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Bayesian analysis of Markov Modulated Bernoulli Processes

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Abstract. We consider Markov Modulated Bernoulli Processes (MMBP) where the success probability of a Bernoulli process evolves over time according to a Markov chain. The MMBP is applied in reliability modeling where systems and components function in a randomly changing environment. Some of these applications include, but are not limited to, reliability assessment in power systems that are subject to fluctuating weather conditions over time and reliability growth processes that are subject to design changes over time. We develop a general setup for analysis of MMBPs with a focus on reliability modeling and present Bayesian analysis of failure data and illustrate how reliability predictions can be obtained.

Key words: Markov modulation, Bernoulli process, System reliability, Bayesian analysis

1 Introduction and overview

In this paper, we consider Markov Modulated Bernoulli Processes (MMBP) which were introduced in Özekici [10]. The MMBP is obtained by assuming that the success probability of a Bernoulli process evolves over time according to a Markov chain. It has applications in reliability modeling where systems and components function in a randomly changing environment. Some of these applications include, but are not limited to, reliability assessment in power systems that are subject to fluctuating weather conditions over time (see, for example, Billinton and Allan [1]), and reliability monitoring of a system that is subject to design changes over time (see, for example, Erkanlı, Mazzuchi and Soyer [5]). The effect of environmental conditions on reliability of power systems have been recognized in earlier papers by Gaver, Montmeat and Patton [6] and Billinton and Bollinger [2]. As these authors have pointed out power systems are exposed to fluctuating weather conditions and the failure rates of equipment and lines increase during severe environmental conditions. More recently, network reliability assessment under a random environment is discussed by Özekici and Soyer [11] where the authors' work was motivated by power system networks. In all of these applications the failure characteristics of the systems or components in question change over time as related to the changes in environments that they operate in. A natural strategy for modeling the failure behavior of such systems is to assume that the environment evolves over time according to a stochastic process. Some of the related work on this area include Çınlar and Özekici [4], Lefèvre and Milhaud [8], Singpurwalla and Youngren [14], Singpurwalla [13] and Özekici [9].

We consider a periodic-review reliability model where a system or a component either fails or survives in a given period. The probability of survival depends on the state of the environment in which the system operates. Using this setup with a Markovian environmental process, Özekici [10] focuses on probabilistic modeling and provides a complete transient and ergodic analysis of MMBPs. In the present setting, we focus on inferential issues in the analysis of MMBPs and present their Bayesian analysis. In so doing, we will develop results relevant to reliability assessment and illustrate how reliability predictions can be obtained from MMBPs in the Bayesian framework. Our approach is based on Markov Chain Monte Carlo methods.

The MMBPs and their analyses are discussed in Section 2. In so doing, results relevant to reliability analysis are presented. Section 3 illustrates how these results can be used if uncertainty about model parameters are described by a prior distribution in a Bayesian setup. We present posterior analysis of failure data under a good-as-new replacement scenario and illustrate how reliability predictions can be obtained in Section 4. This analysis is presented both for the case when the environment is observable and unobservable. We illustrate the implementation of the Bayesian approach via an example in Section 5 using simulated data.

2 Markov Modulated Bernoulli Process

Consider a system observed periodically at times $t = 1, 2, \ldots$ and the state of the system at time t is described by a Bernoulli random variable

$$X_t = \begin{cases} 1, & \text{if system is not functioning at time } t \\ 0, & \text{if system is functioning at time } t. \end{cases}$$

We assume that the reliability of the system depends on a randomly changing environment and let Y_t denote the state of the environment at time t. Given that the environment is in some state i at time t, the probability of failure in the period is

$$P[X_t = 1 \mid Y_t = i] = \pi(i) \tag{1}$$

for some $0 \leq \pi(i) \leq 1$. We assume that the environmental process $Y = \{Y_t; t \geq 1\}$ is a Markov chain with some transition matrix P on a discrete state space E. The states of the system at different points in time constitute a Bernoulli process $X = \{X_t; t = 1, 2, \ldots\}$ where the success probability is a function of the environmental process Y.

Given the environmental process Y, the random quantities $X_1, X_2, \dots$ represent a conditionally independent sequence, that is,

$$P[X_1 = x_1, X_2 = x_2, \dots, X_n = x_n \mid Y] = \prod_{k=1}^{n} P[X_k = x_k \mid Y]. \tag{2}$$

In the above setup, the reliability of the system is modulated by the environmental process Y which is assumed to be a Markov process and thus the model is referred to as the MMBP. If the system fails in a period, then it is replaced immediately by an identical one at the beginning of the next period.

It may be possible to think of the environmental process Y as a random mission process such that Y_n is the nth mission to be performed. The success and failure probabilities depend on the mission itself. If the device fails during a mission, then the next mission will be performed by a new and identical device. We now review in this section some of the results presented in Özekici [10] that have direct relevance and importance in reliability theory. The main contribution of this article will be on Bayesian analysis aimed at these results when the model parameters are unknown.

If we denote the lifetime of the system by L, then we can set $L = T_1$ so that the conditional life distribution is

$$P[L = m \mid Y] = P[T_1 = m \mid Y] = \begin{cases} \pi(Y_1), & \text{if } m = 1 \\ \pi(Y_m) \prod_{j=1}^{m-1} (1 - \pi(Y_j)) & \text{if } m \geq 2. \end{cases} \tag{3}$$

Note that if $\pi(i) = \pi$ for all $i \in E$, that is, the system reliability is independent of the environment, then (3) is simply the geometric distribution $P[L = m \mid Y] = \pi(1 - \pi)^{m-1}$. We can also write

$$P[L > m \mid Y] = P[T_1 > m \mid Y] = (1 - \pi(Y_1))(1 - \pi(Y_2)) \dots (1 - \pi(Y_m)) \tag{4}$$

for $m \geq 1$.

We represent the initial state of the Markov chain by Y_1, rather than Y_0, as it is customarily done in the literature, so that it represents the first environment that the system operates in. Thus, most of our analysis and results will be conditional on the initial state Y_1 of the Markov chain. Therefore, for any event A and random variable Z we set $P_i[A] = P[A \mid Y_1 = i]$ and $E_i[Z] = E[Z \mid Y_1 = i]$ to express the conditioning on the initial state.

The life distribution satisfies the recursive expression

$$P_i[L > m + 1] = (1 - \pi(i)) \sum_{j \in E} P(i, j) P_j[L > m] \tag{5}$$

with the obvious boundary condition $P_i[L > 0] = 1$ and this directly leads to the explicit solution

$$P_i[L > m] = P_i[T_1 > m] = \sum_{j \in E} Q_0^m(i, j) \tag{6}$$

where $Q_0(i,j) = (1 - \pi(i))P(i,j)$. Using (6), the conditional expected lifetime can be obtained as

$$E_i[L] = E_i[T_1] = \sum_{m=0}^{+\infty} \sum_{j \in E} Q_0^m(i,j) = \sum_{j \in E} R_0(i,j) \tag{7}$$

where $R_0(i,j) = \sum_{m=0}^{+\infty} Q_0^m(i,j) = (I - Q_0)^{-1}(i,j)$ is the potential matrix corresponding to Q_0.

The total number of failures of the system in m time periods is

$$N_m = X_1 + X_2 + \cdots + X_m \tag{8}$$

and the time of the kth failure is

$$T_k = \inf\{m \geq 1 : N_m \geq k\} \tag{9}$$

for $m, k \geq 1$. We also set $N_0 = T_0 = 0$ for completion.

In reliability modeling we are interested in assessing the state of the system at a given point in time. Given that the initial state is $Y_1 = i$, the probability that the system will fail in the mth period is

$$P_i[X_m = 1] = \sum_{j \in E} P^{m-1}(i,j)\pi(j) \tag{10}$$

where P^m is the m-step transition matrix of the Markov chain. The expected number of system failures over m time periods is

$$E_i[N_m] = \sum_{k=1}^{m} \sum_{j \in E} P^{k-1}(i,j)\pi(j) = \sum_{j \in E} R_m(i,j)\pi(j) \tag{11}$$

where $R_m(i,j) = \sum_{k=1}^{m} P^{k-1}(i,j)$ for $m \geq 1$. Here, (11) simply states the fact that the expected number of failures in the first m time periods is simply the sum of the expected number of visits to environmental states multiplied by the probability of failure of the system in that state.

By conditioning on the initial state $Y_1 = i$, one can obtain the following recursive expression to determine the distribution of N_m

$$P_i[N_{n+1} = k] = (1 - \pi(i)) \sum_{j \in E} P(i,j) P_j[N_n = k] + \pi(i) \sum_{j \in E} P(i,j) P_j[N_n = k-1] \tag{12}$$

for $n, k \geq 0$ and all $i, j \in E$. The boundary conditions are $P_j[N_0 = 0] = 1$, $P_j[N_0 = k] = 0$ for all $k \geq 1$ and $j \in E$, and $P_j[N_n = -1] = 0$ for all $n \geq 0$ and $j \in E$.

The distribution function of N_m can also be characterized by the generating function $E_i[\alpha^{N_m}]$ which is given by

$$E_i[\alpha^{N_m}] = \sum_{j \in E} Q_\alpha^m(i,j) \tag{13}$$

for any $i \in E$, $m \geq 1$ and $0 \leq \alpha \leq 1$ where Q_α is the matrix

$$Q_\alpha(i, j) = (1 + (\alpha - 1)\pi(i))P(i, j). \tag{14}$$

For $k > 1$, the distribution of T_k can be determined using the distribution of N_m by the basic relationship $\{T_k \leq m\} = \{N_m \geq k\}$ so that $P_i[T_k \leq m] = P_i[N_m \geq k]$. Moreover, conditional on $Y_1 = i$, one can obtain the following recursive expression

$$P_i[T_{k+1} > n + 1] = (1 - \pi(i)) \sum_{j \in E} P(i, j)P_j[T_{k+1} > n] + \pi(i) \sum_{j \in E} P(i, j)P_j[T_k > n] \tag{15}$$

for $n, k \geq 0$, and $i, j \in E$ with the boundary conditions $P_j[T_k > 0] = 1$ for all $k \geq 0$ and $P_j[T_0 > n] = 0$ for all $n \geq 1$. Once the conditional distribution of T_k is found, one can determine the conditional distribution of T_{k+1} by using (15) and iterating on $n = 0, 1, 2, \ldots$ in increasing order. Similarly, conditional on $Y_1 = i$, recursive equations for the mean can be obtained as

$$E_i[T_{k+1}] = 1 + (1 - \pi(i)) \sum_{j \in E} P(i, j)E_j[T_{k+1}] + \pi(i) \sum_{j \in E} P(i, j)E_j[T_k] \tag{16}$$

for any $k \geq 0$, $i \in E$ and the boundary condition is $E_i[T_0] = 0$. Using these recursions, once $E_i[T_k]$ is obtained, $E_i[T_{k+1}]$ can be computed by solving the resulting system of linear equations.

The processes N and T no longer have independent and stationary increments due to their dependence on the modulating process Y. However, these properties hold conditional on the Y process, that is,

$$P[N_{n+m} - N_n = k \mid N_l, Y_l; l \leq n] = P_{Y_n}[N_m = k] \tag{17}$$

and

$$P[T_{l+k} - T_l = n \mid T_m, Y_m; m \leq T_l] = P_{Y_{T_l}}[T_k = n]. \tag{18}$$

The properties stated in (17) and (18) are extremely useful from the standpoint of Bayesian analysis. Once we have obtained data and determined the posterior distribution of the uncertain model parameters, the future increments of the processes X, N and T depend only on the present state of the environment.

3 Prior reliability analysis

The results presented for MMBPs in the previous section can be used in reliability analysis if the model parameters are known. In what follows, we consider the case where these parameters are unknown and present a Bayesian analysis by describing our uncertainty about the unknown parameters probabilistically. This requires description of our uncertainty about the elements of the parameter vector $\Theta = (P, \pi)$. For the components of the transition matrix P of the environmental process, we assume that the ith row $P(i) = \{P(i,j); j \in E\}$ has a Dirichlet prior

$$p(P(i)) \propto \prod_{j \in E} P(i,j)^{\alpha_j^i - 1} \tag{19}$$

and that the $P(i)$'s are independent of each other. For a given environment $i \in E$, we assume that $\pi(i)$ has a beta density

$$p(\pi(i)) \propto \pi(i)^{a(i)-1}(1 - \pi(i))^{b(i)-1} \tag{20}$$

denoted as $\pi(i) \sim \text{Beta}(a(i), b(i))$. Also, we assume that $\pi(i)$'s are independent of each other for all $i \in E$ and they are independent of the components of P. We will denote the prior distribution of Θ as $p(\Theta)$.

We note that the reliability function $P_i[L > m]$ given in (6) is dependent on the matrix Q_0 which is a function of $\Theta = (\pi, P)$. To express this dependence clearly, we will denote its entries by $Q_0(i, j \mid \Theta)$. Thus, the probability of failure free performance for m periods is written conditional on the parameter vector Θ. To make this explicit, we will write this probability as

$$f(i, m \mid \Theta) = P_i[L > m \mid \Theta] = \sum_{j \in E} Q_0^m(i, j \mid \Theta). \tag{21}$$

Prior to observing any data, we can make reliability predictions as

$$f(i, m) = P_i[L > m] = \int f(i, m \mid \Theta) p(\Theta)\, d\Theta. \tag{22}$$

This integral can not be evaluated analytically, but it can be approximated via simulation as a Monte Carlo integral

$$f(i, m) \approx \frac{1}{G} \sum_g f(i, n \mid \Theta^{(g)}) \tag{23}$$

by generating G realizations from the prior distribution $p(\Theta)$.

It is clear that similar analysis can be developed for the other quantities of interest involving the distributions and means of L, N_n and T_k given in the previous section. We will not discuss all of them to avoid repetition. Obviously, the formulas presented in Section 2 are needed in this analysis. As another illustration, suppose we want to analyze the mean time to failure $E_i[L]$. Now, we only need to take

$$f(i|\Theta) = E_i[L|\Theta] = \sum_{j \in E} R_0(i, j \mid \Theta) \tag{24}$$

in (7) where the matrix $R_0(i, j \mid \Theta)$ is obtained from $Q_0(i, j \mid \Theta)$ as a function of the parameters Θ. The prior estimate of the mean time to failure is

$$f(i) = E_i[L] = \int f(i|\Theta) p(\Theta)\, d\Theta \tag{25}$$

and this can be approximated via simulation as a Monte Carlo integral

$$f(i) \approx \frac{1}{G}\sum_{g} f(i|\Theta^{(g)}) \tag{26}$$

by generating G realizations from the prior distribution $p(\Theta)$.

4 Posterior reliability analysis

If the system is observed for n time periods, then we can write the likelihood function of the unknown parameters $\Theta = (P, \pi)$. The observed data consists of $\mathbf{X}^n = (X_1, X_2, \ldots, X_n)$ and in some applications it may also include $\mathbf{Y}^n = (Y_1, Y_2, \ldots, Y_n)$ if the environmental process is observable. We recall whenever the system fails it is replaced by a new and identical one.

Once the system is observed for n periods, our uncertainty about Θ is revised as described by the posterior distribution $p(\Theta|D)$ where D denotes the observed data. In developing the posterior analysis under the good-as-new replacement scenario, we will consider two cases based on what the observed data D consists of. Specifically, in the first case we will assume that the environmental process Y will also be observed for n periods. This may be appropriate for situations where the environment such as weather conditions will be observable and components such as temperature, wind velocity, etc. can be measured during each period. In this case, the observed data will consist of both the component failure history $\mathbf{X}^n$ and the history $\mathbf{Y}^n$ of the Y process over n periods. In the second case, we consider the situation where the environment may not be observable due to its complexity and thus the history of the Y process will not be available for the posterior analysis. In this case the Bayesian analysis of the MMBP presents a structure similar to the *hidden Markov models* which were considered by a host of authors such as Robert, Celeux and Diebolt [12] for analyzing mixture models.

We can write the likelihood function under the good-as-new replacement scenario as

$$\mathscr{L}(\Theta; \mathbf{X}^n, \mathbf{Y}^n) \propto \prod_{t=1}^{n} P(Y_{t-1}, Y_t)\pi(Y_t)^{X_t}(1 - \pi(Y_t))^{1-X_t} \tag{27}$$

where we set $P(Y_0, Y_1) = 1$ when $t = 1$ and we observe only $n - 1$ transitions of Y. For the case where Y is not observable, the joint likelihood function of Θ and Y^n is obtained from (27) by excluding the term $P(Y_{t-1}, Y_t)$.

4.1 *Environmental process is observable*

In this case we assume that both failure data $\mathbf{X}^n$ and the history of the environmental process Y are observed, that is, $D = \{\mathbf{X}^n, \mathbf{Y}^n\}$. Thus, the likelihood function of Θ is given conditional on both $\mathbf{X}^n$ and $\mathbf{Y}^n$.

We can show that given D, the posterior distribution of $P(i)$'s can be obtained as independent Dirichlets given by

$$(P(i) \mid D) \sim \text{Dirichlet}\left\{ \alpha_j^i + \sum_{t=1}^{n-1} 1(Y_t = i,\, Y_{t+1} = j);\, j \in E \right\} \tag{28}$$

where 1(.) is the indicator function. Similarly, the posterior distributions of $\pi(i)$'s are obtained as independent beta densities given by

$$(\pi(i) \mid D) \sim \text{Beta}[a^*(i), b^*(i)] \tag{29}$$

with

$$a^*(i) = a(i) + \sum_{t=1}^{n} 1(Y_t = i)X_t \tag{30}$$

and

$$b^*(i) = b(i) + \sum_{t=1}^{n} 1(Y_t = i)(1 - X_t). \tag{31}$$

We note that posteriori elements of $\pi(i)$'s and P are independent of each other for all $i \in E$. We also note that the conditional independence of the elements of $\pi(i)$ is preserved posteriori when the history of the Y process is known. Thus, the posterior distribution $p(\Theta|D)$ can be easily evaluated using this independence structure.

Once the posterior analysis is completed we can make reliability predictions conditional on the observed data $D = \{\mathbf{Y}^n, \mathbf{X}^n\}$. In this case, the data consists of both the failure data and the history of the environmental process. Conditional on Θ, we know that the reliability is given by $f(i, m \mid \Theta) = P_i[L > m \mid \Theta]$ in (21). Using this solution, we can obtain the posterior reliability prediction as

$$P[\hat{L} > m \mid \mathbf{Y}^n, \mathbf{X}^n] = \int P[\hat{L} > m \mid \Theta, \mathbf{Y}^n, \mathbf{X}^n] p(\Theta \mid \mathbf{Y}^n, \mathbf{X}^n)\, d\Theta \tag{32}$$

$$= \int f(Y_n, m \mid \Theta) p(\Theta \mid \mathbf{Y}^n, \mathbf{X}^n)\, d\Theta \tag{33}$$

where $\hat{L}$ is the remaining lifetime of the system in use at time n. Note that (33) follows from the conditional independence of the increments of N as expressed by (17) since this implies

$$P[\hat{L} > m \mid \Theta, \mathbf{Y}^n, \mathbf{X}^n] = P_{Y_n}[\hat{L} > m \mid \Theta] = f(Y_n, m \mid \Theta). \tag{34}$$

The integral in (33) can not be evaluated analytically, but it can be approximated via simulation as a Monte Carlo integral

$$P[\hat{L} > m \mid \mathbf{Y}^n, \mathbf{X}^n] \approx \frac{1}{G} \sum_g f(Y_n, m \mid \Theta^{(g)}) \tag{35}$$

by generating G realizations from the posterior distribution $p(\Theta|D)$.

Once again, other relevant quantities can be approximated in a similar manner. For example, if we are interested in the number of failures $\hat{N}_m = N_{n+m} - N_n$ in the next m periods, we only need to take

$$f(i, m \mid \Theta) = \sum_{j \in E} R_m(i, j \mid \Theta)\pi(j) \tag{36}$$

in our analysis using the explicit result (11) where the matrix $R_m(i, j \mid \Theta) = \sum_{k=1}^{m} P^{k-1}(i, j)$ trivially depends on $\Theta = (P, \pi)$. The posterior estimate of the number of failures in the next m period is

$$E[\hat{N}_m \mid \mathbf{Y}^n, \mathbf{X}^n] = \int f(Y_n, m \mid \Theta) p(\Theta \mid \mathbf{Y}^n, \mathbf{X}^n)\, d\Theta \tag{37}$$

which can also be approximated using (35).

4.2 Environmental process is unobservable

When the history of Y is not observable, we can no longer obtain an analytically tractable posterior analysis as in the previous section. Thus, in what follows we will develop posterior analysis using Markov Chain Monte Carlo methods and, more specifically, using Gibbs sampling (see, for example, Gelfand and Smith [7]). Assuming the same priors for components of Θ as in the previous section and defining observed data $D = \mathbf{X}^n$, a Gibbs sampler can be developed to iteratively draw from the full posterior conditional distributions of all unknown quantities to obtain a sample from, $p(\Theta, \mathbf{Y}^n \mid D)$, the joint posterior distribution of $(\Theta, \mathbf{Y}^n)$. We note that when the environmental process is unobservable, as previously pointed out, the likelihood function (27) is adjusted by excluding the term $P(Y_{t-1}, Y_t)$ to obtain the joint likelihood function of Θ and $\mathbf{Y}^n$.

The implementation of the Gibbs sampler requires the full posterior conditional distributions $\{p(P(i) \mid D, \mathbf{Y}^n); i \in E\}$ that are independent Dirichlet densities as given by (28), $\{p(\pi(i) \mid D, \mathbf{Y}^n); i \in E\}$ that are independent beta densities with parameters given by (30), and $p(Y_t \mid D, \mathbf{Y}^{(-t)}, \pi(Y_t), P)$ where $\mathbf{Y}^{(-t)} = \{Y_\tau; \tau \neq t\}$.

For the given replacement scenario, it can be shown that the full conditional distributions of Y_t's are given by

$$p(Y_t \mid D, Y^{(-t)}, \pi(Y_t), P) \propto P(Y_{t-1}, Y_t)\pi(Y_t)^{X_t}(1 - \pi(Y_t))^{1-X_t} P(Y_t, Y_{t+1}) \tag{38}$$

with constant of proportionality

$$\left[\sum_{j \in E} P(Y_{t-1}, j)\pi(j)^{X_t}(1 - \pi(j))^{1-X_t} P(j, Y_{t+1})\right]^{-1}. \tag{39}$$

For the boundary cases of $t = 1$ and $t = n$, one should set $P(Y_0, Y_1) = P(Y_n, Y_{n+1}) = 1$ in the above terms. Given the above forms, a posterior sample from $p(\Theta, \mathbf{Y}^n \mid D)$ can be easily obtained by iteratively drawing from the given full posterior conditionals. As we have noted before, the Bayesian analysis we have presented in the above shares common features with the Bayesian analysis of hidden Markov models considered by Robert, Celeux and Diebolt [12].

The data now consists only of $\mathbf{X}^n$ in this analysis and the posterior distribution is given by $p(\Theta, \mathbf{Y}^n \mid \mathbf{X}^n)$. To obtain posterior reliability predictions, we want to evaluate

$$P[\hat{L} > m \mid \mathbf{X}^n] = \sum_{\mathbf{Y}^n} \int f(Y_n, m \mid \Theta) p(\Theta, \mathbf{Y}^n \mid \mathbf{X}^n)\, d\Theta \tag{40}$$

where $f(i, m \mid \Theta) = \sum_{j \in E} Q_0^m(i, j)$ as given in (6). The above integral can be approximated as

$$P[\hat{L} > m \mid \mathbf{X}^n] \approx \frac{1}{G} \sum_g f(Y_n^{(g)}, m \mid \Theta^{(g)}) \tag{41}$$

by generating G realizations from the posterior distribution $p(\Theta,\, Y_n \mid \mathbf{X}^n)$.

As another illustration of the application of the results in Section 2, consider the generating function of the number of failures $\hat{N}_m$ in the next m time periods. Using (13), we can now take

$$f(i|\Theta) = \sum_{j \in E} Q_\alpha^m(i, j \mid \Theta) \tag{42}$$

in the posterior analysis. Clearly, $Q_\alpha(i, j \mid \Theta) = (1 + (\alpha - 1)\pi(i))P(i, j)$ is a function of the parameters $\Theta = (P, \pi)$. The posterior generating function then is

$$E[\alpha^{\hat{N}_m} | \mathbf{X}^n] = \sum_{\mathbf{Y}^n} \int f(Y_n|\Theta) p(\Theta, \mathbf{Y}^n \mid \mathbf{X}^n)\, d\Theta \tag{43}$$

which can be approximated through Monte Carlo simulation. The approach in the Bayesian analysis is demonstrated only for a number of quantities of interest, but this demonstration illustrates how the analysis can be done for all of the results presented in Section 2.

5 Numerical illustration

To illustrate the use of Bayesian inference procedures for reliability assessment, we considered a scenario consisting of two environments, which can be regarded as normal and severe environments, and simulated the probabilistic structure and the system failure/survival process discussed in Section 2. The system was simulated for 360 time periods assuming the good-as-new replacement scenario. In so doing, the transition matrix of

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix} \tag{44}$$

was used in simulating the environmental process with the initial state given as $Y_1 = 1$. Actual failure probabilities $\pi(1) = 0.05$ and $\pi(2) = 0.8$ were used to simulate the failure data under environments 1 and 2. The simulated failure data is not given here due to space limitations.

In what follows, we will focus on posterior analysis of the simulated data for both the observable and unobservable environmental process cases. In so doing, we will discuss how the posterior distributions and posterior reliability predictions can be obtained and compare results in the two cases. In both cases we use diffused prior distributions for the elements of Θ. More specifically, for the Dirichlet priors in (19) we assume that $\alpha_j^i = 1$ for all (i, j) implying that $E[P_i] = 0.5$ for $i = 1, 2$. For the beta priors of $\pi(i)$'s in (20) we assume that $a(i) = b(i) = 1$ for $i = 1, 2$ implying that $E[\pi(i)] = 0.5$. The above choice of parameters implies uniform prior distributions over $(0, 1)$ for each of the parameters and represents a high degree of prior uncertainty about the components of Θ. In a situation where prior information exists, the prior parameters can be specified by eliciting best guess values for elements of Θ and uncertainties about these values from reliability analysts. Methods for prior elicitation in beta and Dirichlet distributions are given in Chaloner and Duncan [3].

In Figure 1 we present the posterior distributions of system failure probabilities for environments 1 and 2. We note that in this case the posterior distributions are given by beta densities and, compared to the uniform prior distributions, the posterior distributions are peaked and they are concentrated in different regions depending on the environment. In fact, the posterior means of the $\pi(i)$'s are very close to the actual values used in simulating the data. For environment 1, the posterior mean of the failure probability is 0.055 and the posterior standard deviation is 0.012. The posterior mean of the failure prob-

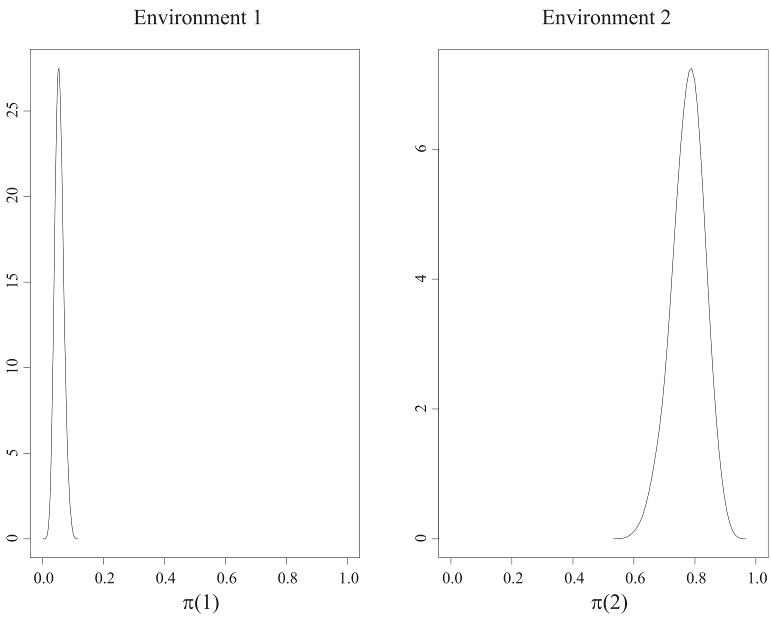

Fig. 1. Posterior Distributions of Failure Probabilities in Environments 1 and 2 when Environments are Observable.

ability is 0.778 with standard deviation 0.052 under environment 2. Note that the posterior distribution under environment 2 has a higher variance than the one under environment 1. This is expected since the environmental process spends more time in state 1 than in state 2.

The posterior distributions of $\pi(i)$'s for the case where the environment is not observable is given in Figure 2. In this case the posterior distributions are not beta densities. Thus, we present the density plots for $\pi(i)$'s in Figure 2. If we compare these densities to those presented in Figure 1, we see that the distributions of $\pi(i)$'s are still peaked in different regions but they exhibit more uncertainty in this case. This is expected since we do not observe the environmental state but only infer about it based on the failure/survival data. The posterior means in this case are 0.071 for environment 1 and 0.738 for environment 2 with standard deviations 0.032 and 0.141 respectively.

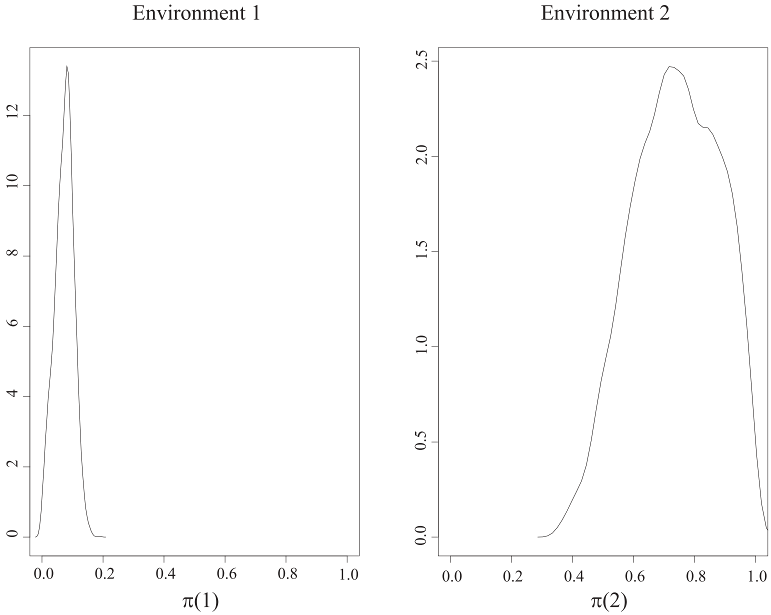

Fig. 2. Posterior Distributions of Failure Probabilities in Environments 1 and 2 when Environments are Unobservable.

We note that in both cases the posterior distribution of $\pi(2)$, the failure probability under environment 2, is more diffused than $\pi(1)$. In the first case where the environmental process is observable, this can be explained by looking at the state occupancy data over time and noting that the environmental process stayed in environment 1 most of the time, specifically 299 out of 360 periods. In the case where environmental process is not observable we can see this by looking at the posterior distribution of the transition probability matrix P given in Table 1. We note that the posterior means are very close to the actual values of the transition probabilities given in (44) implying that the environmental process will be in state 1 most of the time. The posterior means are very similar when the environmental process is observable, but with smaller standard deviations.

Table 1. Posterior means and standard deviations for transition probabilities when environmental process is unobservable

$P(i, j)$	$E[P(i, j) \mid D]$	$\sqrt{V[P(i, j) \mid D]}$
$P(1, 1)$	0.932	0.035
$P(1, 2)$	0.068	0.035
$P(2, 1)$	0.369	0.120
$P(2, 2)$	0.631	0.120

In the case where the environmental process is unobservable, the Gibbs sampler enables us to obtain the posterior distributions of the environmental states $Y_1, \ldots, Y_n$. In Table 2 we present these posterior distributions for selected states as well as their actual (but unobserved) values used in the simulation of the data. For example, at time period 50 the actual value of the unobserved environment is 1 and based on our approach posterior probability that the environmental process is in state 1 is obtained as 0.985. Similarly, at time period 180 the actual value of the environmental state is 2 and the posterior probability of this state is 0.939. Table 2 illustrates the posterior probabilities for 6 periods. The posterior distributions are available for all periods and in almost all cases the posterior probability of the correct state is very high. Thus, the approach is able to infer the unobserved environment correctly for the data.

Table 2. Posterior distributions of selected Y_t's and their actual values

t	$P[Y_t = 1 \mid D]$	$P[Y_t = 2 \mid D]$	Y_t
5	0.899	0.101	1
18	0.014	0.986	2
50	0.985	0.015	1
83	0.203	0.797	2
180	0.061	0.939	2
360	0.978	0.022	1

Once the posterior samples are available we can make posterior reliability predictions using the Monte Carlo approximations (35) for the observed environmental process and (41) for the case of unobserved environment. In the first case, the last observed environmental state is 1 and in the case of unobserved environment the posterior distribution of environmental state for time period 360 is given in Table 2. We note that in evaluating (41) the term $f(Y_n, m \mid \Theta)$ is evaluated for each realization of Y_n and Θ from the posterior distribution. In Figure 3 we show the posterior reliability functions for both

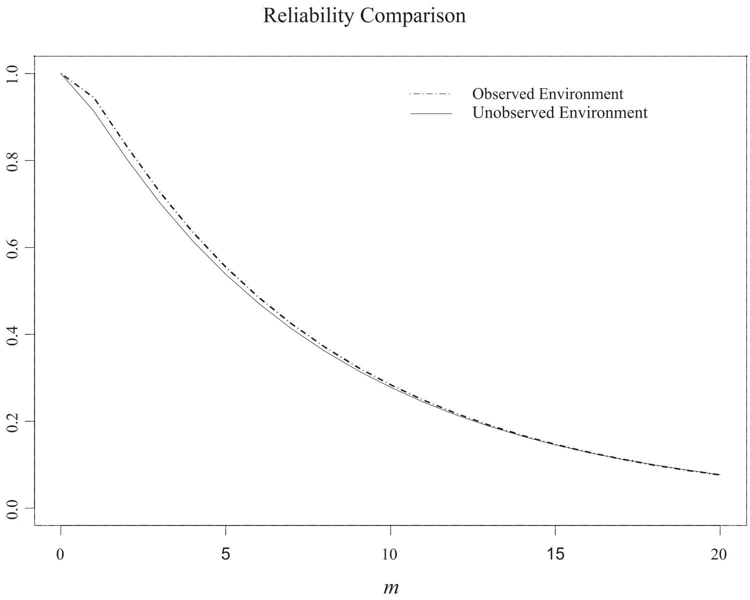

Fig. 3. Comparison of Posterior Reliability Functions for Observed and Unobserved Environments.

cases and note that the posterior reliability functions are very close to each other. We can also evaluate the expected posterior lifetime of the system after period 360. For the observed environment case we can use Monte Carlo approximation

$$E_{Y_n}[\hat{L}|\mathbf{X}^n] \approx \frac{1}{G}\sum_g E_{Y_n}[\hat{L}|\Theta^{(g)}) \tag{45}$$

by generating G realizations from the posterior distribution $p(\Theta \mid \mathbf{Y}^n, \mathbf{X}^n)$ and for the unobserved environment case we use the approximation

$$E[\hat{L}|\mathbf{X}^n] \approx \frac{1}{G}\sum_g E_{Y_n^{(g)}}[\hat{L}|\Theta^{(g)}) \tag{46}$$

by generating G realizations from the posterior distribution $p(\Theta, Y_n \mid \mathbf{X}^n)$. In our example, we calculated these as 8.62 periods for the observable environment case and 8.48 periods for the unobservable environment case.

In the Bayesian setup we can also make probability statements about reliability at a given mission time m; that is, given data we can look at the posterior distribution of the random variable $\sum_{j \in E} Q_0^m(Y_n, j \mid \Theta)$ for fixed m using the posterior samples of Θ in the observed environment case and posterior samples of Y_n and Θ in the unobserved environment case. In Figure 4 we show the posterior distributions of reliability at mission times $m = 1$ and 4 for both cases. Again as expected, the posterior distributions for the unobserved environment case shows more dispersion as implied by longer tails. In both

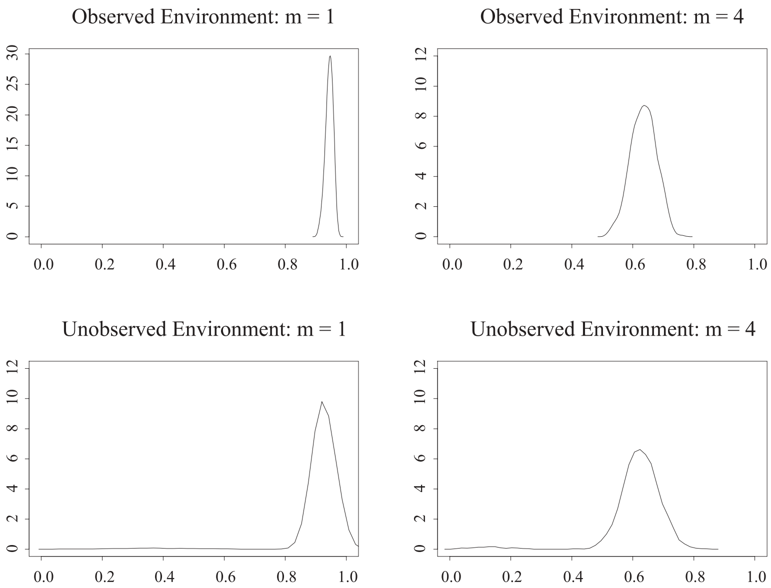

Fig. 4. Posterior Distributions of Reliability for Mission Times 1 and 4 for Observed and Unobserved Environments.

cases the distributions are concentrated around high values for mission time 1 and they shift to the left as mission time increases. We note that the posterior reliability functions presented in Figure 3 represents the means of the posterior reliability distributions at different mission times.

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