite) simple graph which contains no cycles, and a *rooted tree* is a tree together with a distinguished element, called the *root*. A *rooted forest* is a graph whose connected components are rooted trees. A subforest (i.e, subgraph) G of a rooted forest F is also rooted, by letting the roots of G be those vertices in G having minimal distance to a root in F. Alternatively, one can define a rooted forest as a finite poset F in which every element is covered by at most one element. The Hasse diagram of such a poset is a forest in the usual sense, having maximal elements as roots. A subforest of a rooted forest F is a disjoint union of convex subposets of F.

If $\mathcal{F}$ is a family of forests which is closed under formation of subforests and disjoint unions, then we can construct the incidence Hopf algebra $H(\mathcal{F})$ arising from the family of distributive lattices $\{J(F) : F \in \mathcal{F}\}$.

The Hopf algebra $H(\mathcal{F})$, for $\mathcal{F}$ the set of all rooted forests, was introduced in [8] by Dür, who mentions applications to the theory of Butcher series and thus to numerical integration of ordinary differential equations. There is a canonical bijection from $\mathcal{F}$ onto the set of all non-empty rooted trees $\mathcal{T}$, given by adding a new vertex v to any forest F, connecting v by an edge to each of the roots of F and declaring v to be the root of the resulting tree. Later, and independently of Dür, Grossman and Larson defined a Hopf algebra having isomorphism classes in $\mathcal{T}$ as a basis, which *via* the above bijection is isomorphic to the graded dual of $H(\mathcal{F})$ (see [9]). They also define analagous Hopf algebras for the family of rooted trees endowed with various additional structures, such as labellings and orderings. In [10], Grossman and Larson give applications of the Hopf algebra $H(\mathcal{F})'$ to differential equations, symbolic algebra, and solutions of non-linear systems.

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