E. Arthur (Robbie) Robinson
(Joint work with Ayse Sahin)

The George Washington University

Talk at KIAS, Seoul, Korea.

September 27, 2010
1. **Introduction**

2. **Finite rank, $\mathbb{Z}$ case**

3. **The formal definition**

4. **The $\mathbb{Z}^2$ case**

5. **Directional entropy**

6. **Directional entropy and rank 1**

7. **More...**

8. **Extras**
1. INTRODUCTION

2. Finite rank, $\mathbb{Z}$ case

3. The formal definition

4. The $\mathbb{Z}^2$ case

5. Directional entropy

6. Directional entropy and rank 1

7. More.

8. Extras
Cutting and Stacking

- **Elementary** method to construct examples in **ergodic theory**.
- Classical version: invertible Lebesgue measure preserving transformation $T : [0, 1) \to [0, 1)$.
- Equivalently, a measure preserving $\mathbb{Z}$ action (**MPZA**).
- Easily generalizes to $\mathbb{Z}^d$ or $\mathbb{R}^d$ to produce **MPZ$^d$A** or **MPR$^d$A**.
- More general than **substitutions**.
**Entropy**

- **Kolmogorov-Sinai, 1959**: entropy $h(T)$ of a measure preserving transformation $T$. Average “information” per time step.
- Straightforward generalization to $d$-dimensional entropy $h(T)$ of $\mathbb{MP\mathbb{Z}}^d A$ $T$.
- **Adler-Konheim-McAndrew, 1965**: Topological entropy $h_{\text{top}}(T)$ of continuous map (or $\mathbb{Z}^d$ action) $T$. Exponential growth in “complexity” $h(T) \leq h_{\text{top}}(T)$.
- **Milnor, 1986**: directional entropy $h_n(V, T)$ of $\mathbb{MP\mathbb{Z}}^d A$, $T$. Here $V \subseteq \mathbb{R}^d$ subspace, $\dim(V) = n$. 
1 Introduction

2 Finite rank, $\mathbb{Z}$ case

3 The formal definition

4 The $\mathbb{Z}^2$ case

5 Directional entropy

6 Directional entropy and rank 1

7 More...

8 Extras
Von Neuman’s “adding machine”

Step 0

0   1/2   1

Step 1

1/2   1

Step 3

1/2   1

Step n

...
Illustrated as block concatenation

$W_1 = 1$

$W_{n+1} = W_n W_n$

Picture shows base step and induction step, illustrating the combinatorial data needed for the construction:

$W_1 = 0, \quad W_{n+1} = W_n W_n.$

The tower is turned on its side, with individual levels blurred.
As $T : [0, 1) \rightarrow [0, 1)$
As Toeplitz sequence

Action together with partition equals process.
Here the combinatorial data is $W_1 = 0$ and $W_{n+1} = W_n W_n 1 W_n$. 
Chacon’s transformation

Step 0

Step 1

Step 2
Definition. $T$ is rank 1 if it can be constructed by cutting and stacking with one large tower in each step.

- Left over interval called a spacer.

Theorem

Rank 1 implies (uniquely) ergodic. (Also minimal if number of adjacent spacers is bounded.)

- Adding machine has discrete spectrum. Chacon’s transformation has continuous spectrum (i.e., is weakly mixing.)
- Any ergodic $T$ with discrete spectrum is rank 1 (e.g., irrational rotation transformation).
(Smorodinski)-Adams (1998) version (see also Ornstein (1968)).

Recurrence relation: $W_1 = 0$, $W_{n+1} = W_n 1 W_n^1 2 \ldots W_n^1 q_n$.

Mixing provided $q_n \uparrow \infty$ sufficiently fast.
The Morse dynamical system

\[ W_1^0 = 0 \]
\[ W_1^1 = 1 \]
Morse sequences

Step 0

Step 1

Step 2

Step 3
In this example, there are 2 towers at each step. We say $T$ has rank $\leq 2$.

- A. del Junco showed this $T$ is not rank 1. Thus $T$ is rank 2.
- The spectrum of $T$ is simple, and mixed (both discrete and continuous).
- Can similarly define rank $\leq r$, rank $r$, and finite rank.

**Theorem (see Queffelec, (1987/2010))**

A substitution on $r$ letters is rank $\leq r$. 
Rank, Spectrum and Entropy

Theorem (Baxter, 1971)

Finite rank implies \( h(T) = 0 \).

Proof.

- Rank \( n \) implies spectral multiplicity \( M_T \leq n \) (Chacon, 1970).
- Positive entropy (\( h(T) > 0 \)) implies \( M_T = +\infty \) (Bernoulli factor) (Sinai’s Theorem).
INTRODUCTION

FINITE RANK, $\mathbb{Z}$ case

THE FORMAL DEFINITION

THE $\mathbb{Z}^2$ case

DIRECTIONAL ENTROPY

DIRECTIONAL ENTROPY AND RANK 1

MORE . . .

EXTRAS
Rohlin Towers

- Let \( T : X \to X \) be a MPZA on a probability space \((X, \mathcal{B}, \mu)\).
- If \( B, TB, T^2 B, \ldots, T^{h-1} B \) are pairwise disjoint, we call it a **Rohlin tower** with height \( h \) and base \( B \).
- The error is \( E = \left( \bigcup_{k=0}^{h-1} T^k B \right)^c \).
- Call \( \xi = \{B, TB, \ldots, T^{h-1} B, E\} \) a Rohlin partition.

**Theorem (Rohlin’s Lemma)**

*If \( T \) is ergodic, then for any \( h \in \mathbb{N} \) and \( \epsilon > 0 \), there is a height \( h \) Rohlin tower with \( \mu(E) < \epsilon \).*
**Rank 1**

- Let $\xi_n$ be a sequence of partitions. Say $\xi_n$ separates ($\xi_n \to \varepsilon$) if for any $A \in \mathcal{B}$ there is $A_n \leq \xi_n$ so that $\mu(A \Delta A_n) \to 0$.

**Definition**

$T$ is rank 1 if there is a sequence $\xi_n$ of Rohlin towers so that $\xi_n \to \varepsilon$.

Cutting and stacking definition of Rank 1 implies this one: $\xi_n \to \varepsilon$ follows from $\text{diam}(B_n) \to 0$.

**Theorem (Baxter, 1971)**

$\xi_n$ may be chosen so that $\xi_n \leq \xi_{n+1}$ and $B_{n+1} \subseteq B_n$.

Thus all these $T$ may be obtained by cutting and stacking.
“Funny” Rank 1

- Call a finite $R \subseteq \mathbb{Z}$ a shape.
- Suppose $\mu(B) > 0$ and $T^k B \cap T^\ell B = \emptyset$ for all $k, \ell \in R$, $k \neq \ell$.
- Call $\xi = \{E, T^k B : k \in R\}$ a funny Rohlin tower.
  - In rank 1, $R = \{0, 1, \ldots, h - 1\}$.
- Define funny rank 1 analogously.

Shape matters! Rank 1 implies “loosely Bernoulli” (Katok, 1977, Ornstein-Rudolph-Weiss 1982), but funny rank 1 does not (Ferenczi, 1985).
1 INTRODUCTION

2 FINITE RANK, $\mathbb{Z}$ CASE

3 THE FORMAL DEFINITION

4 THE $\mathbb{Z}^2$ CASE

5 DIRECTIONAL ENTROPY

6 DIRECTIONAL ENTROPY AND RANK 1

7 MORE . . .

8 EXTRAS
Actions of $\mathbb{Z}^d$

- Let $(X, \mathcal{B}, \mu)$ be a probability space.
- Let $T_1, T_2 : X \to X$ be MP$\mathbb{Z}$As that commute: $T_1 T_2 = T_2 T_1$.
- For $n = (n_1, n_2) \in \mathbb{Z}^2$, define $\text{MP}\mathbb{Z}^2 A \ T^n = T_1^{n_1} T_2^{n_2}$.
- Similar definition for $\text{MP}\mathbb{Z}^d A$, (i.e., $T_1, T_2, \ldots, T_d$ commute).
- Call a finite $R \subseteq \mathbb{Z}^d$ a shape.

**Definition.** A shape-$R$ Rohlin tower consists of disjoint sets $T^n B$, $n \in R$. The partition $\xi = \{E, T^n B : n \in R\}$ is a Rohlin partition.
**Definition**

A MP \( \mathbb{Z}^d \) A \( T \) is **rank 1** if there is a sequence \( \xi_n \) of shape \( R_n \) Rohlin partitions so that \( \xi_n \rightarrow \varepsilon \).

**Proposition (R-Sahin, 2010)**

*Rank 1 (any shape) implies ergodic and simple spectrum.*

**Corollary**

*Rank 1 (any shape) implies \( h(T) = 0 \).*
**Definition**

Suppose $T$ is a MP\(\mathbb{Z}^dA\) there are shapes $R^j_n$ and positive measure sets $B^j_n$, for $j = 1, \ldots, r$ and $n \in \mathbb{N}$, so that

$$\xi_n = \{T^n B^j_n : n \in R^j_n, j = 1, \ldots, n\} \cup \{X \setminus \bigcup_{j=1}^n \bigcup_{n \in R^j_n} T^n B^j_n\}$$

is a partition, and $\xi_i \to \varepsilon$. We say $T$ is rank $\leq r$ for shapes $\{R^1_n, R^2_n, \ldots, R^j_n\}$.

Rank $r$ if rank $\leq r$ and not rank $\leq r - 1$.

**Proposition**

$\text{Rank } \leq r$ implies $M_T \leq r$ and $h(T) = 0$, but not necessarily ergodic.
A sequence $\mathcal{R} = \{R_k\}$ of shapes in $\mathbb{Z}^2$ is a Følner sequence (van Hove sequence) if for any $n \in \mathbb{Z}^2$

$$\lim_{k \to \infty} \frac{|R_k \triangle (R_k + n)|}{|R_k|} = 0,$$

- A natural choice is rectangles

$$R_k = [0, \ldots, w_k - 1] \times [0, \ldots, h_k - 1],$$

where $w_k, h_k \to \infty$. 

Følner sequences
Types of rank 1

- **Rank 1**: no shape restriction.
- **Følner rank 1**: $R_n$ a Følner sequence.

**Proposition (R-Sahin, 2010)**

If Følner, can get $\xi_n \leq \xi_{n+1}$ with the same $\mathcal{R} = \{R_n\}$.

- Cutting and stacking works!
- **Rectangular rank 1**: rectangles
- **Geometric restrictions** (on rectangular Rank 1):
  - Bounded eccentricity: $1/K \leq w_k/h_k \leq K$.
  - Subexponential eccentricity: $\log(w_k)/h_k \to 0$ ($w_k \geq h_k$).
The $\mathbb{Z}^2$ case

**Chacon $\mathbb{Z}^2$ actions**

\[
W_1 = 0 \\
W_{n+1} = W_n W_n W_n W_n W_n W_n W_n W_n
\]

Weak mixing, not strong mixing, & “MSJ” (R-Park, 1991).

**Note.** $w_n/h_n = 1$: “bounded” eccentricity.
Rudolph’s example

One of \((wn)(hn)\) blocks. \(N_n\) of these blocks in a row.
A block consisting of all possible 

\[ ((\Delta w_n)(\Delta h_n))^{N_n} \]

rows, in some particular order.

There are 

\[ \left( ((\Delta w_n)(\Delta h_n))^{N_n} \right)! \]

of these.
All \( \left( ((\Delta w_n)(\Delta h_n))^N \right)^n \)
blocks (every possible order) stacked.

\[ w_{n+1} = ((\Delta w_n)(\Delta h_n))^N \times (w_n + \Delta w_n). \]

\[ h_{n+1} = \left( \left( ((\Delta w_n)(\Delta h_n))^N \right)^n \times ((\Delta w_n)(\Delta h_n))^N \times (h_n + \Delta h_n). \right. \]
Properties of Rudolph’s example

- Requires appropriate choice of $\Delta w_n \to \infty$, $\Delta h_n \to \infty$ and $N_n \to \infty$.

- Side lengths
  
  $w_{n+1} = ((\Delta w_n)(\Delta h_n))^{N_n} (w_n + \Delta w_n)$, and
  
  $h_{n+1} = \left(\frac{((\Delta w_n)(\Delta h_n))^{N_n}!}{((\Delta w_n)(\Delta h_n))^{N_n} (h_n + \Delta h_n)}\right)$.

- Sides satisfy $\log(h_n)/w_n \to \infty$. Super exponential eccentricity.

**Theorem** (Rudolph, 1978)

*Horizontal $T_1$ is Bernoulli shift with arbitrary finite entropy $0 < h(T_1) < \infty$.***
1. Introduction

2. Finite rank, $\mathbb{Z}$ case

3. The formal definition

4. The $\mathbb{Z}^2$ case

5. Directional entropy

6. Directional entropy and rank 1

7. More . . .

8. Extras
Before defining directional entropy, we briefly review the ordinary (\(d\)-dimensional) entropy of a MP \(\mathbb{Z}^d\) A \(T\).

- Let \(\xi\) be a finite partition. The entropy of \(\xi\) is
  \[
  H(\xi) = - \sum_{A \in \xi} \mu(A) \log \mu(A).
  \]
- Define \(\xi_n = \bigvee_{n \in [0, \ldots, n]^d} T^{-n} \xi\)
- The \(\xi\)-entropy of \(T\) is
  \[
  h(T, \xi) = \lim_{n \to \infty} \frac{1}{n^d} H(\xi^n).
  \]
- The entropy of \(T\) is given by
  \[
  h(T) = \sup_\xi h(T, \xi).
  \]

This gives usual entropy of transformation \(T\) when \(d = 1\).
Subspace $V \subseteq \mathbb{R}^d$, $n = \dim(V) < d$.

$Q \subseteq V$, $Q' \subseteq V^\perp$ unit cubes, and $S(V, t, m) = (tQ + mQ')$ (we call it a window.)
Directional entropy (Milnor, 1986)

Let $T$ be a MPZ$^d$A, with $\xi$ a finite partition, and dim$(V) = n$.

- $\xi_{V,t,m} := \bigvee_{n \in S(V,t,m)} T^{-n}\xi$.

- $h_n(T, V, \xi, m) := \limsup_{t \to \infty} \frac{1}{tn} H(\xi_{V,t,m})$.

- $h_n(T, V, \xi) := \sup_{m > 0} h_n(T, V, \xi, m)$

**Definition (Milnor, 1986)**

If $1 \leq n < d$, $n$-dimensional directional entropy in direction $V$ is

$$h_n(T, V) = \sup_{\xi} h_n(T, \xi, V).$$

If $n = d$, then $h_d(T, V) = h(T)$, (where $V = \mathbb{R}^d$).
**Directional entropy (\(\mathbb{Z}^2\) case)**

- \(h_1(V, T) < \infty\) for some \(V\), implies \(h_2(T) = 0\).
  - Ledrappier's \(\mathbb{Z}^2\) shift \(T\) has \(h_1(T, V) > 0\) for all \(V\).
  - K. Park (unpublished, c 1987) Chacon MP\(\mathbb{Z}^2\)A \(T\) has \(h_1(T, V) = 0\) for all \(V\).
- \(h_1(T, V) = ||(p, q)||^{-1}h(T^{q,p}), V = (p, q)\mathbb{R}, p/q \in \mathbb{Q}\).
  - Rudolph rank 1 \(\mathbb{Z}^2\) has \(h_1(V, T) > 0\) where \(V = e_1\mathbb{R}\).
- (K. Park, 1999) If \(V = v\mathbb{R}, ||v|| = 1\), then \(h_1(T, V) = h(F^{tv})\) for the unit \(\mathbb{R}^2\) suspension \(F^t\) of \(T\).
- (K. Park, 1999) The function \(h(v) = h(T, v\mathbb{R}), ||v|| = 1\), is upper semicontinuous, and \(\{v : h(v) = 0\}\) is \(G_\delta\).
1. **Introduction**

2. **Finite rank, $\mathbb{Z}$ case**

3. **The formal definition**

4. **The $\mathbb{Z}^2$ case**

5. **Directional entropy**

6. **Directional entropy and rank 1**

7. **More...**

8. **Extras**
The first result has no assumptions beyond rectangular rank 1.

**Theorem 1. (R-Sahin, 2010)**

Let $T$ be a rectangular rank-1 $MP\mathbb{Z}^d A$. Then there is a 1-dimensional subspace $V \subseteq \mathbb{R}^d$ so that $h_1(T, V) = 0$.

With addition hypotheses on the eccentricity, we can say more.

**Theorem 2. (R-Sahin, 2010)**

Let $T$ be a rectangular rank-1 $MP\mathbb{Z}^d A$ with subexponential eccentricity. If $V \subseteq \mathbb{R}^d$ is an $n$-dimensional subspace, $1 \leq n \leq d$, then $h_n(T, V) = 0$. 
Two lammas

**Lemma (Milnor, 1988)**

The formulas that define directional entropy simplify to

\[
h_n(T, V, \xi, m) = \lim_{t \to \infty} \frac{1}{t^n} H(\xi_{V,t,m}), \quad \text{and} \]

\[
h_n(T, V, \xi) = \lim_{m \to \infty} h_n(T, V, \xi, m).
\]

**Theorem (Boyle-Lind, 1997)**

If \( \xi_k \leq \xi_{k+1} \) and \( \xi_k \to \epsilon \) then

\[
h_n(T, V) = \lim_{k \to \infty} h_n(T, V, \xi_k).
\]
**Lemma**

Suppose $\xi_k \leq \xi_{k+1}$ and $\xi_k \to \varepsilon$. If $t_j \to \infty$, and

$$\lim_{j \to \infty} \frac{1}{(t_j)^n} H((\xi_k)_{V,t_j,m}) = 0,$$

for all $k$ and all $m > 0$, then $h_n(T, V) = 0$.

We will use this lemma in the proofs of both theorems.
Proofs (set-up)

We do the case $d = 2$.

Let $V \subseteq \mathbb{R}^2$ be a 1-dimensional subspace (to be specified later for Theorem 1), and let $\xi_k$ be a sequence of shape-$R_k$ Rohlin towers for $T$.

Assume WOLOG:

1. $\xi_k \leq \xi_{k+1}$ (Baxter’s Theorem),
2. $R_k$ is $w_k \times h_k$ where $h_k \leq w_k$ for all $k$.

Note. There are no eccentricity assumptions in Theorem 1.

Let $t_j \rightarrow \infty$ be a slowly increasing sequence, to be specified later.

Ultimate Goal. For fixed $m, k$, show that $H((\xi_k)_{V,t_j,m})/t_j \rightarrow 0$. 
Proofs (labels)

- Let $j > k$.
- Call a level $T_n B_j$ in $\xi_j$ **good** if $S(V, t_j, m) \subseteq R_j - n$.
- Let $G_j \subseteq \mathbb{Z}^2$ be the set of good levels.
- Let $F_j = (\bigcup_{n \in G_j} T_n B_j)^c$.
- And, recall $E_j = (\bigcup_{n \in R_j} T_n B_j)^c$. 
Proofs (Good Levels)
Proofs (New partitions)

- \( \xi_j^* := \{ T^n B_j : n \in G_j \} \cup \{ F_j \} \).
- \( \eta_j := (\xi_k)_{T,t_j,m} \lor \xi_j^* \).
- Note that \( (\xi_k)_{T,t_j,m} \leq \eta_j \).
- Thus \( H((\xi_k)_{T,t_j,m}) \leq H(\eta_j) \).
- So it suffices to show \( H(\eta_j)/t_j \to 0 \).
- (This will achieve our **Ultimate Goal**.)
**Proofs (Relations among partitions)**

**Key observation:** Each of the sets $T^n B_j$ for $n \in G_j$ belong to the partition $\eta_j$.

“Goodness” insures the partition $(\xi_k)_{V,t_j,m}$ is “constant” on levels $T^n B_j$, for $n \in G_j$. In other words, each $T^n B_j$ is a subset of some $A \in (\xi_k)_{V,t_j,m}$.

$$H(\eta_j)/t_j = -\frac{1}{t_j} \sum_{A \in \eta_j} \mu(A) \log \mu(A)$$

$$= -\frac{1}{t_j} \left( \sum_{n \in G_j} \mu(T^n B_j) \log \mu(T^n B_j) + \sum_{A \in \eta'_j} \mu(A) \log \mu(A) \right)$$

$$= -\frac{1}{t_j} \left( |G_j| \mu(B_j) \log \mu(B_j) - \sum_{A \in \eta'_j} \mu(A) \log \mu(A) \right).$$
Proofs (Left term Goal)

\[- \frac{1}{t_j} |G_j| \mu(B_j) \log \mu(B_j) \leq - \frac{1}{t_j} |R_j| \mu(B_j) \log \mu(B_j) \]

\[= - \left( \frac{w_j h_j}{t_j} \right) \left( \frac{1 - \epsilon_j}{w_j h_j} \right) \log \left( \frac{1 - \epsilon_j}{w_j h_j} \right) \]

\[\leq \frac{\log(w_j h_j) - \log(1 - \epsilon_j)}{t_j}, \]

where \(\epsilon_j = \mu(E_j)\).

Left Term Goal. Show \(\log(w_j h_j)/t_j \to 0\). (Insubstantial entropy from (uniformly covered) good set)
Local entropy lemma

**Theorem (Shields, 1996)**

Suppose $\xi$ is a partition, $\xi' \subseteq \xi$ and $\beta = \mu(\bigcup_{A \in \xi'} A)$. Then

$$- \sum_{A \in \xi'} \mu(A) \log \mu(A) \leq \beta \log |\xi'| - \beta \log \beta.$$
Proofs (Right Term)

- \(|\xi'_j| \leq (|R_k| + 1)|S(V, t_j, m)|.\)
- \(\log |\xi'_j| = |S(V, t_j, m)| \log (|R_k| + 1) \leq 2|S(V, t_j, m)| \log |R_k|.\)
- \(|S(V, t_j, m)| \leq 2t_jm.\)
- \(\log |R_k| = K.\)

Thus

\[\log |\xi'_j| \leq 2Kt_jm.\]
Proofs (Right Term Goal)

Also

\[ \beta = \mu(F_j) = |B_j \setminus G_j| \mu(B_j) + \mu(E_j) \leq \frac{|B_j \setminus G_j|}{w_j h_j} + \epsilon_j. \]

So by the local entropy lemma

\[ -\frac{1}{t_j} \sum_{A \in \xi'} \mu(A) \log \mu(A) \leq 2Km \left( \frac{|B_j \setminus G_j|}{w_j h_j} + \epsilon_j \right) - \frac{\beta \log \beta}{t_j}. \]

(\(t_j/t_j\) cancels in the first term). Since \(\beta < 1\), \((\beta \log \beta)/t_j \to 0\).

Right Term Goal. \(\frac{|B_j \setminus G_j|}{w_j h_j} \to 0\). (This is essentially that measure of bad part, \(\beta \to 0\).)
Proof of Theorem 1 (Left Term Goal)

- Assume $w_j \geq h_j$ for all $j$.
- Take $V = e_1 \mathbb{R}$.
- We want $t_j \to \infty$ so that $\frac{\log(w_j)}{t_j} \to 0$ and $\frac{t_j}{w_j} \to 0$.

Define $t_j = \sqrt{w_j \log w_j}$.

\[
\frac{\log(w_j h_j)}{t_j} \leq \frac{2 \log(w_j)}{t_j} \to 0.
\]

Left Term Goal Achieved.
Proof of Theorem 1 (Right Term Goal)

We have, \( |R_j \setminus G_j| \leq h_j t_j + mw_j \).

Since \( \frac{t_j}{w_j} = \sqrt{\frac{w_j \log w_j}{w_j}} = \sqrt{\frac{\log w_j}{w_j}} \to 0 \), Right Term Goal Achieved.
Proof of Theorem 2 (Left Term Goal)

- Take $V \subseteq \mathbb{R}^2$, $\dim(V) = 1$.
- Assume $w_j \geq h_j$ and define $t_j = \sqrt{h_j \log w_j}$.

$$\frac{\log w_j}{t_j} = \frac{\log w_j}{\sqrt{w_j \log(w_j)}} = \sqrt{\frac{\log w_j}{w_j}} \to 0$$

$$\frac{t_j}{h_j} = \sqrt{\frac{h_j \log w_j}{h_j \log w_j}} = \sqrt{\frac{\log w_j}{h_j}} \to 0$$

(by subexponential eccentricity).

$$\frac{\log(w_j h_j)}{t_j} \leq \frac{2 \log(w_j)}{t_j} \to 0.$$  

Left Term Goal achieved.
Proof of Theorem 2 (Right Term Goal)

We have, \( |R_j \setminus G_j| \leq h_j (t_j + m) \cos \theta + w_j (t_j + m) \sin \theta \).

\[
\frac{|R_j \setminus G_j|}{w_j h_j} = \frac{t_j + m}{w_j} \cos \theta + \frac{t_j + m}{h_j} \sin \theta \to 0,
\]

since \( \frac{t_j}{h_j} \to 0 \), (and \( \frac{t_j}{w_j}, \frac{m}{h_j}, \frac{m}{w_j} \to 0 \).) Right Term Goal achieved.
1 INTRODUCTION

2 FINITE RANK, $\mathbb{Z}$ CASE

3 THE FORMAL DEFINITION

4 THE $\mathbb{Z}^2$ CASE

5 DIRECTIONAL ENTROPY

6 DIRECTIONAL ENTROPY AND RANK 1

7 MORE...
**Rank** $r$

Here is what we can prove in rank $r$. For simplicity, we discuss only the case $T$ is an ergodic rectangular rank $\leq 2 \text{MP} \mathbb{Z}^2 A$. Let $R_n^1$ be $w_n^1 \times h_n^1$ and $R_n^2$ be $w_n^2 \times h_n^2$.

**Theorem A.** If $w_n^1 \geq h_n^1$ and $w_n^2 \geq h_n^2$ for infinitely many $n$ then there exists $V$ so that $h_1(T, V) = 0$ (i.e., $h(T_1) = 0$).

**Theorem B.** Under the same hypotheses as above, if $\log(w_n^1)/h_n^1 \to 0$, and $\log(w_n^2)/h_n^2 \to 0$, then $h_1(T, V) = 0$ for all 1-dimensional $V$.

**Theorem C.** If $w_n^1 \geq h_n^1$ and $w_n^2 \leq h_n^2$ for all $n$, and $\log(w_n^1)/h_n^1 \to 0$, and $\log(h_n^2)/w_n^2 \to 0$, then $h_1(T, V) = 0$ for all 1-dimensional $V$. 
As mentioned before, a substitution on \( r \) letters has rank \( \leq r \). This is also true for a substitution tiling with \( r \) distinct prototiles. The eccentricity is bounded. This implies a substitution tiling system has all directional entropies zero.

Another way to prove this is to note that the complexity of a substitution tiling satisfies \( c(n) \leq Kn^e \) (where \( e = d \) in the self similar case).

A. Julien (2009) proved \( c(n) \leq Kn^e \) for a cut and project tiling where the acceptance domain is polyhedral and “almost canonical”. This implies all directional entropies zero.

More generally a model set with a topologically and measure theoretically regular acceptance domain has discrete spectrum, so is rank 1. This implies all directional entropies zero.
Other examples

- Ledrappier’s shift has $c(n) = Ke^{2n}$ (exponential complexity in smaller dimension). It has positive directional entropy in every direction.
- Radin showed that any uniquely ergodic $\mathbb{Z}^2$ SFT has $c(n) \leq Ke^{\ell n}$. Can it have positive directional entropy.
- Not for the examples that come from substitutions and model sets!
Loosely Bernoulli

Say $\text{MP} \mathbb{Z}^d A \ T$ with $h_d(T) = 0$ is entropy zero loosely Bernoulli (LB) if a suspension of $T$ (to a $\text{MP} \mathbb{R}^d A$) can be time changed to a suspension of some $R$ discrete spectrum (action by rotations on a compact group).

Theorem (Johnson-Sahin, 1998)

A rectangular rank 1 $\text{MP} \mathbb{Z}^2 A \ T$ with bounded eccentricity is loosely Bernoulli.

- This $T$ can be chosen to have $T_1$ be non LB.
- Johnson-Sahin (1998) prove that the same result holds for rank $r > 1$ provided towers have uniformly bounded eccentricity.
Loosely Bernoulli

**Theorem (R-Sahin 2011?)**

If $T$ is a loosely Bernoulli MP $\mathbb{Z}^d A$ with $h_d(T) = 0$ then $h_n(T, V) = 0$ for all $V$.

Implications:
- Ledrappier’s shift is not loosely LB (a “folk theorem”).
- Rudolph’s rank 1 is not LB.
INTRODUCTION

FINITE RANK, $\mathbb{Z}$ CASE

THE FORMAL DEFINITION

THE $\mathbb{Z}^2$ CASE

DIRECTIONAL ENTROPY

DIRECTIONAL ENTROPY AND RANK 1

MORE . . .

EXTRAS
**Z^d Rohlin lemma**

- Say the Rohlin lemma holds for a shape \( R \subseteq \mathbb{Z}^d \) if for any ergodic \( \mathbb{Z}^d \) action \( T \), and \( \epsilon > 0 \), there exists \( B \in \mathcal{B} \) so that \( X \) is partitioned by \( \xi = \{ E, T^nB : \mathbf{n} \in R \} \) and \( \mu(\bigcup_{n \in R} T^nB) > 1 - \epsilon \).

- A shape \( R \) tiles \( \mathbb{Z}^d \) if there exists \( C \subseteq \mathbb{Z}^d \) so that \( \{ T^nR : \mathbf{n} \in C \} \) is a partition of \( \mathbb{Z}^n \).

**Theorem (Ornstein-Weiss, 1980)**

A Rohlin lemma holds for a shape \( R \) if and only if \( R \) tiles \( \mathbb{Z}^d \).