Chapter 13

RELIABILITY MODELING AND ANALYSIS IN RANDOM ENVIRONMENTS

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Abstract We consider a number of models where the main emphasis is on the effects of random environmental changes on system reliability. They include complex hardware and software systems which operate under some set of environmental states that affect the failure structure of all components. Our discussion will be of an expository nature and we will review mostly existing and ongoing research of the authors. In so doing, we will present an overview of continuous and discrete-time models and their statistical analyses in order to provide directions for future research.

Keywords: Reliability models, random environment, Markov modulation, operational profile, Bayesian analysis.

1. Introduction and Overview

In this expository paper, we consider complex reliability models that operate in a randomly changing environment which affects the model parameters. Here, complexity is due not only to the variety in the number of components of the model, but also to the fact that these components are interrelated through their common environmental process. For example, a complex device like an airplane consists of a large number of components where the failure structure of each component depends very much on the set of environmental conditions that it is subjected to during flight. The levels of vibration, atmospheric pressure, temperature, etc. obviously change during take-off, cruising and landing. Component lifetimes and reliabilities depend on these random environmental variations. Moreover, the components have dependent lifetimes since they operate in the same environment. A similar observation holds in software systems. For example, an airline reservation system consists of many modules where failures may be experienced due to the faults or bugs that are still present. In this case, the way that the user operates the system, or the so-called operational profile, plays a key role in software reliability assessment. Failure probabilities of the modules and system reliability all depend on the random sequence of operations it performs. The operational profile in this setting provides the random environment for the software system.

The term "environment" is used in the generic sense in this papers so that it represents any set of conditions that affect the stochastic structure of the model investigated. The concept of an "environmental" process, in one form or another, has been used in the literature for various purposes. Neveu [25] provides an early reference to paired stochastic processes where the first component is a Markov process while the second one has conditionally independent increments given the first. Ezhov and Skorohod [8] refer to this as a Markov process with homogeneous second component. In a more modern setting, Çınlar ([3], [4]) introduced Markov additive processes and provided a detailed description on the structure of the additive component. The environment is modelled as a Markov process in all these cases and the additive process represents the stochastic evolution of a quantity of interest.

The use of an environmental process to modulate the deterministic and stochastic parameters of operations research models is not limited to reliability applications only. Özekici [27] discusses other applications in inventory and queueing. In inventory models, the stochastic structure is depicted by the demand and the lead-time processes. Song and Zipkin [41] argue that the demand for the product may be affected by a randomly changing "state-of-the-world", which we choose to call the "environment" in our exposition. A periodic review model in a random environment with uncertain supply is analyzed in Özekici and Parlar [31].

Queueing models also involve stochastic and deterministic parameters that are subject to variations depending on some environmental factors. The customer arrival rate as well as the service rate are not necessarily constants that remain intact throughout the entire operation of the queueing system. A queueing model where the arrival and service rates depend on a randomly changing two-state environment was first introduced by Eisen and Tainiter [7]. This line of modelling is later extended by other authors such as Neuts ([23], [24]) and Purdue [36]. A comprehensive discussion on Markov modulated queueing systems can be found in Prabhu and Zhu [35].

Although the literature cited above clearly illustrate the use of random environments in inventory and queueing models, the concept is much more applicable in reliability and maintenance models. It is generally assumed that a device always works in a given fixed environment. The probability law of the deterioration and failure process thus remains intact throughout its useful life. The life distribution and the corresponding failure rate function is taken to be the one obtained through statistical life testing procedures that are usually conducted under ideal laboratory conditions by the manufacturer of the device. Data on lifetimes may also be collected while the device is in operation to estimate the life distribution. In any case, the basic assumption is that the prevailing environmental conditions either do not change in time or, in case they do, they have no effect on the deterioration and failure of the device. Therefore, statistical procedures in estimating the life distribution parameters and decisions related with replacement and repair are based on the calendar age of the item.

There has been growing interest in recent years in reliability and maintenance models where the main emphasis is placed on the so-called intrinsic age of a device rather than its real age. This is necessitated by the fact that devices often work in varying environments during which they are subject to varying environmental conditions with significant effects on performance. The deterioration and failure process therefore depends on the environment, and it no longer makes much sense to measure the age in real time without taking into consideration the different environments that the device has operated in. There are many examples where this important factor can not be neglected or overlooked. Consider, for example, the jet engine of an airplane which is subject to varying atmospheric conditions like pressure, temperature, humidity, and mechanical vibrations during take-off, cruising, and landing. The changes in these conditions cause the engine to deteriorate, or age, according to a set of rules which may well deviate substantially from the usual one that measures the age in real time irrespective of the environment.

As a matter of fact, the intrinsic age concept is being used routinely in practice in one form or another. In aviation, the calendar age of an airplane since the time it was actually manufactured is not of primary importance in determining maintenance policies. Rather, the number of take-offs and landings, total time spent cruising in fair conditions or turbulence, or total miles flown since manufacturing or since the last major overhaul are more important factors.

Another example is a machine or a workstation in a manufacturing system which may be subject to varying loading patterns depending on the production schedule. In this case, the atmospheric conditions do not necessarily change too much in time, and the environment is now represented by varying loading patterns so that, for example, the workstation ages faster when it is overloaded, slower when it is underloaded, and not at all when it is not loaded or kept idle. Therefore, the term "environment" is used in a loose sense here so that it represents any set of conditions that affect the deterioration and aging of the device.

In what follows, we assume that the system operates in a randomly changing environment depicted by $Y = \{Y_t; t \in T\}$ where Y_t is the state of the environment at time t. The environmental process Y is a stochastic process with time-parameter set T and some state space E which is assumed to be discrete to simplify the notation.

In Section 2, we consider continuous-time models applicable to hardware systems. This will focus mainly on the intrinsic aging concept. Section 3 is on continuous-time software reliability models where the operational profile plays the key role in testing as well as reliability assessment. Discrete-time periodic models are considered in Section 4 where we first discuss Markov modulated Bernoulli processes in the context of reliability applications and extend this discussion later to networks.

2. Continuous Time Models with Intrinsic Aging

An interesting model of stochastic component dependence was introduced by Çınlar and Özekici [5] where stochastic dependence is introduced by a randomly changing common environment that all components of the system are subjected to. This model is based on the simple observation that the aging or deterioration process of any component depends very much on the environment that the component is operating in. They propose to construct an intrinsic clock which ticks differently in different environments to measure the intrinsic age of the device. The environment is modelled by a semi-Markov jump process and the intrinsic age is represented by the cumulative hazard accumulated in time during the operation of the device in the randomly varying environment. This is a rather stylish choice which envisions that the intrinsic lifetime of any device has an exponential distribution with parameter 1. There are, of course, other methods of constructing an intrinsic clock to measure the intrinsic age. Also, the random environment model can be used to study reliability and maintenance models involving complex devices with many interacting components. The lifetimes of the components of such complex devices are stochastically dependent due to the common environment they are all subject to.

2.1. Intrinsic Aging in a Fixed Environment

The concept of random hazard functions is also used in Gaver [10] and Arjas [1]. The intrinsic aging model of Çınlar and Özekici [5] is studied further in Çınlar et al. [6] to determine the conditions that lead to associated component lifetimes, as well as multivariate increasing failure rate (IFR) and new better than used (NBU) life distribution characterizations. It was also extended in Shaked and Shanthikumar [38] by discussions on several different models with multicomponent replacement policies. Lindley and Singpurwalla [17] discuss the effects of the random environment on the reliability of a system consisting of components which share the same environment. Although the initial state of the environment is random, they assume that it remains constant in time and components have exponential life distributions in each possible environment. This model is also studied by Lefèvre and Malice [15] to determine partial orderings on the number of functioning components and the reliability of k-out-of-n systems, for different partial orderings of the probability distribution on the environmental state. The association of the lifetimes of components subjected to a randomly varying environment is discussed in Lefèvre and Milhaud [16]. Singpurwalla and Youngren [40] also discuss multivariate distributions that arise in models where a dynamic environment affects the failure rates of the components.

For a complex model with m components, intrinsic aging in Çınlar and Özekici [5] is described by the basic relationship

$$\frac{dA_t}{dt} = f(Y_t, A_t) \tag{13.1}$$

where $A_t = (A_t^1, A_t^2, \cdots, A_t^m)$ is the intrinsic age of the system at time t that consists of the intrinsic ages of the m components, $Y = (Y_t^1, Y_t^2, \cdots, Y_t^d)$ is the environmental process with state space E that reflects the states of various environmental factors and f is the intrinsic hazard rate function. For example, Y_t^1 can be the calendar time t, Y_t^2 could be the temperature at t, Y_t^3 could be the pressure at time t, etc.. Moreover, f is of the form $f(i,x) = (f_1(i,x), f_2(i,x), \cdots, f_m(i,x))$ where $f_k(i,x)$ is the intrinsic aging rate of component k in environment i if the intrinsic ages of the components are given by the vector $x = (x_1, x_2, \cdots, x_m)$.

In this exposition, we will further specialize on this basic model by adapting the notation and terminology of Özekici [26] who analyzed the optimal maintenance problem of a single-component device operating in a random environment. In particular, we suppose that the state space E is discrete and let L denote the lifetime of the device. Suppose, for now, that the environment remains fixed at some state $i \in E$ so that $Y_t = i$ for all $t \geq 0$. In any environment $i \in E$, the life distribution is given by the cumulative distribution function

$$F_i(t) = P[L \leq t | Y = i] \tag{13.2}$$

with failure rate function $r_i(t)$ and hazard function $R_i(t) = \int_0^t r_i(s)ds$ so that the survival probability function $\overline{F}_i = 1 - F_i$ can be written as

$$\overline{F}_i(t) = P[L > t | Y = i] = \exp(-R_i(t)). \tag{13.3}$$

Relationship (13.3) allows us to construct an intrinsic clock to measure the intrinsic age of the device at time t as $A_t = R_i(t)$ and the real lifetime is characterized by

$$L = \inf\{s \geq 0; A_s > \hat{L}\} \tag{13.4}$$

where $\hat{L}$ is a random variable representing the intrinsic lifetime of component k. Moreover, it has an exponential distribution with parameter 1 since

$$F_i(t) = P[L \leq t | Y = i] = P[\hat{L}_k < A_t | Y = i] = 1 - e^{-A_t}\ . \tag{13.5}$$

Therefore, in the fixed environment $i \in E$, it follows that if the intrinsic age is measured by the hazard function, then component k has an exponentially distributed intrinsic lifetime with parameter 1. Moreover, its intrinsic clock ticks at the rate $r_i(t)$ at time t. If the real time is t, then the intrinsic clock shows time $R_i(t)$. Similarly, when the intrinsic time is x, the corresponding real time is given by the inverse function

$$\mathcal{R}_i(x) = \inf\{t \geq 0; R_i(t) > x\}. \tag{13.6}$$

In other words, it takes $\mathcal{R}_i(x)$ units of real time operation to age a brand new component to intrinsic age x in environment i.

2.2. Intrinsic Aging in a Random Environment

Suppose now that the environmental process is not fixed but described as the minimal semi-Markov process associated with a Markov renewal process. Let T_n denote the time of the n'th environment change and X_n denote the n'th environmental state for $k \geq 0$ with $T_0 \equiv 0$. The main assumption is that the process $(X, T) = \{(X_n, T_n); n \geq 0\}$ is a Markov renewal process on the state space $E \times R_+$ with some semi-Markov kernel Q where $R_+ = [0, +\infty)$. Moreover, $Y = \{Y_t; t \geq 0\}$ is the minimal semi-Markov process associated with (X, T). More precisely, $Y_t = X_n$ whenever $T_n \leq t < T_{n+1}$. For any $i, j \in E$ and $t \geq 0$,

$$Q(i, j, t) = P[X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i] \tag{13.7}$$

and it is well-known that X is a Markov chain on E with transition matrix $P(i, j) = P[X_{n+1} = j | X_n = i] = Q(i, j, +\infty)$. We further assume that the Markov renewal process has infinite lifetime so that $\sup_n T_n = +\infty$.

A stylish choice to extend the construction of the intrinsic aging process in this setting is to measure the age by the total hazard accumulated during the operation of the device in the randomly varying environment. Therefore, the age process $A = \{A_t; t \geq 0\}$ is the continuously increasing stochastic process defined by

$$\frac{dA_t^k}{dt} = r_{X_n}(\mathcal{R}_{X_n}(A_{T_n}) + (t - T_n)) \tag{13.8}$$

for $T_n \leq t < T_{n+1}$. To simplify the notation, it is convenient to set

$$H_i(x,t) = R_i(\mathcal{R}_i(x) + t) \tag{13.9}$$

so that this represents the amount of aging caused by operating component k of initial age x in environment i for t real time units. If the device is initially at age x at the beginning of environment i, then the amount of real time operation required to age it u time units in this environment is given by

$$\tau_i(x,u) = \mathcal{R}_i(x+u) - \mathcal{R}_i(x) \; . \tag{13.10}$$

To observe the relationship between the intrinsic failure rate f in (13.1) and the ordinary failure rate function r in the present setting, please note that (13.8) implies $f(i,x) = r_i(\mathcal{R}_i(x))$ in compact notation.

This intrinsic aging model simply combines the hazard functions of the components in the environmental states. Given the failure rate functions $\{r_i(\cdot); i \in E\}$ and a realization of the environmental process Y, the age process A is completely defined by (13.8). If the initial age is $A_0 = x$ and the initial environment is $X_0 = i$ with $T_0 \equiv 0$, then the initial real age of the component is $\mathcal{R}_i(x)$ and it ages as $A_s = H_i(x,s)$ for $s \leq T_1$. At some time $T_1 = u$, the environment jumps to state $j \in E$ with some probability $Q(i,j,du)$ and the age is now the accumulated hazard given by $A_{u+s} = H_j(A_u, s)$ for $u + s \leq T_2$. The sample path of A is constructed similarly in time as the environmental process evolves so that, in general, if the environment jumps to some state $l \in E$ at the n'th jump time $T_n = t$, then the age evolves as $A_{t+s} = H_l(A_t, s)$ so long as $t + s \leq T_{n+1}$.

3. Continuous Time Models in Software Reliability Engineering

An undesirable feature of almost all software reliability models is that the parameters of the software failure process, as well as the costs, are independent of the operations that the software performs. This assumes that the software is used for a single operation or that there are no differences between the model parameters under different operations. Moreover, it is generally assumed that there is only one test case. In practice, however, a number of different test cases are run during the testing procedure before the software is released. The model parameters should therefore depend on the test case as well.

Musa [20], [21] argues that a consideration of the software's operational profile should reduce system risk. Moreover, it also makes the testing procedure faster and more efficient. Optimal testing problems involving operational profiles are discussed in detail by Özekici et al. ([30], [29]) who present computational procedures to determine the optimal testing durations for the operations. The notion of an operational profile was introduced by Musa et al. [22]. An operation is an externally initiated task performed by a system "as built". A software system is usually designed to perform a set of well-defined operations or tasks. The operational profile of any software describes how users employ the system. It is a quantitative and probabilistic characterization of how a system will be used. An operational profile is defined as a set of operations and the probabilities of their occurrence. Formally, let X be a generic random variable that represents the operation to be performed by the system and E denote the set of all possible operations. Suppose that the probability distribution function of X is $\pi(n) = P[X = n]$. Then, the pair (E, π) is the operational profile which contains both the possible operations and the probability of the occurrences of these operations.

We claim that the concept of an "operational profile" should be expanded to that of an "operational process", as a more meaningful approach to model software usage. In such a model, E will still denote the set of all operations but the kth operation performed by the system will be denoted by the random variable X_k. The operational process will then be the stochastic process $X = \{X_k; k \geq 0\}$ with state space E. The operational process in software reliability engineering clearly plays the role of the environmental process in hardware reliability. It is simply a stochastic process that modulated the model parameters.

Musa [20] describes how operational profiles can be built and states that there is substantial benefit to be gained by applying it. It can increase user satisfaction by capturing their needs more precisely, satisfy important user needs faster, reduce costs with reduced operation software, speed up the development and improve productivity by allocating resources in relation to use and criticality, reduce the system risk by more realistic testing, and make testing faster and more efficient. Wohlin and Runeson [44] also discuss the effect of usage modelling in software certification. A stochastic model of software usage involving Markov chains is employed in Whittaker and Poore [42] and Whittaker and Thomason [43]. In their approach, the sequence of "inputs" provided by the user is modelled as a Markov chain. This results in a model at the micro level involving all possible values of input variables with a huge state space. An operational process, on the other hand, provides a stochastic model at a more refined macro level because an operation corresponds to a specific task which usually involves ranges of values for many input variables at the same time. The operational profile model concentrates on the user-initiated tasks performed by the system rather than the sequence of user-supplied input values.

3.1. Optimal Testing Strategies

An important decision problem in software engineering is the determination of the optimal release times of software. In life critical software, the most important attribute of the system is its reliability. Therefore, no effort is spared in making the software as reliable as is possible before releasing it for operation. In most of the business related software the total cost of the software is as important as its reliability. This fact gives a trade-off between the reliability of the software and its cost. A consequence of this consideration is the development of the stopping rules for the testing procedure that will minimize the total expected cost or maximize the total expected benefit. It is assumed that there are a random number of faults in the software before testing. The objective of testing is to remove as many of these faults as is possible by debugging. The important issue to note is that all economic and stochastic parameters of the model depend on the test case and the specific operation performed. In Özekici et al. [30], the time to failure caused by each fault is exponential and there are a total of K operations that the software is required to perform. The software is to be tested in all of these operations and the decision problem is to determine the durations. Note that in the testing problem the operational process is controlled and it is not stochastic.

One of the models discussed leads to the following optimization problem

$$\min_{\substack{0\leq t_k<+\infty\\ k=1,\cdots,K}} \sum_{k=1}^{K}\left[c_k t_k + \mu(f_k - f_{k+1})\exp\Big(-\sum_{m=1}^{k}\lambda_m t_m\Big)\right] \qquad (13.11)$$

where t_k is the duration of testing for operation k, c_k is the cost of testing per unit time for operation k, f_k is the benefit of testing for operation k and λ_k is the rate of failures caused by each fault during operation k. Under reasonable assumptions, this problem leads to the explicit solution

$$t_k^* = \frac{1}{\lambda_k}\ln\left[\left(\frac{b_k - b_{k+1}}{a_k - a_{k+1}}\right) / \left(\frac{b_{k-1} - b_k}{a_{k-1} - a_k}\right)\right] \qquad (13.12)$$

with $a_k = c_k/\lambda_k$ and $b_k = \mu f_k$.

The optimal testing model is extended recently by Özekici and Soyer [33] using a Bayesian framework. Uncertainty about model parameters are described probabilistically using available prior information on them. The information gathered during each test is used to update the model parameters and determine the testing durations sequentially. In particular, λ_k is assumed to have a Gamma (α_k, β_k) prior and the initial number of faults have apriori Poisson distribution. The operational process is still controlled but the durations of the operations are not fixed at the beginning of testing. They are recalculated using data obtained for each operation. When $K = 1$, we obtain the Jelinski and Moranda [13] model for which Bayesian analysis has been done by Meinhold and Singpurwalla [18] and more recently by Kuo and Yang [14] by using a Gibbs sampler.

At the nth stage of testing, having obtained data $\mathcal{D}^{n-1}$ testing for the first $n-1$ operations, one is now faced with the optimization problem

$$\min_{\substack{0 \le t_k < +\infty \\ k=n,\cdots,K}} \sum_{k=n}^{K} \left[c_k t_k + E[N_n | \mathcal{D}^{n-1}](f_k - f_{k+1}) \prod_{j=n}^{k} \left(\frac{\beta_j}{\beta_j + t_j} \right)^{\alpha_j} \right] \tag{13.13}$$

where N_n is the number of faults that still remain in the software at the beginning of the nth stage. To solve this problem sequentially, Gibbs sampling is used to do posterior analysis on the number of faults remaining. Although (13.13) does not necessarily have an explicit optimal solution as (13.12), it can be solved using various optimization procedure. Under reasonable conditions, the objective function is convex. The details of the Bayesian analysis will be presented in Section 3.3.

3.2. Software Reliability Assessment

Once testing is completed, the software is released to the users. This is done in an uncontrolled setting and the sequence of operations as well as their durations are now random. This operational process or the environmental process now modulates the parameters of the reliability model and play a crucial role in software reliability assessment. Now, the environmental state Y_t at time t represents the operation performed by the user. The analysis of the software failure process obviously depend on the stochastic structure of the operational process. In Özekici and Soyer [32], Y is assumed to be a Markov process. Briefly, this means that the sequence of operations performed is a Markov chain and the amount of time spent on each operation is exponentially distributed. More precisely, we let X_n denote the nth operation that the system performs and T_n be the time at which the nth operation starts. It is well-known that X is a Markov chain with some transition matrix

$$P(i,j) = P[X_{n+1} = j | X_n = i] \tag{13.14}$$

and

$$P[T_{n+1} - T_n > t | X_n = i] = e^{-\mu(i)t} \tag{13.15}$$

so that the duration of the nth operation is exponentially distributed with rate $\mu(i)$ if this operation is i. The probabilistic structure of the operational process is given by the generator $A(i,j) = \mu(i)(P(i,j) - I(i,j))$ where I is the identity matrix.

An overview of software failure models is presented in Singpurwalla and Soyer [39]. Perhaps the most important aspect of these models is related to the stochastic structure of the underlying failure process. This could be a "times-between-failures" model which assumes that the times between successive failures follow a specific distribution whose parameters depend on the number of faults remaining in the program after the most recent failure. One of the most celebrated failure models in this group is that of Jelinski and Moranda [13] where the basic assumption is that there are a fixed number of initial faults in the software and each fault causes failures according to a Poisson process with the same failure rate. After each failure, the fault causing the failure is detected and removed with certainty so that the total number of faults in the software is decreased by one. In the present setting, the time to failure distribution for each fault in the software is exponentially distributed with parameter $\lambda(k)$ during operation k and this results in an extension of the Jelinski-Moranda model.

In dealing with software reliability, one is interested in the number of faults N_t remaining in the software at time t. Then, N_0 is the initial number of faults and the process $N = \{N_t; t \geq 0\}$ depicts the stochastic evolution of the number of faults. If there is perfect debugging, then N decreases as time goes on, eventually to diminish to zero. Defining the bivariate process $Z_t = (Y_t, N_t)$, it follows that $Z = (Y, N)$ is a Markov process with discrete state space $F = E \times \{0, 1, 2, \cdots\}$. This follows by noting that Y is a Markov process and N is a process that decreases by 1 after an exponential amount of time with a rate that depends only on the state of Y. In particular, if the current state of Z is (i, n) for any $n > 0$, then the next state is either (j, n) with rate $\mu(i)P(i, j)$ or $(i, n-1)$ with rate $n\lambda(i)$. If $n = 0$, then the next state is $(j, 0)$ with rate $\mu(j)$. Note that 0 is an absorbing state for N.

This implies that the sojourn in state (i, n) is exponentially distributed with rate

$$\beta(i, n) = \mu(i) + n\lambda(i) \tag{13.16}$$

and the generator Q of Z is

$$Q((i, n), (j, m)) = \begin{cases} -(\mu(i) + n\lambda(i)), & j = i, m = n \\ \mu(i)P(i, j), & j \neq i, m = n \\ n\lambda(i), & j = i, m = n-1 \end{cases} . \tag{13.17}$$

Reliability is defined as the probability of failure free operation for a specified time. We will denote this by the function

$$R(i, n, t) = P[L > t | Y_0 = i, N_0 = n] = P[N_t = n | Y_0 = i, N_0 = n] \tag{13.18}$$

defined for all $(i, n) \in F$ and $t \geq 0$. Note that this is equal to the probability that there will be no arrivals until time t in a Markov modulated Poisson process with intensity function $\hat{\lambda}(t) = n\lambda(Y_t)$. Thus, using the matrix generating function (22) in Fischer and Meier-Hellstern [9] with $z = 0$, we obtain the explicit formula

$$R(i, n, t) = \sum_{j \in E} \left[e^{(A - n\Lambda)t} \right]_{ij} \tag{13.19}$$

where

$$e^{(A-n\Lambda)t} = \sum_{k=0}^{+\infty} \frac{t^k}{k!}(A-n\Lambda)^k \tag{13.20}$$

is the exponential matrix and $\Lambda(i,j) = \lambda(i)I(i,j)$.

3.3. Bayesian Analysis of Software Reliability Models

In the software reliability model of Section 3.2, it is assumed that the parameters are given. A Bayesian analysis of this model can be developed as in Özekici and Soyer [32] by specifying prior probability distributions to describe uncertainty about the unknown parameters. In Özekici and Soyer [32] independent gamma priors are assumed on each $\lambda(i)$ with shape parameter $a(i)$ and scale parameter $b(i)$, denoted as $\lambda(i) \sim$ Gamma $(a(i), b(i))$ for all $i \in E$. Similarly, independent gamma priors are assumed for the components of μ as $\mu(i) \sim$ Gamma $(c(i), d(i))$ for all $i \in E$. A Poisson distribution with parameter γ, denoted as $N_0 \sim$ Poisson (γ), is assumed as the prior for initial number of faults N_0. For the components of the transition matrix, the ith row $P_i = \{P(i,j); j \in E\}$ has a Dirichlet prior

$$p(P_i) \propto \prod_{j\in E} P(i,j)^{\alpha_j^i - 1} \tag{13.21}$$

denoted as Dirichlet $\{\alpha_j^i; j \in E\}$ and P_i's are independent for all $i \in E$. Furthermore, it is assumed that apriori λ, μ, P and N_0 are independent. We denote the joint prior distribution of the parameters by $p(\Theta)$ where $\Theta = (\lambda, \mu, P, N_0)$.

During the usage phase as debugging is performed, the failure times during each operation as well as the operation types and their durations are observed. Assuming that during a usage phase of τ units of time K operations are performed and $K-1$ of those are completed, the observed data is given by $D = \{(X_k, S_k), (U_1^k, U_2^k, \ldots, U_{M_k}^k); k = 1, \cdots, K\}$ where X_k is the kth operation performed, while S_k is the time at which the kth operation starts and U_j^k is the time (since the start of the kth operation) of jth failure during the kth operation. Defining $N_k = N_{S_k}$ to denote the total number of faults remaining in the software just before the kth operation, M_k as the number of failures observed during the kth operation and assuming that the initial operation is $X_1 = i$ for some operation i starting at $S_1 = 0$, $\mathcal{L}(\Theta|D)$, the likelihood function of Θ is obtained as

$$\mathcal{L}_K \left\{ \prod_{k=1}^{K-1} P(X_k, X_{k+1})\mu(X_k)e^{-\mu(X_k)(S_{k+1}-S_k)} \right. \tag{13.22}$$

$$\left. \cdot\frac{N_k!}{(N_k-M_k)!}\lambda(X_k)^{M_k}\, e^{-\lambda(X_k)[\sum_{j=1}^{M_k} U_j^k + (N_k-M_k)(S_{k+1}-S_k)]} \right\}$$

where $\mathcal{L}_K$ is the contribution of the Kth operation to the likelihood given by

$$\mathcal{L}_K = e^{-\mu(X_K)(\tau-S_K)}\frac{N_K!}{(N_K-M_K)!} \cdot\lambda(X_K)^{M_K}\, e^{-\lambda(X_K)[\sum_{j=1}^{M_K} U_j^K+(N_K-M_K)(\tau-S_K)]}. \tag{13.23}$$

Given the independent priors, the posterior distribution of P_i's can be obtained as independent Dirichlets given by

$$(P_i|D) \sim \text{Dirichlet } \{\alpha_j^i + \sum_{k=1}^{K-1} 1(X_k = i, X_{k+1} = j); j \in E\} \tag{13.24}$$

where $1(\cdot)$ is the indicator function. Similarly, the posterior distributions of $\mu(i)$'s are obtained as independent gamma densities given by

$$(\mu(i)|\, D) \sim \text{Gamma}\,(c(i)+\sum_{k=1}^{K-1} 1(X_k=i),\, d(i)+\sum_{k=1}^{K}(S_{k+1}-S_k)1(X_k=i)) \tag{13.25}$$

where $S_{K+1} = \tau$. We note that posteriori μ and P are independent of λ and N_0 as well as each other.

A tractable Bayesian analysis for λ and N_0 is not possible due to the infinite sums involved in the posterior terms, but the Bayesian analysis can be made by using a Gibbs sampler [see Gelfand and Smith [12]]. The implementation of the Gibbs sampler requires the full posterior conditionals $p(N_0|\lambda,\, D)$ and $p(\lambda(i)|\lambda^{(-i)},\, N_0,\, D)$ for all $i \in E$ where $\lambda^{(-i)} = \{\lambda(j);\, j \neq i\,, j \in E\}$. Using the fact that $N_1 = N_0$, it can be shown that

$$(N_0 - M\,|\lambda,\, D) \sim \text{Poisson}\,(\gamma e^{-\sum_{k=1}^{K}\lambda\,(X_k)(S_{k+1}-S_k)}), \tag{13.26}$$

where $M = \sum_{k=1}^{K} M_k$. The full conditionals, $p(\lambda(i)|\lambda^{(-i)},\, N_0,\, D)$'s are obtained as

$$\text{Gamma}\,(a(i) + \sum_{k=1}^{K} M_k\, 1(X_k = i), b(i) + \sum_{k=1}^{K} W_k\, 1(X_k = i)) \tag{13.27}$$

where $W_k = \sum_{j=1}^{M_k} U_j^k + (N_k - M_k)(S_{k+1} - S_k)$. Thus all of the posterior distributions can be evaluated by recursively simulating from the full conditionals in a straightforward manner. It is important to note that using the independent priors, given N_0, aposteriori the $\lambda(i)$'s are independent.

We note that in the controlled testing setup of Özekici and Soyer [33], presented in Section 3.2, the operations and their durations are deterministic. Thus, the Bayesian inference in the controlled testing setup can be obtained as a special case by using (13.26) and (13.27) above with $t_k = (S_{k+1} - S_k)$ and $\beta_k = b(k)$ and the expected cost term (13.13) can be evaluated and optimized sequentially after each testing stage.

Once uncertainty about Θ is revised to $p(\Theta|D)$, it is of interest to make posterior reliability predictions as $P[L > t|D]$. Note that both A and Λ are functions of Θ. Conditional on Θ, using the Markov property of the Z process and (13.19), we obtain

$$P[L > t|\Theta, D] = \sum_{j \in E} \left[e^{(A(\Theta)-(N_0-M)\Lambda(\Theta))t} \right]_{X_K,j}. \tag{13.28}$$

Conditional on Θ, $e^{(A(\Theta)-(N_0-M)\Lambda(\Theta))t}$ can be computed from the matrix exponential form using one of the available methods, for example, in Moler and van Loan [19]. Then the posterior reliability prediction can be approximated as a Monte Carlo integral

$$P[L > t|D] \approx \frac{1}{G} \sum_g P[L > t|\Theta^{(g)}, D] \tag{13.29}$$

using G realizations from the posterior distribution $p(\Theta|D)$. Similarly, prior to observing any data, reliability predictions can be made by replacing $(N_0 - M)$ with N_0 and the index (X_K, j) with (i, j) in (13.28) and using (13.29) with realizations from the prior distribution $p(\Theta)$.

4. Discrete Time Models

We now consider discrete-time models for hardware systems where a device is observed periodically at discrete time points. The device survives each period with a probability that depends on the state of the prevailing environment in that period. Since each period ends with a failure or survival, one can model this system as a Bernoulli process where the success probability is modulated by the environmental process. Using this setup with a Markovian environmental process, Özekici [28] focuses on probabilistic modeling and provides a complete transient and ergodic analysis. We suppose throughout the following discussion the sequence of environmental states $Y = \{Y_t; t = 1, 2, \cdots\}$ is a Markov chain with some transition matrix P on a discrete state space E.

4.1. Markov Modulated Bernoulli Process

Consider a system observed periodically at times $t = 1, 2, \cdots$ and the state of the system at time t is described by a Bernoulli random variable

$$X_t = \begin{cases} 1, & \text{if system is not functioning at time } t \\ 0, & \text{if system is functioning at time } t. \end{cases}$$

Given that the environment is in some state i at time t, the probability of failure in the period is

$$P[X_t = 1|Y_t = i] = \pi(i) \tag{13.30}$$

for some $0 \le \pi(i) \le 1$. The states of the system at different points in time constitute a Bernoulli process $X = \{X_t; t = 1, 2, \cdots\}$ where the success probability is a function of the environmental process Y.

Given the environmental process Y, the random quantities $X_1, X_2, \cdots$ represent a conditionally independent sequence, that is,

$$P[X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n|Y] = \prod_{k=1}^{n} P[X_k = x_k|Y]. \tag{13.31}$$

In the above setup, the reliability of the system is modulated by the environmental process Y which is assumed to be a Markov process and thus the model is referred to as the Markov Modulated Bernoulli Process (MMBP). If the system fails in a period, then it is replaced immediately by an identical one at the beginning of the next period. It may be possible to think of the environmental process Y as a random mission process such that Y_t is the tth mission to be performed. The success and failure probabilities depend on the mission itself. If the device fails during a mission, then the next mission will be performed by a new and identical device.

If we denote the lifetime of the system by L, then the conditional life distribution is

$$P[L = m|Y] = \begin{cases} \pi(Y_1), & \text{if } m = 1 \\ \pi(Y_m) \prod_{j=1}^{m-1} (1 - \pi(Y_j)) & \text{if } m \ge 2. \end{cases} \tag{13.32}$$

Note that if $\pi(i) = \pi$ for all $i \in E$, that is, the system reliability is independent of the environment, then (13.32) is simply the geometric distribution $P[L = m|Y] = \pi(1 - \pi)^{m-1}$. We can also write

$$P[L > m|Y] = (1 - \pi(Y_1))(1 - \pi(Y_2)) \cdots (1 - \pi(Y_m)) \tag{13.33}$$

for $m \ge 1$.

We represent the initial state of the Markov chain by Y_1, rather than Y_0, as it is customarily done in the literature, so that it represents the first environment that the system operates in. Thus, most of our analysis and results will be conditional on the initial state Y_1 of the Markov chain. Therefore, for any event A and random variable Z we set $P_i[A] = P[A|Y_1 = i]$ and $E_i[Z] = P[Z|Y_1 = i]$ to express the conditioning on the initial state.

The life distribution satisfies the recursive expression

$$P_i[L > m+1] = (1 - \pi(i)) \sum_{j \in E} P(i,j) P_j[L > m] \tag{13.34}$$

with the obvious boundary condition $P_i[L > 0] = 1$. The survival probabilities can be explicitly computed via

$$P_i[L > m] = \sum_{j \in E} Q_0^m(i,j) \tag{13.35}$$

where $Q_0(i,j) = (1 - \pi(i))P(i,j)$. Using (13.35), the conditional expected lifetime can be obtained as

$$E_i[L] = \sum_{m=0}^{+\infty} \sum_{j \in E} Q_0^m(i,j) = \sum_{j \in E} R_0(i,j) \tag{13.36}$$

where $R_0(i,j) = \sum_{m=0}^{+\infty} Q_0^m(i,j) = (I - Q_0)^{-1}(i,j)$ is the potential matrix corresponding to Q_0.

4.2. Network Reliability Assessment

Özekici and Soyer [34] consider networks that consist of components operating under a randomly changing common environment in discrete time. Their work is motivated by power system networks that are subject to fluctuating weather conditions over time that effect the performance of the network. The effect of environmental conditions on reliability of power networks have been recognized in earlier papers by Gaver, Montmeat and Patton [11] and Billinton and Bollinger [2] where the authors pointed out that power systems networks are exposed to fluctuating weather conditions and that the failure rates of equipment and lines increase during severe environmental conditions.

Consider a network with K components with an arbitrary structure function ϕ and reliability function h. The components of the network are observed periodically at times $t = 1, 2, \cdots$ and the probability that the kth component survives the period in environment i with probability $\pi_k(i)$.

It follows that the life distribution of component k is characterized by

$$P[L_k > n | Y] = \prod_{t=1}^{n} \pi_k(Y_t) \tag{13.37}$$

since the component must survive the first n time periods. Moreover, we assume that, given the environment, the component lifetimes are conditionally independent so that

$$P[L_1 > n, L_2 > n, \cdots L_K > n | Y] = \prod_{k=1}^{K} \prod_{t=1}^{n} \pi_k(Y_t). \tag{13.38}$$

We will denote the set of all components that are functioning prior to period t by Z_t such that Z_1 is the set of all functioning components at the outset and

$$Z_{t+1} = \{k = 1, 2, \cdots, K; X_t(k) = 1\} \tag{13.39}$$

is the set of components that survive period t for all $t \geq 1$. The state space of the stochastic process $Z = \{Z_t; t = 1, 2, \cdots\}$ is the set of all subsets of the component set $\mathcal{K} = \{1, 2, \cdots, K\}$. Although it is not required in the following analysis, it is reasonable to assume that $Z_1 = \mathcal{K}$. Moreover, it follows from the stochastic structure explained above that

$$P[Z_{t+1} = M | Z_t = S, Y_t = i] \equiv Q_i(S, M) = \prod_{k \in M} \pi_k(i) \prod_{k \in (S \cap M^c)} (1 - \pi_k(i)) \tag{13.40}$$

for any subsets M, S of $\mathcal{K}$ with $M \subseteq S$. In words, $Q_i(S, M)$ is the probability that the set of functioning components after one period will be M given that the environment is i and the set of functioning components is S. This function will play a crucial role in our analysis of the network. The stochastic structure of our network reliability model is made more precise by noting that, in fact, the bivariate process (Y, Z) is a Markov chain with transition matrix

$$P[Z_{t+1} = M, Y_{t+1} = j | Z_t = S, Y_t = i] \equiv P(i, j) Q_i(S, M) \tag{13.41}$$

for any $i, j \in E$ and subsets M, S of $\mathcal{K}$ with $M \subseteq S$. In many cases, it is best to analyze network reliability and other related issues using the Markov property of the chain (Y, Z).

Denote the set structure function Ψ by

$$\Psi(M) = \phi(m) \tag{13.42}$$

where $m = (m_1, m_2, \cdots, m_K)$ is the binary vector with $m_k = 1$ if and only if $k \in M$. Then,

$$p_i(S) = \sum_{M \subseteq S,\ \Psi(M) = 1} Q_i(S, M) \tag{13.43}$$

is the conditional probability that the network will survive one period in environment i given that the set of functioning components is S. The characterization in (13.43) can also be written in terms of the path-sets of the network. Let $\mathcal{P}$ denote the set of all combinations of components that makes the network functional. In other words,

$$\mathcal{P} = \{M \subseteq \mathcal{K}; \Psi(M) = 1\} \tag{13.44}$$

then (13.43) becomes

$$q_i(S) = \sum_{M \subseteq S,\ M \in \mathcal{P}} Q_i(S, M) \tag{13.45}$$

and

$$q(i) = q_i(\mathcal{K}) = \sum_{M \in \mathcal{P}} Q_i(\mathcal{K}, M) = \sum_{M \in \mathcal{P}} \prod_{k \in M} \pi_k(i) \tag{13.46}$$

is the probability that the network, with all components functioning, will survive one period in environment i.

In assessment of network reliability, we are interested in failure free operation of the network for n time periods. More specifically, we want to evaluate $P[L > n]$ for any time $n \geq 0$. Note that we can trivially write

$$P[L > n] = \sum_{i \in E} P[L > n | Y_1 = i, Z_1 = S] P[Y_1 = i, Z_1 = S] \tag{13.47}$$

that requires computation of the conditional probability $P[L > n | Y_1 = i, Z_1 = S]$ given any initial state i and S. We will denote the conditional network survival probability by

$$f(i, S, n) = P[L > n | Y_1 = i, Z_1 = S] \tag{13.48}$$

which is simply the probability that the network will survive n time periods given the set S of initially functioning components and the initial state i of the environment. Similarly, we define the conditional mean time to failure (MTTF) as

$$g(i, S) = E[L | Y_1 = i, Z_1 = S] = \sum_{n=0}^{+\infty} P[L > n | Y_1 = i, Z_1 = S]. \tag{13.49}$$

We will now exploit the Markov property of the process (Z, Y) to obtain computational results for f and g. Once they are computed, it is clear that we obtain the desired results as $f(i, \mathcal{K}, n)$ and $g(i, \mathcal{K})$ since it is reasonable to assume that $Z_1 = \mathcal{K}$ initially.

Minimal and Maximal Repair Models If we assume that there is minimal repair and all failed components are replaced only if the whole system fails, then the Markov property of (Z, Y) at the first transition yields the recursive formula

$$f(i, S, n+1) = \sum_{j \in E} \sum_{M \subseteq S,\ \Psi(M) = 1} P(i, j) Q_i(S, M)\ f(j, M, n). \tag{13.50}$$

The recursive system (13.50) can be solved for any (i, S) starting with $n = 1$ and the boundary condition $f(i, S, 0) = 1$. A further simplification of (13.50) is obtained by noting that we only need to compute $f(i, S, n)$ for $S \in \mathcal{P}$. The definition of $\mathcal{P}$ in (13.44) implies that we can rewrite (13.50) as

$$f(i,S,n+1) = \sum_{j\in E} \sum_{M\subseteq S,\, M\in\mathcal{P}} P(i,j)Q_i(S,M)\ f(j,M,n) \qquad (13.51)$$

for $S \in \mathcal{P}$ since $f(j,M,n) = 0$ whenever $M \notin \mathcal{P}$.

Similarly, using (13.49) or the Markov property directly we obtain the system of linear equations

$$g(i,S) = \ q_i(S)\ + \sum_{j\in E} \sum_{M\subseteq S,\, \Psi(M)=1} P(i,j)Q_i(S,M)\ g(j,M) \qquad (13.52)$$

which can be solved easily since both E and $\mathcal{K}$ are finite. Once again, the dimension of the system of linear equations in (13.52) can be reduced by noting that $g(j,M) = 0$ whenever $M \notin \mathcal{P}$ and we only need to compute $g(i,S)$ for $S \in \mathcal{P}$. The reader should bear in mind that this computational simplification applies in all expressions with $\Psi(M) = 1$ since this is true if and only if $M \in \mathcal{P}$.

If we assume that there is maximal repair and all failed components are replaced at the beginning of each period, then this implies that all components are functioning at the beginning of a period and we can take $Z_1 = S = \mathcal{K}$. Now (13.50) can be written as

$$f(i,\mathcal{K},n+1) = \sum_{j\in E} \sum_{M\subseteq\mathcal{K},\, \Psi(M)=1} P(i,j)Q_i(\mathcal{K},M)\, f(j,\mathcal{K},n) \qquad (13.53)$$

with the same boundary condition $f(i,\mathcal{K},0) = 1$. Note that (13.53) is dimensionally simpler than (13.50) since it can be rewritten as

$$\begin{aligned} f(i,n+1) &= \left[\sum_{\Psi(M)=1} Q_i(\mathcal{K},M)\right] \sum_{j\in E} P(i,j)\, f(j,n) && (13.54)\\ &= q(i) \sum_{j\in E} P(i,j)\, f(j,n) && (13.55)\end{aligned}$$

after suppressing $\mathcal{K}$ in f.

A similar analysis on the MTTF yields the system of linear equations

$$g(i) = q(i) + q(i) \sum_{j\in E} P(i,j)\, g(j). \qquad (13.56)$$

Defining the matrix $R(i,j) = q(i)P(i,j)$, (13.56) can be written in compact form as $g = q + Rg$ with the explicit solution

$$g = (I - R)^{-1} q. \qquad (13.57)$$

4.3. Bayesian Analysis of Discrete Time Models

The results presented for the MMBP and the network reliability assessment are all conditional on the specified parameters. In what follows we will consider the case where the parameters are treated unknown and present a Bayesian analysis. In so doing, we will present the Bayesian inference for the network reliability model and show that results for the MMBP can be obtained as a special case.

Under the network reliability setup of Section 4.2, we describe our uncertainty about the elements of the transition matrix P and the elements of the vector $\pi(i) = (\pi_1(i), \ldots, \pi_K(i))$. Thus, in terms of our previous notation we have $\Theta = (P, \pi(i), i \in E)$. As in Section 3.3, for the ith row of P we assume the Dirichlet prior given by (13.21) with P_i's are independent for $i \in E$. For a given environment, we assume that $\pi(i)$ has independent components with beta densities denoted as $\pi_k(i) \sim$ Beta $(a_k(i), b_k(i))$. Also, we assume that $\pi(i)$'s are independent of each other for all $i \in E$ and they are independent of the components of P.

If the network is observed for n time periods, then the observed data consists of $D = \{\mathbf{X}_t;\, t = 1, \ldots, n\}$ where $\mathbf{X}_t = (X_t(1), X_t(2), \ldots, X_t(K))$. The failure data also provides the values $Z^n = \{Z_t;\, t = 1, \ldots, n+1\}$ since $Z_{t+1} = \{k = 1, 2, \cdots, K; X_t(k) = 1\}$. It is assumed that the environmental process is unobservable. In this case the Bayesian analysis of the network reliability presents a structure similar to the *hidden Markov models* which were considered by Robert, Celeux and Diebolt [37].

In the minimal repair model, we can write the likelihood function as

$$\mathcal{L}(\Theta, Y^n; D) \propto \prod_{t=1}^{n} P(Y_{t-1}, Y_t) \left\{ \prod_{k \in Z_{t+1}} \pi_k(Y_t) \prod_{k \in (Z_t \cap Z_{t+1}^c)} [1 - \pi_k(Y_t)] \right\}, \tag{13.58}$$

where $Z_1 \supseteq \cdots \supseteq Z_{n+1}$with $Z_1 = \mathcal{K}$ and $Y^n = (Y_1, ..., Y_n)$. In the maximal repair model, the likelihood function is given by

$$\mathcal{L}(\Theta, Y^n; D) \propto \prod_{t=1}^{n} P(Y_{t-1}, Y_t) \left\{ \prod_{k \in \mathcal{K}} [\pi_k(Y_t)]^{X_t(k)} \, [1 - \pi_k(Y_t)]^{1 - X_t(k)} \right\}. \tag{13.59}$$

Note that in (13.58) and (13.59), we set $P(Y_0, Y_1) = 1$ when $t = 1$ and we observe only $n - 1$ transitions of Y.

As pointed out in Özekici and Soyer [34], when the history of Y process is not observable, there is no analytically tractable posterior analysis. Thus, as in Section 3.3 the posterior analysis can be developed using the Gibbs sampler.

The full conditional distributions of P_i's are obtained as independent Dirichlet densities

$$(P_i|D, Y^n) \sim \text{Dirichlet } \{\alpha_j^i + \sum_{t=1}^{n} 1(Y_t = i, Y_{t+1} = j); j \in E\}. \quad (13.60)$$

The full conditionals of $\pi_k(i); i \in E$, $k = 1, ..., K$ are independent beta densities given by $(\pi_k(i)|\, D, Y^n) \sim$ Beta $(a_k^*(i), b_k^*(i))$ with

$$a_k^*(i) = a_k(i) + \sum_{t=1}^{n} 1(Y_t = i)1(k \in Z_{t+1}), \quad (13.61)$$

$$b_k^*(i) = b_k(i) + \sum_{t=1}^{n} 1(Y_t = i)1(k \in (Z_t \cap Z_{t+1}^c)) \quad (13.62)$$

for the minimal repair model and with

$$a_k^*(i) = a_k(i) + \sum_{t=1}^{n} 1(Y_t = i)\ X_t(k), \quad (13.63)$$

$$b_k^*(i) = b_k(i) + \sum_{t=1}^{n} 1(Y_t = i)\ (1 - X_t(k)) \quad (13.64)$$

for the maximal repair model. We note that posteriori elements of $\pi(i)$'s and P_i's are independent of each other for all $i \in E$. The full conditional distributions of the environmental process, $p(Y_t|D, Y^{(-t)}, \pi(Y_t), P)$ where $Y^{(-t)} = \{Y_\tau; \tau \neq t\}$ is obtained for the minimal repair model as

$$p(Y_t|D, Y^{(-t)}, \pi(Y_t), P) \propto \quad (13.65)$$
$$P(Y_{t-1}, Y_t) \left\{ \prod_{k \in Z_{t+1}} \pi_k(Y_t) \prod_{k \in Z_t \cap Z_{t+1}^c} [1 - \pi_k(Y_t)] \right\} P(Y_t, Y_{t+1})$$

and for the maximal repair case as

$$p(Y_t|D, Y^{(-t)}, \pi(Y_t), P) \propto \quad (13.66)$$
$$P(Y_{t-1}, Y_t) \left\{ \prod_{k \in \mathcal{K}} [\pi_k(Y_t)]^{X_t(k)}\ [1 - \pi_k(Y_t)]^{1 - X_t(k)} \right\} P(Y_t, Y_{t+1}).$$

Thus, for both repair scenarios, a posterior sample from $p(\Theta, Y^n|D)$ can be easily obtained by iteratively drawing from the given full posterior conditionals.

Once the posterior distribution is obtained, posterior reliability predictions can be made by evaluating $P[\hat{L} > m|D]$, where $\hat{L} = L - n$ is the time remaining to network failure. For the minimal repair case, using (13.51) and the Markov property of the chain (Y, Z), by generating G realizations from the posterior distribution $p(\Theta, Y_n|D)$ we can approximate the posterior network reliability as a Monte Carlo integral

$$P[\hat{L} > m|D] \approx \frac{1}{G} \sum_{g} \sum_{j \in E} P(Y_n^{(g)}, j) f(j, Z_{n+1}, m-1|\Theta^{(g)}), \qquad (13.67)$$

where $f(j, Z_{n+1}, m|\Theta)$ is obtained as the solution of (13.51). In the maximal repair model, similar results can be obtained by using (13.55) to compute f for each realization g.

For the MMBP we can obtain the Bayesian inference by considering the special case $K = 1$ in the maximal repair model. By setting $a_k^*(i) = a^*(i)$, $b_k^*(i) = b^*(i)$, $\pi_k(i) = \pi(i)$, and $X_t(k) = X_t$ in (13.63), (13.64) and (13.66) we can obtain posterior analysis for the MMBP.

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