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Estimates on Level Set Integral Operators in dimension two

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ABSTRACT. Both oscillatory integral operators and level set operators appear naturally in the study of properties of degenerate Fourier integral operators (such as generalized Radon transforms). The properties of oscillatory integral operators have a longer history and are better understood. On the other hand, level set operators, while sharing many common characteristics with oscillatory integral operators, are easier to handle.

We study L^2-estimates on level set operators in dimension two and compare them with what is known about oscillatory integral operators. The cases include operators with non-degenerate phase functions and the level set version of Melrose-Taylor transform (as an example of a degenerate phase function). The estimates are formulated in terms of the Newton polyhedra and type conditions.

1. Introduction

Properties of generalized Radon transformations [PS86a] and more generally oscillatory and Fourier integral operators associated to degenerate canonical relations have been attracting continuous interest since the paper by Melrose and Taylor [MT85] appeared in 1985 (see the reviews [Pho95] and [GSW00, GS02]). Such operators often appear in the theory of partial differential equations (scattering theory [MT85], trace regularity [Tat98]) and in the integral geometry (restricted X-ray transforms [GG68], [GU89], [GS94], averages over curves, generalized Radon transformations).

Let $X = Y = \mathbb{R}^n$, $n \geq 1$. Consider the Fourier integral operator

$$\mathfrak{F}u(x) = \int_{\mathbb{R}^N \times Y} e^{i\phi(x,\theta,y)} a(x,\theta,y) u(y) d\theta\, dy, \tag{1.1}$$

where the phase function $\phi(x,\theta,y)$ is smooth and homogeneous of degree 1 in θ, and

$$a \in S^d_{\mathrm{cl.}}(X \times \mathbb{R}^N \times Y) \equiv S^d_{\mathrm{phg.}}(X \times \mathbb{R}^N \times Y)$$

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is a classical (polyhomogeneous) symbol of order d, introduced by Hörmander [Hör71]. For simplicity, we assume that $a(x,\theta,y)$ has a compact support in both x and y and is homogeneous in θ (of degree d):

$$a(x,\theta,y) = |\theta|^d a(x,\theta/|\theta|,y), \qquad \text{for some} \quad d\in\mathbb{R}.$$

To this Fourier integral operator we associate the canonical relation

$$\mathbf{C} = \{(x, d_x\phi(x,\theta,y), y, -d_y\phi(x,\theta,y)) : d_\theta\phi(x,\theta,y) = 0\} \subseteq T^*X\times T^*Y. \tag{1.2}$$

If this canonical relation is non-degenerate (the projections from $\mathbf{C}$ onto $T^*X\backslash 0$ and $T^*Y\backslash 0$ are locally diffeomorphisms), then the continuity of $\mathfrak{F}$ in Sobolev spaces $H^s \equiv W^{s,2}$ follows from Hörmander's paper [Hör71], and estimates in L^p-based Sobolev spaces $W^{s,p}$ follow from the paper of Seeger, Sogge, and Stein [SSS91]. The case of degenerate canonical relations is less understood.

Let $\varphi\in C_0^\infty(\mathbb{R})$ be identically equal to 1 in a neighborhood of the origin. We define $\mathfrak{F}_\lambda$ by cutting off large values of θ:

$$\begin{aligned}\mathfrak{F}_\lambda u(x) \quad &= \int_{\mathbb{R}^N\times Y} |\theta|^d\varphi(|\theta|/\lambda)e^{i\phi(x,\theta,y)}\,a(x,\theta/|\theta|,y)\,u(y)\,d\theta\,dy\\ &= \lambda^{d+N}\int_{\mathbb{S}^{N-1}\times\mathbb{R}_+\times Y} t^{d+N-1}\varphi(t)\,e^{i\lambda t\phi(x,\alpha,y)}\,a(x,\alpha,y)\,u(y)\,d^{N-1}\alpha\,dt\,dy,\end{aligned} \tag{1.3}$$

where $\alpha=\theta/|\theta|\in\mathbb{S}^{N-1}$ and $t=|\theta|/\lambda$ (so that $\theta=\alpha|\theta|=\lambda\alpha t$).

To study the properties of the Fourier integral operator $\mathfrak{F}$ one estimates the λ-decay of the $L^2\to L^2$-norm of the oscillatory integral operators of the form

$$T_\lambda u(x) = \int_Y e^{i\lambda\Phi(x,y)}\,a(x,y)\,u(y)\,dy, \qquad \Phi(x,y)\in C^\infty(X\times Y). \tag{1.4}$$

Such operators were treated extensively by Phong and Stein [PS82, PS83, PS86a, PS86b, PS89, PS91, PS92, PS94a, PS94b, PS94c, PS97, PS98, PSS99], Seeger [See93, See98], Greenleaf and Seeger [GS94, GS98, GS99], Cuccagna [Cuc97], Comech [Com97, Com98a, Com98b, Com99], Cuccagna and Comech [CC00], and many others.

The results are optimal in one dimension. They are sparse for higher dimensions; essentially, what is known are the estimates for the cases when the projections from the canonical relation $\mathbf{C}$ onto T^*X and T^*Y have singularities of sufficiently low type, such as folds and cusps.

The properties of Fourier integral operators are intrinsically related to the properties of level set integral operators. The latter operators come into play in the following way. Rewrite (1.3) as

$$\mathfrak{F}_\lambda u(x) = \lambda^{d+N}\int_{\mathbb{S}^{N-1}\times Y}\psi_0(\lambda\phi(x,\alpha,y))\,a(x,\alpha,y)\,u(y)\,d^{N-1}\alpha\,dy, \tag{1.5}$$

where $\psi_0(\tau)=\int_0^\infty t^{d+N-1}\varphi(t)e^{it\tau}dt$ is a Schwartz function, $\psi_0\in\mathscr{S}(\mathbb{R})$. Note that if there is no dependence on α (e.g. $N=1$), then

$$\mathfrak{F}_\lambda u(x) = \lambda^{d+N}\int_Y \psi_0(\lambda\phi(x,y))\,a(x,y)\,u(y)\,dy, \tag{1.6}$$

which is a level set integral operator (sublevel sets where $\phi(x,y) \approx \lambda^{-1}$). Thus, another way to study Fourier integral operators is to consider integral operators associated to sublevel sets

$$\mathfrak{L}_\lambda : \; L^1(Y) \to L^\infty(X), \qquad \mathfrak{L}_\lambda u(x) = \int_Y a(x,y)\psi(\lambda\Phi(x,y))u(y)\,dy. \tag{1.7}$$

For simplicity, we assume that $a \in L^\infty_{comp}(X \times Y)$, $\psi \in \mathscr{S}(\mathbb{R})$ and Φ is a phase function. Thus, we are interested in the decay rate of the $L^p \to L^q$-norm of $\mathfrak{L}_\lambda$ for large values of λ. The $L^p \to L^q$ estimates for particular cases in dimension $n = 1$ were considered in [CCW99] and [PSS01]. In the first reference, Christ, Carbery and Wright gave estimates on level set integral operators when the phase function is smooth, non-degenerate (i.e., $\Phi_{xy} \neq 0$) and its Newton diagram consists of one point (see [CCW99]). Then $\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \le C\,\lambda^{-1/2}\ln^{1/2}\lambda$. In the second reference, Phong, Stein and Sturm considered the case when the phase function is a polynomial with an arbitrary Newton diagram. They obtained a sharp estimate (without the logarithmic factor).

The purpose of this paper is to consider the $L^2 \to L^2$ estimates on operators associated to sublevel sets in dimension $n = 2$ with the non-degenerate phase function Φ (Theorem 1.1 below) and a specific example of a degenerate phase function (of type $(1,k)$) which comes from the Melrose-Taylor transform (Theorem 1.2).

Theorem 1.1. *(Non-degenerate case in 2D) Assume that $\Phi \in C^\infty(X \times Y)$, $\dim X = \dim Y = 2$, and $\det \Phi_{x_i y_j} \neq 0$ on the support of $a(x,y)$. Then*

$$\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \le C\,\lambda^{-\frac{1}{2d}}\ln^4\lambda, \qquad \text{where} \quad d = 1/2.$$

We prove this theorem in Section 3. We also show there that the logarithmic factor can not be completely removed in dimension 2; in particular, we examine the case $\Phi(x,y) = x \cdot y$ (example 3.6) where the sharp estimate is

$$\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \le C\,\lambda^{-1}\ln\lambda. \tag{1.8}$$

Next we study the level set version of the Melrose-Taylor transform [MT85]. This model contains a rich set of examples, namely, operators associated to canonical relations with a fold singularity in one of the projections and an arbitrary singularity in the other projection. The estimates that hold for the Melrose-Taylor transform with particular singularities of the projections from the associated canonical relation also hold for arbitrary oscillatory integral operators associated to canonical relations with the same singularities (see [Com98b] and [Com99]). We hope this is also the case for the level set operators (arbitrary level set integral operators vs. level set version of Melrose-Taylor transform).

Theorem 1.2. *(Melrose-Taylor transform) Let K be a compact domain in $\mathbb{R}^3$ with the smooth boundary ∂K. Then for any fixed points $\boldsymbol{r}_0 \in \partial K$ and $\boldsymbol{\omega}_0 \in \mathbb{S}^2$, the level set version of the Melrose-Taylor transform*

$$\mathfrak{L}_\lambda u(\boldsymbol{r}) = \int_{\mathbb{S}^2} \psi(\lambda(\boldsymbol{r}-\boldsymbol{r}_0)\cdot(\boldsymbol{\omega}-\boldsymbol{\omega}_0))\,a(r,\omega)\,u(\boldsymbol{\omega})\,d\boldsymbol{\omega}, \quad \boldsymbol{r} \in \partial K, \quad \boldsymbol{\omega} \in \mathbb{S}^2, \quad a \in L^\infty(\partial K \times \mathbb{S}^2)$$

satisfies the bound

$$\|\mathfrak{L}_\lambda\|_{L^2(\mathbb{S}^2)\to L^2(\partial K)} \le C\,\lambda^{-1+1/(4+2k^{-1})}\ln^4\lambda, \tag{1.9}$$

where $k \in \mathbb{N}$ is the maximal order of contact of tangent lines with the boundary of K (see [Com98b]); if the Gaussian curvature of the boundary is strictly positive, then $k = 1$.

The paper is constructed as follows: next section provides definitions, notation and a simple generalization of Varchenko's result for a non-degenerate phase function for all dimensions but it is not always sharp, in a sense it gives just an a priori bound. In Section 3 we study the non-degenerate case in dimension 2 following by the asymptotically degenerate case in Section 4. Section 5 contains some necessary description of the Melrose-Taylor transform and proof of Theorem 1.2. We end the paper with an observation on smooth kernels and connection between the level set integral operators and oscillatory integral operators in Section 6.

To conclude the introduction, we remark that since φ is compactly supported, the function $\psi_0(\tau) = \int_0^\infty t^{d+N-1}\varphi(t)e^{it\tau}\,dx$ would have certain cancellation properties. Therefore, it is natural to expect that the operator in (1.7) with this particular function ψ_0 will have better regularity properties than $\mathfrak{L}_\lambda$ with a generic $\psi \in \mathscr{S}(\mathbb{R})$. However, it turns out that in several important cases the regularity of $\mathfrak{F}$ that one recovers from considering (1.7) with *arbitrary* $\psi \in \mathscr{S}(\mathbb{R})$ is optimal.

2. Notation and Definitions

Let $\mathbb{R}_+$ denote the set of non-negative real numbers, and $\partial_x^\alpha \Phi(x)$ is the standard multi-index notation for the higher-order partial derivatives of Φ:

$$\partial_x^\alpha \Phi(x) = \frac{\partial^{|\alpha|}\Phi}{\partial^{\alpha_1}x_1 \dots \partial^{\alpha_n}x_n}, \qquad |\alpha| = \sum_{j=1}^n \alpha_j.$$

For reader's convenience we recall the Newton polyhedron associated to a function at a particular point (Varchenko [Var76]).

Definition 2.1. Let U be an open neighborhood of p and let $\Phi \in C^\infty(U)$. The Newton polyhedron $\Gamma_p(\Phi)$ that corresponds to Φ at the point p is defined as the convex hull of the union of sets $(\mathbb{R}_+)^n \subseteq \mathbb{R}^n$ translated to the points $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, where the points $(\alpha_1, \dots, \alpha_n)$ are such that $\partial_x^\alpha \Phi(x)$ is different from zero at p.

Definition 2.2. The distance $d[\Gamma]$ to the Newton polyhedron Γ is defined as a positive rational number $d \in \mathbb{Q}_+$ such that $(d, \dots, d)$ is the coordinate of the point of intersection of the line $x_1 = \dots = x_n$ with the boundary $\partial\Gamma$ of Γ.

In order to get sharp estimates we need to distinguish *long* and *short* Newton distances.

Definition 2.3. The long Newton distance $D_p[\Phi]$ that corresponds to the function $\Phi(z)$ is defined as

$$D_p[\Phi] = \sup_{\{z_j\}} d[\Gamma_p(\Phi(z))], \qquad D[\Phi] = \sup_{p \in X \times Y} D_p[\Phi],$$

where the supremum is taken over all real analytic local coordinates $\{z_j\}$ in an open neighborhood of the point $p \in X \times Y$.

Definition 2.4. The short Newton distance $d_p[\Phi]$ that corresponds to the function $\Phi(x, y) \in C^\infty(X \times Y)$ is defined as

$$d_p[\Phi] = \sup_{\{x_i\}} \sup_{\{y_j\}} d[\Gamma_p(\Phi(x, y))], \qquad d[\Phi] = \sup_{p \in X \times Y} d_p[\Phi],$$

where the supremum is taken over all real analytic local coordinates $\{x_i\}$ and $\{y_i\}$ in open neighborhoods of $\pi_X(p) \in X$ and $\pi_Y(p) \in Y$, respectively.

Remark 2.5. The difference between Definition 2.4 and Definition 2.3 is that when taking the supremum we no longer allow changes of local coordinates that mix coordinates in X and Y (see example 2.7). Clearly, $D_p[\Phi] \geq d_p[\Phi]$. The equality $D_p[\Phi] = d_p[\Phi]$ holds when $\Gamma_p(\Phi)$ consists of a single vertex.

Consider the operator

$$\mathfrak{L}_{\Phi,\lambda} u(x) = \int_{\mathbb{R}} a(x, y)\, \psi(\lambda\Phi(x, y))\, u(y)\, dy, \qquad \lambda \gg 1, \tag{2.1}$$

where $a \in L^\infty_{comp}(\mathbb{R} \times \mathbb{R})$, $\Phi \in C(\mathbb{R} \times \mathbb{R})$, and $\psi \in L^1(\mathbb{R})$.

Using the results of Alexander Varchenko [Var76], we immediately obtain the following estimate.

Proposition 2.6. *Assume that the phase Φ is real analytic and that the distance d to the Newton polyhedron is greater than 1. Then*

$$\|\mathfrak{L}_\lambda\|_{L^2 \to L^2} \leq C\, \lambda^{-\frac{1}{2D[\Phi]}} \ln^p \lambda, \qquad 0 \leq p < \infty.$$

Furthermore, if the point $(d, \ldots, d)$ is inside some face of the Newton polyhedron, then $p = 0$.

This follows from the estimate on the decay of oscillatory integrals, i.e., $I_{\Phi,\lambda} = \int_{\mathbb{R}^n} \psi(x) e^{i\lambda\Phi}\, dx$, where $\psi \in C_0^\infty(\mathbb{R})$. Assume that the phase Φ is real-analytic, that $D[\Phi] = \sup_{p \in \operatorname{supp} a(x)} D[\Gamma_p(\Phi)] > 1$, and that the line $x(t) = (t, \ldots, t)$ meets $\Gamma_p(\Phi)$ at the interior of the face of $\Gamma_p(\Phi)$ at the points where $d[\Gamma_p(\Phi)] = d$. Then, according to [Var76], we obtain $|I_{\Phi,\lambda}| \leq C\, \lambda^{-\frac{1}{D[\Phi]}}$; the result now follows from Hölder's inequality.

Example 2.7. Consider the phase function $\Phi(x, y) = x - y$, $x, y \in \mathbb{R}$. When computing the short Newton distance $d[\Phi]$, we do not mix x and y; one computes that $d[\Phi] = 1/2$. On the other hand, when computing $D[\Phi]$, we can choose the local coordinates $x' = x - y$, $y' = y$. Then $\Phi = x'$ and the Newton diagram consists of the single point $(1, 0)$; hence, the distance to the Newton polyhedron is $D[\Phi] = 1$. Next, the level set integral

$$I_\lambda = \int_{\mathbb{R}} \psi(\lambda\Phi(x, y)) a(x, y)\, dx\, dy, \qquad a(x, y) \in C^\infty_{comp}(\mathbb{R} \times \mathbb{R})$$

has the bound $|I_\lambda| \leq C\, \lambda^{-1} = C\, \lambda^{-\frac{1}{D[\Phi]}}$, with $D[\Phi] = 1$. For the level set operator

$$\mathfrak{L}_\lambda u(x) = \int_{\mathbb{R}} \psi(\lambda\Phi(x, y)) u(y)\, dy,$$

Proposition 2.6 gives the bound $\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \le \lambda^{-\frac{1}{2D[\Phi]}}$. On the other hand, if we apply the Schur lemma, we obtain the stronger bound $\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \le C\,\lambda^{-1} = C\,\lambda^{-\frac{1}{2d[\Phi]}}$, with $d[\Phi]=1/2$.

Remark 2.8. Note that we consider the *complete Newton polyhedron*, as opposed to the *reduced Newton polyhedron* (points (m,n) with $m=0$ or $n=0$ are excluded) that appeared in [PS97]. The difference is the following: adding terms like x^a or y^b to the phase function $\Phi(x,y)$ (that does not affect the reduced Newton polyhedron) only leads to the unitary factors in oscillatory integral operators, not affecting $L^2\to L^2$ estimates. On the other hand, the terms like x^a or y^b could improve the estimates on operators associated to the sublevel sets.

3. Non-degenerate case

In this section we prove Theorem 1.1 and study further consequences. We consider the level set integral operator

$$\mathfrak{L}_\lambda u(x) = \int_Y \psi(\lambda\Phi(x,y))\,u(y)\,dy, \qquad x\in X, \quad y\in Y, \tag{3.1}$$

where X and Y are small open neighborhoods of the origin in $\mathbb{R}^2$ and $\psi\in C_{comp}(\mathbb{R})$, $\Phi\in C^\infty(X\times Y)$.

Proof. (of Theorem 1.1) We introduce the localization

$$\mathfrak{L}_\lambda = \sum_{\sigma,\rho,m,n} \mathfrak{L}^{\sigma,\rho}_{m,n}, \tag{3.2}$$

where $m\in\mathbb{N}^2$ $(=\mathbb{N}\times\mathbb{N})$, $n\in\mathbb{N}^2$, $\sigma\in\{-1,1\}^2$, $\rho\in\{-1,1\}^2$, and

$$\mathfrak{L}^{\sigma,\rho}_{m,n}u(x) = \int_Y \prod_{i=1}^2 \beta(\sigma_i 2^{m_i}\Phi_{y_i}) \prod_{j=1}^2 \beta(\rho_j 2^{n_j}\Phi_{x_j})\,\psi(\lambda\Phi(x,y))\,u(y)\,dy, \tag{3.3}$$

and the localization function $\beta\in C_0^\infty([1/2,2])$ satisfies

$$\sum_{m\in\mathbb{N}} \beta(2^m t) = 1 \qquad \text{for any } t\in(0,1/2).$$

In other words, on the support of the integral kernel of $\mathfrak{L}^{\sigma,\rho}_{m,n}$, the following inequalities are satisfied

$$2^{-m_i-1} \le \sigma_i\Phi_{y_i} \le 2^{-m_i+1}, \qquad 2^{-n_j-1} \le \rho_j\Phi_{x_j} \le 2^{-n_j+1}.$$

The proof is based on the following 3 lemmas. First, we get a general estimate on $\mathfrak{L}^{\sigma,\rho}_{m,n}$ for all m and n, then we show that the case of large m and n can be discarded.

Lemma 3.1. *For any $\sigma\in\{-1,1\}^2$, $\rho\in\{-1,1\}^2$ and for any $m\in\mathbb{N}^2$, $n\in\mathbb{N}^2$,*

$$\|\mathfrak{L}^{\sigma,\rho}_{m,n}\|_{L^2\to L^2} \le C\,\lambda^{-1}.$$

Proof. To obtain this estimate, we apply the Schur lemma. Denote by $L(x,y)$ the integral kernel of $\mathfrak{L}^{\sigma,\rho}_{m,n}$. Changing the variables of integration from (x_1,x_2) to (Φ_{y_1},Φ), we get

$$\int_X |L(x,y)|\,d^2x \le \int_X \prod_{i=1}^{2}\beta(\sigma_i 2^{m_i}\Phi_{y_i})\prod_{j=1}^{2}\beta(\rho_j 2^{n_j}\Phi_{x_j})\psi(\lambda\Phi(x,y))\frac{d\Phi_{y_1}\,d\Phi}{\left|\frac{\partial(\Phi_{y_1},\Phi)}{\partial(x_1,x_2)}\right|} \tag{3.4}$$

$$\le \frac{C\,\lambda^{-1}2^{-m_1}}{\inf\limits_{\operatorname{supp} L}\left|\frac{\partial(\Phi_{y_1},\Phi)}{\partial(x_1,x_2)}\right|}.$$

Similarly, interchanging y_1 and y_2, we get

$$\int_X |L(x,y)|\,d^2x \ \le \frac{C\,\lambda^{-1}2^{-m_2}}{\inf\limits_{\operatorname{supp} L}\left|\frac{\partial(\Phi_{y_2},\Phi)}{\partial(x_1,x_2)}\right|}. \tag{3.5}$$

Thus,

$$\begin{aligned}\int_X |L(x,y)|\,d^2x \quad &\le \quad C\,\lambda^{-1}\min\left(\frac{2^{-m_1}}{\inf\limits_{\operatorname{supp} L}\left|\frac{\partial(\Phi_{y_1},\Phi)}{\partial(x_1,x_2)}\right|},\frac{2^{-m_2}}{\inf\limits_{\operatorname{supp} L}\left|\frac{\partial(\Phi_{y_2},\Phi)}{\partial(x_1,x_2)}\right|}\right) \\ &\le \quad C\,\frac{2^{-m_1}+2^{-m_2}}{\max\left(\inf\limits_{\operatorname{supp} L}\left|\frac{\partial(\Phi_{y_1},\Phi)}{\partial(x_1,x_2)}\right|,\inf\limits_{\operatorname{supp} L}\left|\frac{\partial(\Phi_{y_2},\Phi)}{\partial(x_1,x_2)}\right|\right)}.\end{aligned} \tag{3.6}$$

Let us denote the denominator of the above expression by D; then

$$2D^2 \ge \begin{vmatrix}\Phi_{x_1y_1} & \Phi_{x_1}\\ \Phi_{x_2y_1} & \Phi_{x_2}\end{vmatrix}^2 + \begin{vmatrix}\Phi_{x_1y_2} & \Phi_{x_1}\\ \Phi_{x_2y_2} & \Phi_{x_2}\end{vmatrix}^2 = \left\|\begin{bmatrix}\Phi_{x_1y_1} & \Phi_{x_2y_1}\\ \Phi_{x_1y_2} & \Phi_{x_2y_2}\end{bmatrix}\begin{bmatrix}\Phi_{x_2}\\ -\Phi_{x_1}\end{bmatrix}\right\|^2. \tag{3.7}$$

We use the following inequality:

Lemma 3.2. *For any matrix $A\neq 0$ and any vector u, we have $\|Au\| \ge \inf\limits_{\lambda\in\sigma(A)}|\lambda|\,\|u\| \ge \frac{|\det A|}{\|A\|^{N-1}}\|u\|$, where $\sigma(A)$ is the set of eigenvalues of A.*

Since $\|[\Phi_{x_1},\Phi_{x_2}]^T\| \ge C(2^{-n_1}+2^{-n_2})$, we deduce from (3.7) that

$$D \ge C\left|\det\Phi_{x_iy_j}\right|(2^{-n_1}+2^{-n_2}).$$

Thus, continuing the proof of the previous lemma, (3.6) becomes

$$\sup_{y\in Y}\int_X |L(x,y)|\,d^2x \le \frac{C\,\lambda^{-1}}{\inf\limits_{\operatorname{supp} L}|\det\Phi_{x_iy_j}|}\,\frac{2^{-m_1}+2^{-m_2}}{2^{-n_1}+2^{-n_2}}. \tag{3.8}$$

Applying the same argument to the integration in y, we get

$$\sup_{x\in X}\int_Y |L(x,y)|\,d^2y \le \frac{C\,\lambda^{-1}(2^{-n_1}+2^{-n_2})}{\inf\limits_{\operatorname{supp} L}|\det\Phi_{x_iy_j}|(2^{-m_1}+2^{-m_2})} \tag{3.9}$$

The Schur lemma finishes the proof of Lemma 3.1. □

For very large values of m_i and n_j, we use a different estimate.

Lemma 3.3. *For any* $\sigma \in \{-1,1\}^2$, $\rho \in \{-1,1\}^2$,

$$\|\mathfrak{L}^{\sigma,\rho}_{m,n}\|_{L^2\to L^2} \le C\, 2^{-\frac{1}{2}(m_1+m_2+n_1+n_2)}.$$

Proof. Changing the variables of integration $(x_1, x_2) \to (\Phi_{y_1}, \Phi_{y_2})$, similarly to the derivation of (3.4) in Lemma 3.1, and then using localizations with respect to Φ_{y_1} and Φ_{y_2}, we obtain

$$\int_X |L(x,y)|\, d^2x = \int_X \dots \frac{d\Phi_{y_1} d\Phi_{y_2}}{\left|\frac{\partial(\Phi_{y_1},\Phi_{y_2})}{\partial(x_1,x_2)}\right|} \le C\, \frac{2^{-m_1-m_2}}{|\det \Phi_{x_i y_j}|}.$$

Similarly,

$$\int_Y |L(x,y)|\, d^2y \le C\, \frac{2^{-n_1-n_2}}{|\det \Phi_{x_i y_j}|}.$$

Now, applying the Schur lemma, we obtain the desired estimate. □

Lemma 3.3 means that in summation (3.2) the terms with $\max(m_i, n_j) \ge 2\ln\lambda$ are bounded by convergent geometric series that is bounded by $C\,\lambda^{-1}$. The rest is bounded by

$$\sum_{\sigma,\rho} \left(\sum_{1\le m_i,n_j\le 2\ln\lambda} \|L^{\sigma,\rho}_{m,n}\| \right) \le C\,\lambda^{-1} \ln^4 \lambda. \tag{3.10}$$

This finishes the proof of Theorem 1.1. □

Remark 3.4. If we track what happens to the estimate when $\inf_{\operatorname{supp} L} |\det \Phi_{x_i y_j}|$ becomes asymptotically small, we get

$$\|\mathfrak{L}_\lambda\| \le \frac{C}{\inf |\det \Phi_{x_i y_j}|} \lambda^{-1} \ln^4 \lambda. \tag{3.11}$$

This asymptotic is not optimal; the rescaling suggests that optimal result could be as good as

$$\|\mathfrak{L}_\lambda\| \le \frac{C}{\inf |\det \Phi_{x_i y_j}|^{1/2}} \lambda^{-1}. \tag{3.12}$$

Indeed, in several important cases we can regain $|\det \Phi_{x_i y_j}|^{1/2}$.

Remark 3.5. Applying Stein's Almost Orthogonality argument, one can try to reduce the power of $\ln\lambda$ in (3.11). However, in the general case, this logarithmic factor can not be completely removed, which we demonstrate in the following example.

Example 3.6. In the case $\Phi(x,y) = x\cdot y$, where x, $y \in \mathbb{R}^2$, we can show that the sharp estimate is

$$\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \le C\,\lambda^{-1} \ln \lambda. \tag{3.13}$$

To obtain (3.13), one considers the partition

$$1 = \sum_{j\in\mathbb{N},k\in\mathbb{N}} \beta(2^j x_1)\beta(2^k x_2),$$

splitting $\mathfrak{L}_\lambda$ into $\mathfrak{L}_\lambda = \sum_{j\in\mathbb{N},\, k\in\mathbb{N}} \mathfrak{L}_{\lambda,i,j}$. One can show that $\|\mathfrak{L}_{\lambda,i,j}\| \le C\,\lambda^{-1}$, then the almost orthogonality considerations lead to the bound (3.13).

The bound (3.13) is sharp, i.e.,

$$\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \ge C\,\lambda^{-1}\ln\lambda. \tag{3.14}$$

To show that, consider

$$u_\lambda(r) = \begin{cases} \frac{c_\lambda}{r}, & r \ge \lambda^{-1}, \\ 0, & r < \lambda^{-1}, \end{cases}$$

where $c_\lambda = \ln^{-1/2}\lambda$ is chosen so that $\|u_\lambda\|_{L^2} = 1$. Then

$$T_\lambda u_\lambda(x) = \int_{\mathbb{R}^2} \chi(\lambda x\cdot y)u(y)\,dy \ge \int_{\lambda^{-1}}^{\infty}\int_0^{2\pi} \chi(\lambda|x|r\cos\theta)\,r\,dr\,d\theta.$$

Now by shifting $\theta \mapsto \theta + \pi/2$ we can replace $\cos\theta$ by $\sin\theta$, and it suffices to consider θ such that $\sin\theta \approx \theta$. Thus, consider

$$\int_{\lambda^{-1}}^{\infty}\int_0^{2\pi} \chi(\lambda|x|r\theta)\frac{c_\lambda}{r}r\,dr\,d\theta = c_\lambda\int_{\lambda^{-1}}^{\infty}\int_0^{2\pi} \chi(\lambda|x|r\theta)\,dr\,d\theta.$$

When integrating in θ, we consider two cases

1. $\lambda|x|r > 1$, so that $\int_0^1 \chi(\lambda|x|r\theta)\,d\theta \approx 1/(\lambda|x|r)$.

2. $\lambda|x|r < 1$, so that $\int_0^1 \chi(\lambda|x|r\theta)\,d\theta \approx 1$.

As it turns out, it suffices to consider the first case; this case alone yields

$$T_\lambda u_\lambda(x) \ge \frac{c_\lambda}{\lambda|x|}\int_{1/(\lambda|x|)}^{1}\frac{dr}{r} = \frac{c_\lambda}{\lambda|x|}\ln r\Big|_{1/(\lambda|x|)}^{1} = \frac{c_\lambda\ln(\lambda|x|)}{\lambda|x|}.$$

The bound from below on the $L^2 \to L^2$ norm of T_λ is given by

$$\|T_\lambda\|^2_{L^2\to L^2} = \int_{\mathbb{R}^2} |T_\lambda u_\lambda(x)|^2\,dx \ge 2\pi\int_{1/\lambda}^{1} \frac{c_\lambda^2\ln^2(\lambda|x|)}{\lambda^2|x|^2}\,|x|\,d|x|$$

$$= \frac{2\pi c_\lambda^2}{\lambda^2}\int_{1/\lambda}^{1}\ln^2(\lambda|x|)\,d(\ln(\lambda|x|)) = \frac{2\pi c_\lambda^2\ln^3\lambda}{3\lambda^2},$$

where $c_\lambda^2 = 1/\ln\lambda$. It follows that

$$\|T_\lambda\|_{L^2\to L^2} \ge C\,\frac{\ln\lambda}{\lambda}.$$

We conclude that we can not get rid of the logarithmic factor in the two-dimensional case.

This example can be generalized for higher dimensions ($n \ge 2$): if $\psi \in C_{comp}$ and is non-negative, then for any $a(x,y) \in L^\infty_{comp}(\mathbb{R}^n\times\mathbb{R}^n)$ the operator

$$\mathfrak{L}_\lambda u(x) = \int_{\mathbb{R}^n} a(x,y)\,\psi(\lambda\Phi(x,y))\,u(y)\,dy,$$

with $\Phi(x,y) = x \cdot y$, is continuous in L^2 with the bound $\lambda^{-1} \ln \lambda$, and there can be no improvement. However, assuming that ψ has at least $M > n/2+1$ vanishing moments so that $\hat{\psi}^{(k)}(0)$ vanishes for $0 \le k \le M-1$ allows to push estimates to match those of non-degenerate oscillatory integral operators. We prove this assuming that $a(x,y)$ is smooth (see Lemma 6.1). For higher dimensions $n > 2$ we expect that in the non-degenerate case $(\partial_x^\alpha \partial_y^\beta \Phi \ne 0)$ the following estimate holds

$$\|\mathfrak{L}_\lambda\|_{L^2 \to L^2} \le C\, \lambda^{-\frac{1}{2d[\Phi]}} \ln^p \lambda, \qquad 0 \le p < \infty,$$

where $d[\Phi] = \max(\max_i \alpha_i, \max_j \beta_j)$ is the *short Newton distance* that corresponds to Φ.

4. Asymptotically degenerate case

We start with a "pure phase" result.

Lemma 4.1. *Assume that $\Phi \in C^\infty(X \times Y)$, $\dim X = \dim Y = 2$,*

$$\Phi(x,y) = \Phi_1(x_1,y_1) + \Phi_2(x_2,y_2),$$

and $\partial_{x_2}\partial_{y_2}\Phi_2 \ne 0$ on the support of $a(x,y)$. Then

$$\|\mathfrak{L}_\lambda\|_{L^2 \to L^2} \le C\, \lambda^{-\frac{1}{2d[\Phi]}} \ln^4 \lambda.$$

This lemma resembles the situation with the oscillatory integral operators; the proof is rather straightforward and we leave it as an exercise for the reader.

Remark 4.2. The same estimate holds for a slightly more degenerate phase function $\Phi(x,y)$ that satisfies $\partial_{x_i}\partial_{y_i}\Phi(x,y) = 0$ for some $i \in \{1,2\}$ and $j \in \{1,2\}$.

Now we consider the asymptotic case. Let $\varkappa > 0$ be asymptotically small.

Proposition 4.3. *Assume that*

$$|\det \Phi_{x_i y_j}| \ge \varkappa, \tag{4.1}$$

$$|\Phi_{x_1y_2}\Phi_{x_2y_1}| \le \frac{1}{4}|\Phi_{x_1y_1}\Phi_{x_2y_2}|. \tag{4.2}$$

Then

$$\|\mathfrak{L}_{\lambda,\varkappa}\| \le C\, \frac{\lambda^{-1}}{\varkappa^{1/2}} \ln^4 \lambda.$$

The condition (4.2) is quite standard (see e.g. [Com97]); it represents our next best approximation to the "pure phase" case.

Proof. First, observe that (4.1) and (4.2) imply that

$$|\Phi_{x_1y_1}\Phi_{x_2y_2}| \ge \frac{3\varkappa}{4}. \tag{4.3}$$

As before, we use the decomposition of $\mathfrak{L}_\lambda$ into

$$\mathfrak{L}_\lambda = \sum_{\sigma,\rho,m,n} \mathfrak{L}^{\sigma,\rho}_{m,n}, \tag{4.4}$$

where $m \in \mathbb{N}^2$, $n \in \mathbb{N}^2$, $\sigma \in \{-1,1\}^2$, $\rho \in \{-1,1\}^2$. Denote the integral kernel of $\mathfrak{L}_{m,n}^{\sigma,\rho}$ by $K(\mathfrak{L}_{m,n}^{\sigma,\rho})(x,y)$. The rest of the proof is based on the following auxiliary lemmas.

Lemma 4.4. *Consider the set of points $U_1 \subset X \times Y$, where*

$$\text{either} \qquad |\Phi_{x_2}\Phi_{x_1y_1}| \geq 2|\Phi_{x_1}\Phi_{x_2y_1}| \qquad \text{or} \qquad |\Phi_{x_2}\Phi_{x_1y_1}| \leq \frac{1}{2}|\Phi_{x_1}\Phi_{x_2y_1}| \tag{4.5}$$

and

$$\text{either} \qquad |\Phi_{y_1}\Phi_{x_2y_2}| \geq 2|\Phi_{y_2}\Phi_{x_2y_1}| \qquad \text{or} \qquad |\Phi_{y_1}\Phi_{x_2y_2}| \leq \frac{1}{2}|\Phi_{y_2}\Phi_{x_2y_1}|. \tag{4.6}$$

The operator $\mathfrak{L}_{U_1}$ with the integral kernel $K_1(x,y) = \chi_{U_1}(x,y)K(\mathfrak{L}_{m,n}^{\sigma,\rho})(x,y)$ satisfies

$$\|\mathfrak{L}_{U_1}\|_{L^2\to L^2} \leq C\,\frac{\lambda^{-1}}{\sqrt{\varkappa}}.$$

Proof. We indicate the changes of variables in order to estimate the integrals and then apply the Schur lemma.

$$\int |K_1(x,y)|\,d^2x \to \int |K_1(x,y)|\left|\frac{\partial(x_1,x_2)}{\partial(\Phi,\Phi_{y_1})}\right| d(\Phi,\Phi_{y_1}) \to \frac{\lambda^{-1}|\Phi_{y_1}|}{|\Phi_{x_2}\Phi_{x_1y_1}|}, \tag{4.7}$$

$$\int |K_1(x,y)|\,d^2y \to \int |K_1(x,y)|\left|\frac{\partial(y_1,y_2)}{\partial(\Phi,\Phi_{x_2})}\right| d(\Phi,\Phi_{x_2}) \to \frac{\lambda^{-1}|\Phi_{x_2}|}{|\Phi_{y_1}\Phi_{x_2y_2}|}. \tag{4.8}$$

Therefore,

$$\|\mathfrak{L}_{U_1}\|_{L^2\to L^2} \leq C\,\frac{\lambda^{-1}}{\inf|\Phi_{x_1y_1}\Phi_{x_2y_2}|^{1/2}} \leq C\,\frac{\lambda^{-1}}{\sqrt{\varkappa}},$$

by (4.3). □

Analogously to Lemma 4.4, we obtain the following result.

Lemma 4.5. *Consider the set of points $U_2 \subset X \times Y$, where*

$$\text{either} \qquad |\Phi_{x_1}\Phi_{x_2y_2}| \geq 2|\Phi_{x_2}\Phi_{x_1y_2}| \qquad \text{or} \qquad |\Phi_{x_1}\Phi_{x_2y_2}| \leq \frac{1}{2}|\Phi_{x_2}\Phi_{x_1y_2}| \tag{4.9}$$

and

$$\text{either} \qquad |\Phi_{y_2}\Phi_{x_1y_1}| \geq 2|\Phi_{y_1}\Phi_{x_1y_2}| \qquad \text{or} \qquad |\Phi_{y_2}\Phi_{x_1y_1}| \leq \frac{1}{2}|\Phi_{y_1}\Phi_{x_1y_2}|. \tag{4.10}$$

The operator $\mathfrak{L}_{U_2}$ with the integral kernel $K_2(x,y) = \chi_{U_2}(x,y)K(\mathfrak{L}_{m,n}^{\sigma,\rho})(x,y)$ satisfies

$$\|\mathfrak{L}_{U_2}\|_{L^2\to L^2} \leq C\,\frac{\lambda^{-1}}{\sqrt{\varkappa}}.$$

The reason we need the above two lemmas is

Lemma 4.6. *The set of points (x,y) where neither (4.5) nor (4.9) is satisfied is empty.*

Similarly, the set of points where neither (4.6) nor (4.10) is satisfied, is empty.

Proof. If the first statement of the lemma was not true, we would have

$$|\Phi_{x_2}\Phi_{x_1y_1}| \approx |\Phi_{x_1}\Phi_{x_2y_1}|, \qquad |\Phi_{x_1}\Phi_{x_2y_2}| \approx |\Phi_{x_2}\Phi_{x_1y_2}|,$$

where $\approx$ means that both sides differ at most by a factor of 2. Multiplying both these equalities and cancelling Φ_{x_1} and Φ_{x_2} (these are nonzero due to localization (3.3)), we obtain

$$|\Phi_{x_1y_1}\Phi_{x_2y_2}| \approx |\Phi_{x_2y_1}\Phi_{x_1y_2}|,$$

where $\approx$ means that both sides differ at most by a factor of 4. This contradicts (4.2).

Similar reasoning proves the second statement of the lemma. □

Thus, we are left to consider the following two sets of points:

- The set $U_3 \subset X \times Y$ where (4.5) and (4.10) are satisfied, while (4.6) and (4.9) are not.
- The set $U_4 \subset X \times Y$ (4.6) and (4.9) are satisfied, while (4.5) and (4.10) are not.

By the symmetry, it suffices to consider the first of these sets:

Lemma 4.7. *Consider the set of points $U_3 \subset X \times Y$ where*

$$\text{either} \qquad |\Phi_{x_2}\Phi_{x_1y_1}| \ge 2|\Phi_{x_1}\Phi_{x_2y_1}| \qquad \text{or} \qquad |\Phi_{x_2}\Phi_{x_1y_1}| \le \frac{1}{2}|\Phi_{x_1}\Phi_{x_2y_1}|, \tag{4.11}$$

$$|\Phi_{y_1}\Phi_{x_2y_2}| \approx |\Phi_{y_2}\Phi_{x_2y_1}|, \tag{4.12}$$

$$|\Phi_{x_1}\Phi_{x_2y_2}| \approx |\Phi_{x_2}\Phi_{x_1y_2}|, \tag{4.13}$$

where $\approx$ means that both sides differ at most by a factor of 2, and where

$$\text{either} \qquad |\Phi_{y_2}\Phi_{x_1y_1}| \ge 2|\Phi_{y_1}\Phi_{x_1y_2}| \qquad \text{or} \qquad |\Phi_{y_2}\Phi_{x_1y_1}| \le \frac{1}{2}|\Phi_{y_1}\Phi_{x_1y_2}|. \tag{4.14}$$

The operator $\mathfrak{L}_{U_3}$ with the integral kernel $K_3(x,y) = \chi_{U_3}(x,y)K(\mathfrak{L}^{\sigma,\rho}_{m,n})(x,y)$ satisfies

$$\|\mathfrak{L}_{U_3}\|_{L^2\to L^2} \le C\,\frac{\lambda^{-1}}{\sqrt{\varkappa}}.$$

Proof. Similarly to previous proofs we only indicate the change of variables in order to get the estimates.

$$\int d^2x \to \int d(\Phi,\Phi_{y_1}) \to \frac{\lambda^{-1}|\Phi_{y_1}|}{|\Phi_{x_2}\Phi_{x_1y_1}|}, \tag{4.15}$$

$$\int d^2y \to \int d(\Phi,\Phi_{x_1}) \to \frac{\lambda^{-1}|\Phi_{x_1}|}{|\Phi_{y_1}\Phi_{x_1y_2}|} \sim \frac{\lambda^{-1}|\Phi_{x_2}\Phi_{x_1y_2}|/|\Phi_{x_2y_2}|}{|\Phi_{y_1}\Phi_{x_1y_2}|}. \tag{4.16}$$

Bounds (4.15) and (4.16) give the desired estimate

$$\|\mathfrak{L}_{U_3}\|_{L^2\to L^2} \le C\,\frac{\lambda^{-1}}{\inf|\Phi_{x_1y_1}\Phi_{x_2y_2}|^{1/2}} \le C\,\frac{\lambda^{-1}}{\sqrt{\varkappa}}.$$

□

This finishes the proof of Proposition 4.3. □

5. Level set version of the Melrose-Taylor transform

We start this section by describing the Melrose-Taylor transform. Let $K \subset \mathbb{R}^n$ be a compact domain with a smooth boundary $B = \partial K$. Melrose and Taylor showed in [MT85] that the operator

$$\mathcal{R}_{MT}u(t, \boldsymbol{r}) = \int_{\mathbb{R}\times\mathbb{S}^{n-1}} \delta(t - s - \boldsymbol{r}\cdot\boldsymbol{\omega})u(s, \boldsymbol{\omega})\, ds\, d\boldsymbol{\omega}, \ \boldsymbol{r} \in B, \ d\boldsymbol{\omega} \in \mathbb{S}^{n-1}, \ s, t \in \mathbb{R}, \quad (5.1)$$

is a Fourier integral operator associated to the degenerate canonical relation

$$\mathbf{C} \subseteq T^*(\mathbb{R} \times B)\backslash 0 \times T^*(\mathbb{R} \times \mathbb{S}^2)\backslash 0.$$

This means that the projections from $\mathbf{C}$ onto $T^*(\mathbb{R} \times B)\backslash 0$ and $T^*(\mathbb{R} \times \mathbb{S}^2)\backslash 0$ become singular. Melrose and Taylor proved that when B is strictly convex (Gaussian curvature is nonzero), $\mathcal{R}_{MT}$ loses $1/6$ of a derivative compared to properties of non-degenerate Fourier integral operators (when both projections from the associated canonical relation are locally diffeomorphisms).

Using the diffeomorphism

$$\imath : (\mathbb{R} \times B) \times (\mathbb{R} \times \mathbb{S}^2) \xrightarrow{\cong} \mathbf{C},$$

we lift projections π_L and π_R onto $(\mathbb{R} \times B) \times (\mathbb{R} \times \mathbb{S}^2)$, keeping the same notations π_L and π_R for the lifted projections:

$$\pi_L : (\mathbb{R} \times B) \times (\mathbb{R} \times \mathbb{S}^2) \to T^*(\mathbb{R} \times B), \qquad \pi_R : (\mathbb{R} \times B) \times (\mathbb{R} \times \mathbb{S}^2) \to T^*(\mathbb{R} \times \mathbb{S}^2).$$

The singular components of these projections are given by the projections

$$\pi_L|_{t,\boldsymbol{r},s} : \mathbb{S}^2 \to T^*_{\boldsymbol{r}}K, \qquad \pi_R|_{t,s,\boldsymbol{\omega}} : K \to T^*_{\boldsymbol{\omega}}\mathbb{S}^2.$$

We use the Euclidean metric to identify tangent and cotangent planes; then the singular components of these projections are given by the orthogonal projections

$$\pi_L|_{t,\boldsymbol{r},s} : \mathbb{S}^2 \to T_{\boldsymbol{r}}K, \qquad \pi_R|_{t,s,\boldsymbol{\omega}} : K \to T_{\boldsymbol{\omega}}\mathbb{S}^2.$$

The first projection, being a projection from a sphere onto a plane, is always a Whitney fold. The second projection is a Whitney fold at all points if and only if B is strictly convex, with non-vanishing Gaussian curvature.

Let us consider an equivalent problem for oscillatory integral operators. Let f be a smooth function that locally represents the boundary $B = \partial K$ of a "scatterer" K, so that

$$\{x \in \mathbb{R} : x_3 = f(x_1, x_2)\} \subseteq B,$$

and let g be a smooth function that locally represents the boundary of the unit sphere

$$g(y_1, y_2) = 1 - \sqrt{1 - y_1^2 - y_2^2}.$$

Instead of the phase function $\Phi(x, y) = \boldsymbol{r}(x) \cdot \boldsymbol{\omega}(y)$ from (5.1) we will consider the phase function

$$\Phi(x, y) = (\boldsymbol{r}(x) - \boldsymbol{r}(0)) \cdot (\boldsymbol{\omega}(y) - \boldsymbol{\omega}(0)) = x_1 g(y_1, y_2) - y_1 f(x_1, x_2) + x_2 y_2, \qquad (5.2)$$

with $x \in X, y \in Y$ and $X = Y = \mathbb{R}^2$. (The phase function (5.1) with $\boldsymbol{r}(0) \neq 0$ and $\boldsymbol{\omega}(0) \neq 0$ satisfies $d_x\Phi \neq 0$ and $d_y\Phi \neq 0$, and the Schur lemma immediately leads to the estimate $\|\mathfrak{L}\| \leq C\,\lambda^{-1}$.)

We would like to know the λ-decay of the $L^2(Y) \to L^2(X)$ norm of the oscillatory integral operator

$$T_\lambda u(x) = \int_{\mathbb{R}\times\mathbb{S}^2} e^{-i\lambda\Phi(x,y)} u(s,\boldsymbol{\omega})\,dy.$$

The associated canonical relation is parameterized by x and y as

$$(x,y) \overset{\cong}{\longmapsto} (x,\Phi_x,y,\Phi_y) \in \mathbf{C} \subseteq T^*X\backslash 0 \times T^*Y\backslash 0 \tag{5.3}$$

The projections $\pi_L : \mathbf{C} \to T^*X$ and $\pi_R : \mathbf{C} \to T^*Y$ become singular on the common variety where the mixed Hessian $\frac{\partial^2\Phi}{\partial x_i \partial y_j}$ becomes degenerate. We denote

$$h(x,y) = \det_{i,j} \frac{\partial^2\Phi}{\partial x_i \partial y_j}, \qquad 1 \leq i,j \leq 2,$$

and define the critical set of Φ by

$$\rho = \{(x,y) \in X \times Y : h(x,y) = 0\}. \tag{5.4}$$

Remark 5.1. The simplest singular case is when both projections

$$\pi_L : (x,y) \mapsto (x, d_x\Phi(x,y)), \qquad \pi_R : (x,y) \mapsto (y, d_y\Phi(x,y))$$

have Whitney fold singularities. The corresponding result for oscillatory integral operators was proved by Pan and Sogge [PS90]. They proved that if the projections π_L and π_R are Whitney folds, then the operator T_λ has the λ-decay of its $L^2 \to L^2$ norm that is less by $1/6$ than in the case when π_L, π_R are non-degenerate (local diffeomorphisms).

Let us remind the definition of the *type* of a smooth map introduced in [Com99]. Let M and N be two C^∞-manifolds of the same dimension. Consider a smooth map $\pi : M \to N$ that drops rank simply by 1: the co-rank of $d\pi$ is at most 1, and at its critical variety the differential $d(\det d\pi)$ does not vanish. The *type* of π at a critical point is defined as the highest order of vanishing of the determinant of its Jacobian in the directions of the kernel of its differential. Let $V \in C^\infty(\Gamma(TM))$ be a kernel vector field

$$V|_{\det d\pi=0} \neq 0, \qquad V|_{\det d\pi=0} \in \ker d\pi.$$

If $\dim\ker d\pi|_{p_o} > 0$, then we define the type of π at p_o as the smallest $k \in \mathbb{N}$ such that $V^k(\det d\pi)|_{p_o} \neq 0$. Thus, for the level-set version of Melrose-Taylor transform in Theorem 1.2 the degenerate phase function is of the type $(1,k)$.

Now we prove Theorem 1.2 which follows from the following two lemmas. In the first lemma, we derive a bound on the operator truncated away from the critical set. In the second lemma, we get the estimate on the part of the operator supported in a small neighborhood of the critical set.

Lemma 5.2. *Consider the level set version of the Melrose-Taylor transform restricted to the set of points where $|\det\Phi_{x_iy_j}| \geq \varkappa$ for some small $\varkappa > 0$*

$$\mathfrak{L}_{\lambda,\varkappa}u(x) = \int (1-\beta_0(\varkappa^{-1}\det\Phi_{x_iy_j}))\psi(\lambda\Phi(x,y))u(y)\,dy.$$

This operator satisfies

$$\|\mathfrak{L}_{\lambda,\varkappa}\|_{L^2(Y)\to L^2(X)} \le C\,\frac{\lambda^{-1}}{\varkappa^{1/2}}\ln^4\lambda.$$

Proof. First, let us compute $\det\Phi_{x_iy_j}$, we obtain $\det\Phi_{x_iy_j} = g_1(y) - f_1(x) + f_2(x)g_2(y)$. For brevity, we used the notations $f_i(x) = \partial_{x_i}f(x)$, $g_j(y) = \partial_{y_j}g(y)$.

Let $K(x,y)$ denote the integral kernel of $\mathfrak{L}_{\lambda,\varkappa}$. We employ the following integration

$$\int K(x,y)\,d^2x \le \int K(x,y)\frac{d(\Phi,\Phi_{y_2})}{\left|\frac{\partial(\Phi,\Phi_{y_2})}{\partial(x_1,x_2)}\right|} \le C\,\frac{\lambda^{-1}|\Phi_{y_2}|}{|\Phi_{x_1}\Phi_{x_2y_2} - \Phi_{x_2}\Phi_{x_1y_2}|}, \tag{5.5}$$

which we supplement with

$$\int K(x,y)\,d^2y \le \int K(x,y)\frac{d(y_1,\Phi)}{\left|\frac{\partial(y_1,\Phi)}{\partial(y_1,y_2)}\right|} \le C\,\frac{\lambda^{-1}|y_1|}{|\Phi_{y_2}|}. \tag{5.6}$$

The denominator in (5.5) is the following expression

$$\begin{aligned}\Phi_{x_1}\Phi_{x_2y_2} - \Phi_{x_2}\Phi_{x_2y_1} &= (g(y) - y_1f_1(x)) - (-y_1f_2(x) + y_2)g_2(y)\\ &= (g(y) - y_1g_1(y) - y_2g_2(y)) + y_1\det\Phi_{x_iy_j}.\end{aligned}$$

If $|g(y)-y_1g_1(y)-y_2g_2(y)|$ differs from $|y_1\det\Phi_{x_iy_j}|$ at least by a factor of 2 at each point (x,y), then (5.5) is bounded from above by $C\,\lambda^{-1}|\Phi_{y_2}|/(|y_1||\det\Phi_{x_iy_j}|)$, and applying the Schur lemma, we get the estimate

$$\|\mathfrak{L}_{\lambda,\varkappa}\|^2_{L^2(Y)\to L^2(X)} \le \frac{C\,\lambda^{-1}}{\inf|\det\Phi_{x_iy_j}|^{1/2}} \le \frac{C\,\lambda^{-1}}{\sqrt{\varkappa}}.$$

Next we consider the case when these two terms differ less than by a factor of two, i.e.,

$$|g(y) - y_1g_1(y) - y_2g_2(y)| \approx |y_1\det\Phi_{x_iy_j}|.$$

Due to the strict convexity of the unit sphere,

$$|g(y) - y_1g_1(y) - y_2g_2(y)| \ge c|y|^2,$$

for some $c > 0$, so that we have

$$\frac{c}{2}|y|^2 \le |y_1||\det\Phi_{x_iy_j}| \le |y||\det\Phi_{x_iy_j}|.$$

This results in the bound on $|y|$,

$$|y| \le \frac{2}{c}|\det\Phi_{x_iy_j}|.$$

Since we also know that $|g_2(y)| \le C\,|y|$ for small values of $|y|$, we have the bound

$$|\Phi_{x_1y_2}\Phi_{x_2y_1}| = |f_2(x)g_2(y)| \le C\,|f_2(x)||\det\Phi_{x_iy_j}|,$$

which is smaller than $|\det\Phi_{x_iy_j}|$, if $|f_2(x)|$ is sufficiently small (this is easily provided by restricting the x-support). Proposition 4.3 becomes applicable, and it finishes the proof. $\square$

Lemma 5.3. *The level set version of the Melrose-Taylor transform, truncated to the set* $|\det \Phi_{x_i y_j}| \le \varkappa$,

$$\mathfrak{L}^0_{\lambda,\varkappa} u(x) = \int \beta_0(\varkappa^{-1} \det \Phi_{x_i y_j})\, \psi(\lambda \Phi(x,y))\, u(y)\, dy,$$

satisfies

$$\|\mathfrak{L}_{\lambda,\varkappa}\|_{L^2(Y)\to L^2(X)} \le C\, \lambda^{-1/2} \varkappa^{\frac{1}{2}+\frac{1}{2\boldsymbol{k}}} \ln \lambda,$$

where $\boldsymbol{k}$ *is the highest order of contact of the tangent lines with the boundary of* K.

Proof. This time we only need the dyadic localization in Φ_{y_2}. Denote the integral kernel of $\mathfrak{L}_{\lambda,\varkappa}$ by $K^0(x,y)$. Using the same notations as in Lemma 5.2, we estimate

$$\int K^0(x,y)\, d^2x \le \int K^0(x,y)\, \beta_0(\varkappa^{-1} \det \Phi_{x_i y_j})\, \frac{dx_1\, d\Phi_{y_2}}{\left|\frac{\partial(x_1,\Phi_{y_2})}{\partial(x_1,x_2)}\right|} \le C\, |\Phi_{y_2}| \varkappa^{1/\boldsymbol{k}}.$$

Let us mention that the integration in x_1, $\displaystyle\int_{\mathbb{R}} \beta_0(\varkappa^{-1} \det \Phi_{x_i y_j})\, dx_1$, yields a factor of $\varkappa^{1/\boldsymbol{k}}$ due to our finite type assumption on the singularity:

$$(\partial_{x_1}|_{\Phi_{y_2}})^{\boldsymbol{k}} \det \Phi_{x_i y_j} \ne 0.$$

We also estimate

$$\int K^0(x,y)\, d^2y = \int \beta_0(\varkappa^{-1} \det \Phi_{x_i y_j})\, \psi(\lambda\Phi)\, \frac{dy_1\, d\Phi}{\left|\frac{\partial(y_1,\Phi)}{\partial(y_1,y_2)}\right|} \le C\, \frac{\lambda^{-1}}{|\Phi_{y_2}|} \varkappa,$$

since $\partial_{y_1} \det \Phi_{x_i y_j} \ne 0$ (because of the Whitney fold, or $\boldsymbol{k} = 1$). The application of the Schur lemma finishes the proof.

Note that since we only used the localization in Φ_{y_2}, we end up with the first power of $\ln \lambda$ in the estimate. □

Now we choose $\varkappa > 0$ such that the estimates from Lemmas 5.2 and 5.3 would match. This produces the estimate (1.9) and finishes the proof of Theorem 1.2.

6. Smooth Kernels

Estimates on the level set operators with smooth kernels, $a(x,y) \in C^\infty_{comp}(\mathbb{R}^n \times \mathbb{R}^n)$, could be deduced from the estimates on oscillatory integral operators, as we state in the following observation.

Lemma 6.1. *Let* $a \in C^\infty_{comp}(\mathbb{R}^n \times \mathbb{R}^n)$, $\Phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, *and assume that the oscillatory integral operator*

$$T_\lambda u(x) = \int_{\mathbb{R}^n} a(x,y)\, e^{i\lambda\Phi(x,y)}\, u(y)\, dy \qquad (6.1)$$

is bounded by $\|T_\lambda\|_{L^2\to L^2} \le C(1+|\lambda|)^{-\nu}$.

Next, assume that $\psi \in C^\infty(\mathbb{R})$ *has* $M \geq 0$ *zero moments, so that* $\hat{\psi}^{(k-1)}(0) = 0$ *for* $1 \leq k \leq M$ *(*$M = 0$ *means there are no zero moments;* $M = 1$ *means that* $\hat{\psi}(0) = \int_{\mathbb{R}} \psi(z)\, dz = 0$*, etc.) Then the level set operator*

$$\mathfrak{L}_\lambda u(x) = \int_{\mathbb{R}^n} a(x,y)\beta(\lambda\Phi(x,y))u(y)dy, \tag{6.2}$$

where $\psi \in C^\infty(\mathbb{R})$*, is bounded by*

$$\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \leq \begin{cases} C\,\lambda^{-\nu}, & 0 \leq \nu < M+1, \\ C\,\lambda^{-(M+1)} \ln\lambda, & \nu = M+1, \\ C\,\lambda^{-(M+1)}, & \nu > M+1. \end{cases}$$

Proof. One can rewrite (6.2) as

$$\mathfrak{L}_\lambda u(x) = \int_{\mathbb{R}} \hat{\psi}(\rho) \left[\int_{\mathbb{R}} e^{i\lambda\rho\Phi(x,y)}\, a(x,y)\, u(y)\, dy \right] d\rho = \int_{\mathbb{R}} \hat{\psi}(\rho)\, T_{\lambda\rho}\, u(x)\, d\rho. \tag{6.3}$$

Using an "old device" (e.g. see p.3 [CCW99]),

$$\|\mathfrak{L}_\lambda u(x)\|_{L^2\to L^2} < \int_{\mathbb{R}} |\hat{\psi}(\rho)| \cdot \|T_{\lambda\rho}u(x)\|_{L^2}\, d\rho < C\, \|u\|_{L^2} \int_{\mathbb{R}} \frac{|\hat{\psi}(\rho)|}{(1+|\lambda\rho|)^\nu}\, d\rho, \tag{6.4}$$

hence,

$$\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \leq C \int_{\mathbb{R}} \frac{|\hat{\psi}(\rho)|}{(1+|\rho\lambda|)^\nu} d\rho. \tag{6.5}$$

If β has at least M vanishing moments, so that $\hat{\psi}^{(k)}(0) = 0$, $0 \leq k \leq M-1$, then there exists $\epsilon > 0$ so that $|\hat{\psi}(\rho)| \leq C\,|\rho|^M$ as long as $|\rho| \leq \epsilon$.

The integral in (6.4) is bounded as follows

$$\|\mathfrak{L}_\lambda\|_{L^2\to L^2} \leq \int_{\mathbb{R}} \frac{|\hat{\psi}(\rho)|}{(1+|\lambda\rho|)^\nu}\, d\rho \leq \int_0^\epsilon \frac{C\,|\rho|^M}{(1+|\lambda\rho|)^\nu}\, d\rho + \frac{1}{(\epsilon\lambda)^\nu} \int_\epsilon^\infty |\hat{\psi}(\rho)|\, d\rho. \tag{6.6}$$

Since $\hat{\psi} \in \mathcal{S}(\mathbb{R})$, the second term in the right-hand side of (6.6) is bounded by $C\,\lambda^{-\nu}$. Our main concern is the first term. If $\nu < M+1$, we change the variable of integration in the first integral in the right-hand side of (6.6) to $z = \lambda\rho$

$$\int_0^\epsilon \frac{C\,|\rho|^M}{(1+|\lambda\rho|)^\nu}\, d\rho \leq \int_0^\epsilon \frac{C\,|\rho|^M}{|\lambda\rho|^\nu}\, d\rho \leq C\,\lambda^{-\nu}.$$

If $\nu \geq M+1$, the first term can be bounded as follows

$$\int_0^\epsilon \frac{C\,|\rho|^M}{(1+|\lambda\rho|)^\nu}\, d\rho \leq \lambda^{-(M+1)} \int_0^{\epsilon\lambda} \frac{C\,z^M}{(1+z)^\nu}\, dz. \tag{6.7}$$

If $\nu > M+1$, the integral in the right-hand side is bounded uniformly in λ; if $\nu = M+1$, the integral is bounded by $C \ln\lambda$. □

In conclusion we would like to point out that in some cases the estimates on level set integral operators enjoy better decay than the oscillatory integral operators.

Example 6.2. Consider the following example. Let

$$\Phi(x,y) = x^2 + y^2, \qquad x, y \in \mathbb{R}^n.$$

Then $T_\lambda u(x) = \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} a(x,y) u(y)\,dy$ has no decay on λ (since T_λ can be represented as $m(e^{i\lambda x^2}) \circ \tilde{T} \circ m(e^{i\lambda x^2})$, with $m(e^{ix^2})$ being multiplications by a unitary factor, and $\tilde{T}$ does not depend on λ). However, using the Schur lemma, one immediately proves that the $L^2 \to L^2$ operator norm on the level set operator $\mathfrak{L}_\lambda u(x) = \int_{\mathbb{R}^n} \psi(\lambda\,\Phi(x,y)) a(x,y) u(y)\,dy$ decays like $\lambda^{-n/2}$.

References

[CC00] Andrew Comech and Scipio Cuccagna, *Integral operators with two-sided cusp singularities*, Internat. Math. Res. Notices (2000), no. 23, 1225–1242.

[CCW99] Anthony Carbery, Michael Christ, and James Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc. **12** (1999), no. 4, 981–1015.

[Com97] Andrew Comech, *Oscillatory integral operators in scattering theory*, Comm. Partial Differential Equations **22** (1997), no. 5-6, 841–867.

[Com98a] ——, *Damping estimates for oscillatory integral operators with finite type singularities*, Asymptot. Anal. **18** (1998), no. 3-4, 263–278.

[Com98b] ——, *Sobolev estimates for the Radon transform of Melrose and Taylor*, Comm. Pure Appl. Math. **51** (1998), no. 5, 537–550.

[Com99] ——, *Optimal regularity of Fourier integral operators with one-sided folds*, Comm. Partial Differential Equations **24** (1999), no. 7-8, 1263–1281.

[Cuc97] Scipio Cuccagna, *L^2 estimates for averaging operators along curves with two-sided k-fold singularities*, Duke Math. J. **89** (1997), no. 2, 203–216.

[GG68] I. M. Gelfand and M. I. Graev, *Line complexes in the space C^n*, Funkcional. Anal. i Priložen. **2** (1968), no. 3, 39–52.

[GS94] Allan Greenleaf and Andreas Seeger, *Fourier integral operators with fold singularities*, J. Reine Angew. Math. **455** (1994), 35–56.

[GS98] ——, *Fourier integral operators with cusp singularities*, Amer. J. Math. **120** (1998), no. 5, 1077–1119.

[GS99] ——, *On oscillatory integral operators with folding canonical relations*, Studia Math. **132** (1999), no. 2, 125–139.

[GS02] ——, *Oscillatory and Fourier integral operators with degenerate canonical relations*, preprint (2002).

[GSW00] Allan Greenleaf, Andreas Seeger, and Stephen Wainger, *Estimates for generalized Radon transforms in three and four dimensions*, Analysis, geometry, number theory: the mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998), Amer. Math. Soc., Providence, RI, 2000, pp. 243–254.

[GU89] Allan Greenleaf and Gunther Uhlmann, *Nonlocal inversion formulas for the X-ray transform*, Duke Math. J. **58** (1989), no. 1, 205–240.

[Hör71] Lars Hörmander, *Fourier integral operators. I*, Acta Math. **127** (1971), no. 1-2, 79–183.

[MT85] Richard B. Melrose and Michael E. Taylor, *Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle*, Adv. in Math. **55** (1985), no. 3, 242–315.

[Pho95] D. H. Phong, *Singular integrals and Fourier integral operators*, Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), Princeton Univ. Press, Princeton, NJ, 1995, pp. 286–320.

[PS82] D. H. Phong and E. M. Stein, *Some further classes of pseudodifferential and singular-integral operators arising in boundary value problems. I. Composition of operators*, Amer. J. Math. **104** (1982), no. 1, 141–172.

[PS83] ——, *Singular integrals related to the Radon transform and boundary value problems*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), no. 24, Phys. Sci., 7697–7701.

[PS86a] ——, *Hilbert integrals, singular integrals, and Radon transforms. I*, Acta Math. **157** (1986), no. 1-2, 99–157.

[PS86b] ——, *Hilbert integrals, singular integrals, and Radon transforms. II*, Invent. Math. **86** (1986), no. 1, 75–113.

[PS89] ——, *Singular Radon transforms and oscillatory integrals*, Duke Math. J. **58** (1989), no. 2, 347–369.

[PS90] Yibiao Pan and Christopher D. Sogge, *Oscillatory integrals associated to folding canonical relations*, Colloq. Math. **60/61** (1990), no. 2, 413–419.

[PS91] D. H. Phong and E. M. Stein, *Radon transforms and torsion*, Internat. Math. Res. Notices (1991), no. 4, 49–60.

[PS92] ——, *Oscillatory integrals with polynomial phases*, Invent. Math. **110** (1992), no. 1, 39–62.

[PS94a] ——, *Models of degenerate Fourier integral operators and Radon transforms*, Ann. of Math. (2) **140** (1994), no. 3, 703–722.

[PS94b] ——, *On a stopping process for oscillatory integrals*, J. Geom. Anal. **4** (1994), no. 1, 105–120.

[PS94c] ——, *Operator versions of the van der Corput lemma and Fourier integral operators*, Math. Res. Lett. **1** (1994), no. 1, 27–33.

[PS97] ——, *The Newton polyhedron and oscillatory integral operators*, Acta Math. **179** (1997), no. 1, 105–152.

[PS98] ——, *Damped oscillatory integral operators with analytic phases*, Adv. Math. **134** (1998), no. 1, 146–177.

[PSS99] D. H. Phong, E. M. Stein, and J. A. Sturm, *On the growth and stability of real-analytic functions*, Amer. J. Math. **121** (1999), no. 3, 519–554.

[PSS01] D. H. Phong, E. M. Stein, and Jacob Sturm, *Multilinear level set operators, oscillatory integral operators, and Newton polyhedra*, Math. Ann. **319** (2001), no. 3, 573–596.

[See93] Andreas Seeger, *Degenerate Fourier integral operators in the plane*, Duke Math. J. **71** (1993), no. 3, 685–745.

[See98] ——, *Radon transforms and finite type conditions*, J. Amer. Math. Soc. **11** (1998), no. 4, 869–897.

[SSS91] Andreas Seeger, Christopher D. Sogge, and Elias M. Stein, *Regularity properties of Fourier integral operators*, Ann. of Math. (2) **134** (1991), no. 2, 231–251.

[Tat98] Daniel Tataru, *On the regularity of boundary traces for the wave equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 1, 185–206.

[Var76] A. N. Varčenko, *Newton polyhedra and estimates of oscillatory integrals*, Funkcional. Anal. i Priložen. **10** (1976), no. 3, 13–38.

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