Mathematical Quasi-Crystallography: The First 30 Years
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I. Quasicrystals discovered.

II. Crystallographic restriction.

III. X-ray diffraction.

IV. Penrose tilings.

V. Three paradigms.
   ✤ Matching rules.
   ✤ Inflation.
   ✤ Projection.

VI. Tiling spaces & dynamics.

VII. Spectral measures.

VIII. Algebraic topology.

Figure 1.1. A fivefold symmetric patch of the rhombic Penrose tiling with matching rule decorations by two types of arrows.

This constitutes a set of local rules with three remarkable properties. The first is that the rules are compatible with at least one (in fact, more than one) gapless, face to face tiling that covers the entire plane. Secondly, they guarantee that none of these tilings has any (non-zero) period. Finally, they also enforce that any two of these tilings are locally indistinguishable. They provide an example of what we will later call a set of perfect aperiodic local rules. There are many other fascinating aspects of the Penrose tiling (such as its self-similarity, which underlies the proof of the claims, and its description as a projection from 4-space), some of which will be explained in this book.

TODO: mention Dürer

Thursday, March 21, 13
I. Quasicrystals Discovered

- Dan Schectman, 1982.
- Materials Scientist visiting NIST (then called NBS) from Technion in Israel.
- Looking at X-ray diffraction patterns of Al-Mn alloys.
- Saw an “impossible” image.
• Spots or “Bragg Peaks” indicate a crystal: atomic long range spatial order (periodic).

• 5-fold (or 10-fold or icosahedral) symmetry, reflects the underlying symmetry of the atoms.

• But! “Crystallographic Restriction” forbids 5-fold symmetry for crystals.
LONG RANGE “APERIODIC” ORDER


- But long-range order must be strong enough to still result in Bragg peaks in diffraction pattern.

- Long-range order must be weak enough that 5-fold symmetry does not violate the Crystallographic Restriction.
Quasicrystals


  Coined the term “quasicrystals”. Suggested a model based on Penrose tilings (i.e., a 3-dimensional version). The same model simultaneously proposed by several others. But many remained skeptical...


  “Twinning” means two ordinary crystals fused together at angle.
Quasicrystals now have widespread acceptance.

*International Union of Crystallography* new 1992 definition of a crystal: a solid with a “discrete diffraction diagram”. (Formerly, this said “periodic atomic structure”).

“Discrete diffraction” is meant to be taken as evidence of “long range order”.

Dan Schectman won the 2011 Nobel Prize in Chemistry.
II. Crystallographic Restriction

A (classical) **crystal** is a “pattern” \( x \) in \( \mathbb{R}^n \) that is **periodic**: i.e., \( T(x) = \{ v \in \mathbb{R}^n : x + v = x \} \) is a full-rank **lattice**.

Periodicity is “long range order”.

The **space group** \( S(x) \) is the group of rigid motions (i.e., isometries of \( \mathbb{R}^n \)) leaving \( x \) invariant. (Symmetry type.)

The **point group**, defined \( H(x) = S(x)/T(x) \), is isomorphic to a finite subgroup of \( O(n) \).
THE CASE $n=2$

- When $n = 2$, there are 17 different space groups, sometimes called **wallpaper groups**. A “folk theorem”: All types occur in Islamic art, and all were illustrated by M. C. Escher.

- $H(X)$ can either be $\mathbb{Z}_n$ or $\mathbb{D}_n$ for $n = 1, 2, 3, 4, \text{ or } 6$. This is the crystallographic restriction for $n = 2$.

![p3m1](Owen Jones 1856)  
![p4g](Escher)  
![p6](Alhambra)
THE CASE $n=3$

- When $n = 3$, there are 230 different space groups. This result is due to Fyodorov and Schönflies (1892).

- There are 14 possible space groups $T(X)$, called Bravais lattices.

- There are 32 possible point groups $H(X)$. No point group is the icosahedral group.
Bieberbach’s Theorem

- Hilbert’s 18th problem from a list of 23, set out in 1900 by David Hilbert (Part 1): Is the number of $n$-dimensional space groups is finite for each $n$?

- **Theorem** (Bieberbach, 1911). The number of space groups in every dimension is finite. Every isomorphism is implemented “geometrically”.

Parts 2 & 3 of Hilbert’s 18th were to prove $S(x)$ is transitive on fundamental domains, and to prove Kepler’s conjecture. Part 2 is False (Reinhardt, $n=3$, 1928 & Heesch, $n=2$, 1935. Part 3 was proved by Hales in 1998.
Let $x \subseteq \mathbb{R}^n$ be discrete and relatively dense (a Delone set).

Let $\Delta_x = \sum_{v \in x} \delta_v$

Autocorrelate: $\Delta_x \ast \Delta_x = \rho \Delta_{x - x}$ & positive definite.

So $\Delta_x \ast \Delta_x = \hat{\sigma}$ where $\sigma$ is a measure on $\mathbb{R}^n$.

Lebesgue decomposition: $\sigma = \sigma_{\text{dis}} + \sigma_{\text{sc}} + \sigma_{\text{ac}}$.

Say $x$ diffracts if $\sigma_{\text{dis}} \neq 0$. 
A (discrete) classical crystal $x$ is a finite union of lattices (cosets). For simplicity, assume just one.

Let $v_1, v_2, \ldots, v_n$, be a basis for $x$, and let $x^*$ be dual lattice (generated by a dual basis: $v_1^*, v_2^*, \ldots, v_n^*$ such that $\langle v_i, v_j^* \rangle = \delta_{ij}$).

For a lattice $x - x = x$ so $\Delta x \otimes \Delta x = \Delta x - x = \Delta x$.

Then $(\Delta x \otimes \Delta x)^\wedge = \Delta x^\wedge = \Delta x^*$ by the Poisson summation formula: \textit{classical crystals diffract}.

What else diffracts?
Penrose tilings


- The “Penrose Pattern” refers to the aperiodic tilings, discovered around 1976 by Roger Penrose, now called *Penrose tilings*.

- Mackay essentially showed that (the vertices of) *Penrose tilings diffract*.

- He did this *experimentally*, by making a tiny Penrose tiling and shining a laser through it!
Roger Penrose is a British mathematical physicist, mostly working on cosmology, but also interested in recreational math.

Shown at Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M

http://en.wikipedia.org/wiki/Penrose_tiling
ORIGIN IN PENTAGONAL SYMMETRY

- Plane can be tiled by equilateral triangles, squares & (regular) hexagons, but not (regular) pentagons.

- **Penrose** (1976) was experimenting with efficiently filling the plane with pentagons.
Various different Penrose tilings

- Kites & darts
- Penrose chickens
- Rhombs
- Robinson triangles

$x = \phi$, the golden ratio!

Thus we get an irrational relative density of kites to darts – which is impossible for a periodic tiling. ($T_h$ is the numerical density. The kite has $\phi$ times the area of the dart, so the total area covered by kites is $\phi^2$ ($= 1 + \phi$) times that covered by darts.)

Jigsaws and beyond

Here is another pair of quadrilaterals which, with suitable matching rules, tiles the plane only non-periodically: a pair of rhombuses as shown in Figure 13. A suitable shading is suggested in Figure 14, where similarly shaded edges are to be matched against each other. In Figure 15, the hierarchical relation to the kites and darts is illustrated.

The rhombuses appear mid-way between one kite-dart level and the next in the related kite-dart level. Many different jigsaw puzzle versions of the kite-dart pair or the rhombus pair can evidently be given. One suggestion for modified kites and darts, in the shape of two birds, is illustrated in Figure 16. Other modifications are also possible, such as alternative matching rules, suggested by Robert Ammann (see Figure 17) which force half the tiles to be inverted.

Many intriguing features of these tilings have not been mentioned here, such as the pentagonally-symmetric rings that the stripes of Figure 14 produce, Conway’s classification of “holes” in kite-dart patterns (i.e. regions surrounded by “legal” tilings but which cannot themselves be legally filled), Ammann’s three-dimensional version of the rhombuses (four solids that apparently fill space only non-periodically), Ammann’s and Conway’s analysis of “empires” (the infinite system of partly disconnected tiles whose positions are forced by a given set of tiles). It is not known whether there is a single shape that can tile the Euclidean plane non-periodically. For the hyperbolic (Lobachevski) plane a single shape can be provided which, in a certain sense, tiles only non-periodically (see Figure 18) – but in another sense a periodicity (in one direction only) can occur. (This remark is partly based on suggestions of John Moussouris.)

References


Magic constant in a third order magic square. Triangular, hexagonal, pentatope and Bell number. Open meandric number. These are the number of meanders in non-self-intersecting oriented curves.
“Rhombs” version of Penrose tiling “matching rule” and by “inflation”.

Aperiodic, not a crystal, but repetitive, long-range order. Has 5-fold “quasisymmetry”.

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A tiling set is **repetitive** if for every patch that occurs in any tiling, there exists $L$ so that a copy of the patch occurs within $L$ of any location. (Patch occurrences relatively dense.).

A tiling set is **uniformly repetitive** if patches occur with uniform positive frequency.

Repetitivity is type of “long range order”. 
**Penrose Tiles as Art**

GWU Math Department mural. Saxe-Patterson, Taos, NM, 1998

Tony Robbin 1994 “Coast” Danish Technical University (destroyed 2003).
Kepler’s “Penrose” tilings

From Kepler’s Harmonices Mundi, 1619.

Called the fused decagons “monsters”.

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Islamic “Penrose” Tilings


Spandrel tiling with girih patterns (left), and with Kites & Darts Penrose tiling (below).

Peter J. Lu under tile spandrels, Darb-i Imam shrine, Isfahan, Iran, built 1453 (560 years).
3-DIMENSIONAL PENROSE TILING

Deitl & Eschenberg, Made with “Zometool”

http://www.math.uni-augsburg.de/~eschenbu/
V. THREE PARADIGMS

- Matching rule.
- Inflation.
- Projection.
**Matching Rule**

*Subshift of finite type* (SFT) \( d = 1 \):

- **Example**: Golden mean shift: \( X \subseteq \{0,1\}^\mathbb{Z} \): tilings
  \( x = \ldots 0100.101\ldots \) of \( \mathbb{Z} \) with **11 FORBIDDEN**, \( T \) shift map.

- \( \# w = 01010\ldots010 \) of length \( n \) is \( f_n \). Entropy \( h(T) = \log \left( \frac{1 + \sqrt{5}}{2} \right) \)

- *Has periodic* \( x = \ldots 0010.01001\ldots \) and **aperiodic** points.

- 1-dim **Wang tiles**:

- \( x = \ldots \)
**Wang Tiles**

**Wang tilings** = 2-dim subshift of finite type:

- **Example:** Golden mean shift: $X \subseteq \{0,1\}^Z$ with 11 & $\frac{1}{1}$ FORBIDDEN.

<table>
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<tr>
<td>0001000010...</td>
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<tr>
<td>1000011100</td>
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- $x = \ldots 0001000010 \ldots 1000011100$

- Has both *periodic* and *aperiodic* points.

- Mixing, $h(T) > 0$ but $h(T) = ?$ (open).

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$h(T) = \lim h_n$, where $h_n$ recursive rational.
Wang’s Conjecture

- **Conjecture** (Hao Wang, 1961) Every finite Wang tile set, $d=2$, that admits a *valid* tiling also admits a *valid periodic tiling*. (The tiling problem is decidable).

- True for $d = 1$ (1-dim. SFT). Often true $d = 2$, e.g. 2-dim Golden mean.


- Current record 13 tiles (K-C tiles), Kari-Culick, 1996.
Tiling patches by KC-Tiles
Penrose tiles as rhombic aperiodic Wang tiles, i.e., \( \mathbb{R}^2 \text{ SFT.} \)

Matching rules don’t necessarily explain aperiodicity but might explain how quasicrystals grow. (Though probably not!)
Morse substitution (Morse sequence): M. Morse, 1921; A. Thue, 1906; E. Pruhet, 1851.

- **Substitution** \( \sigma: 0 \rightarrow 01, \ 0 \rightarrow 01 \)
- **Iteration:** \( 0 \rightarrow 01 \rightarrow 0110 \rightarrow 01101001 \rightarrow 0110100110010110 \)
- 1-sided limit (fixed point): \( \sigma^\infty(0) = 1001011010010110... \)
- 2-sided fixed point:
  \( x = \sigma^\infty(1)^R . \sigma^\infty(0) = ...10011001.0110100110... \)

**Substitution dynamical system:** shift \( T \) on \( X = \text{cl}\{T^n(x) : n \in \mathbb{Z}\} \).

- **Properties:** aperiodic, hierarchical, repetitive (minimal), uniquely ergodic, *infinite* type, mixed spectrum, \( h=0 \).
Dynamical spectrum

- Transformation $T$ on $X$ (e.g., substitution shift):

- **Eigenvalue-eigenfunction** $f(Tx) = \lambda f(x)$

- Spectral measure $\hat{\gamma}_f(n) = \langle f(T^n x), f(x) \rangle$ (positive definite). Choose $f$ so $\gamma_f$ is maximal.

- Lebesgue decomposition: $\gamma_f = \delta_0 + \gamma_{\text{dis}} + \gamma_{\text{sc}} + \gamma_{\text{ac}}$.

- For an action of $\mathbb{R}^d$ one has $\hat{\gamma}_f(v) = \langle f(T^v x), f(x) \rangle$

- Can detect discrete spectrum, mixed spectrum, weak mixing, etc.
**Other well known Substitutions**

- **Morse**: $0 \rightarrow 01$, $1 \rightarrow 10$ “bijective”, constant length.
- **Period doubling**: $0 \rightarrow 01$, $1 \rightarrow 00$ constant length.
  
  $0 \rightarrow 01 \rightarrow 01 00 \rightarrow 0100 1010 \rightarrow \ldots \rightarrow 0100010100010001\ldots$  

- **Fibonacci**: $0 \rightarrow 01$, $1 \rightarrow 0$  
  
  $0 \rightarrow 01 \rightarrow 01 0 \rightarrow 010 01 \rightarrow 01001 010 \rightarrow 01001010 01001\ldots$  

- **Dekking-Keane**: $0 \rightarrow 0111$, $1 \rightarrow 0$ (weakly mixing).  
  
  $0 \rightarrow 0111 \rightarrow 0111 000 \rightarrow \ldots \rightarrow 0111000011101110111011100001\ldots$  

- **Rudin-Shapiro**: $0 \rightarrow 01$, $1 \rightarrow 02$, $2 \rightarrow 31$, $0 \rightarrow 32$ (Lebesgue spectrum).  

- **Cantor**: $0 \rightarrow 010$, $1 \rightarrow 111$ not primitive.  
  
  $0 \rightarrow 010 \rightarrow 010 111 010 \rightarrow 10111010 11111111 010111010 \ldots$
Morse tiling

2-dimensional Morse substitution: inflation.
**Inflation** on Penrose tiles

\[ e = \left( \frac{1 + \sqrt{5}}{2} \right)^2 \]

Inflation explains aperiodicity, repetitivity, patch frequencies. Also explains 5-fold (quasi-)symmetry. Might explain **diffraction**.
Perfecting the substitution:

\[ e = \frac{5 + \sqrt{5}}{2} \]

\[ d_H = \frac{\log(2)}{\log(2 \cos \frac{\pi}{10})} = \frac{2 \log(2)}{\log\left(\frac{5+\sqrt{5}}{2}\right)} \approx 1.078 \]
Chair tiling

Figure 6.15. Inflation rule for the chair tiling (upper left), and a fourfold symmetric patch of the tiling (right), obtained by three inflation steps from the legal configuration shown in the lower left corner.

It does not possess aperiodic local rules. Breaking the remaining reflection symmetry is thus the essential step to obtain a class with local rules.

So far, all examples had an irrational inflation multiplier, which immediately implied aperiodicity by Theorem 6.2. Let us now turn our attention to tilings that are aperiodic for a different reason.

6.4. Planar tilings with integer inflation multiplier

An interesting and widely used planar tiling with a single prototile and an integral inflation multiplier is the chair tiling [GS87, Sol97, BMS98, Rob99]. The L-shaped prototile is scaled by a factor of 2 and then dissected into four congruent copies, as shown in Figure 6.15. The finite patch shown in the same figure converges (under iterated inflation) to a plane-filling tiling with global $D_4$ symmetry.

This inflation rule is a geometric reformulation of the block substitution introduced in Eq. (4.25) on page 114. The identification is usually done [BMS98, Rob99] by means of four oriented squares as follows...
Only once (i.e., to allow rotations of the prototiles). With an example of an einstein was discovered by Schmitt and Conway; in the general case it is called a similarity; in the general case it is called a
word. There is no einstein up to translation.

(4.1) \( \text{supp}(C) \) in Figure 4(a) and its four rotations. The set \( C \) is called a perfect dec
omposition. We can get around this difficulty by decomposing it into four rotations, two of which are self-similar, the other two are self-affine. In all cases shown in Figure 4,
other polyomino examples are easily devised.

Example

4.1. Perfect decompositions.

The mapping \( \lambda \) for \( L \) in Figure 4(b), is a perfect decomposition. To get around this difficulty, we can use a magic folding table.

Example

4.2 (Raphael Robinson’s triangular Penrose tilings)

Alternate (with coincidence)

Self affine, not self similar.

Alternate (with coincidence)
Zoo of inflation tilings

Ammann-Beenker (Octagonal)

R. Ammann 1977
Beenker 1982
**Zoo of inflation tilings**

"Mixing": Tyler White (March 2012 Dissertation)

Topological mixing

(Case: $q=3, r=1$)
For substitution tilings, local finiteness follows from "two-tile closure property" [6]. For a fairly general class of two-dimensional tiling substitutions, such that no tiling admitted by the substitution is locally finite. We also give a simple example.

\[
p(x) = x^2 - x - 3
\]

\[
\beta = (1 + \sqrt{13})/2 \approx 2.3027756377
\]
Zoo of Inflation Tilings - 3

Pinwheel tiling
Conway/Radin
1994

Rotational symmetry T.
Not finite local complexity (FLC).

\[ \tan(\theta) = \frac{1}{2} \]
**Theorem.** (S. Mozes, 1989) Any $\mathbb{Z}^2$ substitution can be “realized” as a Wang tiling (2-dim SFT).

**Theorem.** (C. Radin, 1994 C. Goodman-Strauss 1998) Any substitution tiling can be “realized” as a tiling with matching rules.

**Comment 1:** Nothing like this can happen in $d=1$!

**Comment 2:** “Realized” means some tiles may need to be replaced by multiple remarked copies. So the result really says a substitution is sofic.
MATCHING RULE FOR CHAIR

Sturmian sequence: Morse & Hedlund, 1938

- Fix $0 \leq \alpha < 1$ irrational, $0 \leq \beta < 1$.
- **Sturmian seq.** $x_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor \in X \subseteq \{0,1\}^\mathbb{Z}$
  
  *(mechanical line, cutting sequence, Beatty sequence).*

  $x = \ldots 010010100.100101001010\ldots$

- Approximates line $y = \alpha x + \beta$.

- **Properties:** aperiodic, repetitive (minimal), uniquely ergodic, infinite type, discrete spectrum, $h=0$.

- Only sometimes hierarchical, (always weakly hierarchical).
Mechanical Lines

\[ S \subseteq \mathbb{Z}^2 \subseteq \mathbb{R}^2 \]

\[ x = \ldots 01001.010010100100\ldots \]

Fibonacci substitution: \(0 \rightarrow 01, 1 \rightarrow 0\)

de Bruijn 1981: this is a 1-dimensional Penrose tiling.
**Theorem.** (de Bruijn, 1981) Penrose tiles come from projecting a hypercubic slice of the lattice $\mathbb{Z}^5$ in $\mathbb{R}^5$ to a symmetric irrational affine plane $L+v$, $L \subseteq \mathbb{R}^2$.

The set of allowed shifts $v$ has codimension-4, so there is actually a 1-parameter family of tilings, only one of which is true Penrose tilings. Each can be obtained as a projection of $\mathbb{Z}^5$ in $\mathbb{R}^5$.

In particular, Penrose tilings are parameterized by $T^4$ up to a “few” parameter values corresponding to more than one tiling.
The geometry of de Bruijn’s idea

Projection explains diffraction (see next topic) and (quasi-)symmetry. But maybe not all quasicrystals come from projection.
Rauzy Substitution

\[0 \rightarrow 01, \ 1 \rightarrow 02, 2 \rightarrow 0\]

\[u = 010201001020010102010010210120101...\]

\[A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\]

\[z^3 = z^2 + z + 1\]

\[\dim(E^s) = 2.\]

Proj. \(P_u\) gives Rauzy quasicrystal

\[\lambda = \frac{1}{3} \left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right) \approx 1.83929\]

\[|\lambda'| = |a \pm ib| < 1\]

Projective limit \(T^3\)

Gerard Rauzy, 1982
Rauzy Substitution

$1 \rightarrow 12, \ 2 \rightarrow 13, \ 3 \rightarrow 1$

$u = 1213121121312121312112131213121...$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
Pisot conjecture

- **Theorem.** (Dekking, 1977) A substitution of constant length has discrete spectrum if and only if it has a coincidence:
  - $0 \rightarrow 01$
  - $1 \rightarrow 10$
  - Morse

- **Pisot conjecture.** (Arnoux & Ito, 2001): An irreducible Pisot substitution has discrete spectrum. (Such a substitution automatically satisfies a generalization of Dekking’s coincidence.)
Pisot Conjecture -2

- **Example.** Fibonacci $0 \rightarrow 01$, $1 \rightarrow 0$ has **incidence matrix**

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$  

$$p_A(\lambda) = \lambda^2 - \lambda - 1. \quad \lambda = \frac{1 \pm \sqrt{5}}{2} \approx 1.61, -0.61$$

- **Pisot number.** Real algebraic integer $> 1$ with all conjugates $< 1$ in modulus. **Irreducible** means $p_A(x)$ irreducible.

- Pisot conjecture is **True** for substitutions on two symbols (Hollander & Solomyak 2003).

- Holds for Rauzy substitution $0 \rightarrow 01$, $1 \rightarrow 02$, $2 \rightarrow 0$ as well.

- Essentially says when a substitution is actually a projection.

We will see later that discrete spectrum is related to diffraction.
Therefore the intersection points lying in a region bounded by thin grid lines (see figure 3(a)) give rise to rhombuses that are arranged periodically. Note, however, that 5-fold symmetry is destroyed by this procedure. An example is shown in figure 3(b).

Figure 2. Pattern with 12-fold bond orientational symmetry. Its Fourier spectrum consists of a 12-fold symmetric arrangement of 8 peaks filling reciprocal space densely.

Figure 3. (a) Example of a grid yielding periodic inclusions. The areas bounded by the thin grid lines generally contain a lot of intersection points of the same type, leading to adjacent tiles of the same type. (b) Tiling with periodic inclusions. The tiles are the Penrose rhombuses.

12-fold quasisymmetry

No rotational symmetry
ZOO OF PROJECTION TILINGS

Projection

"Oblique" tiling

Dual to grid

Oguey, Duneau & Katz, Comm Math Phys 118 1998
Rudolph tiles of $\mathbb{R}^2$.

**Tiling topology:** $x, y$ close if, after small translation, agree on large ball around 0.

**Lemma:** (Rudolph, 1988). Set $X$ of all locally finite tilings is compact metric and translation action $T$ is continuous.
**Theorem.** (RR, 1996) Penrose tiling is almost 1:1 extension of a rotation of $T^4$. Spectrum $\mathbb{Z}[\xi]$, $\xi^5 - 1 = 0$. Inflation map realized as hyperbolic toral automorphism (tiles are unstable manifolds intersect Markov partition).

**Theorem.** (RR, 1999) Chair tiling is almost 1:1 extension of a rotation on $(\mathbb{Z}_2)^2$ ($\mathbb{Z}_2 = 2$-adic integers = “adding machine”). Spectrum $\mathbb{Z}[1/2]^2$.

Hyperbolic toral automorphism & Markov partition $\Rightarrow$ tiling. But what do partition elements look like?

- **Connected?** Yes in “Pisot” case: Furukado 2005, otherwise open
**Model Sets-1 (algebraic part)**

Cut and project scheme:

\[ \mathbb{R}^d = \text{“physical” space} \]

\[ H = \text{“internal” space} \]

\[ L \subseteq \mathbb{R}^d \oplus H = \text{lattice} \]

**Kronecker action:** \( \mathbb{R}^d \)

\[ G = (\mathbb{R}^d \oplus H)/L \text{ compact} \]

Strictly ergodic, discrete spectrum.

Can get essentially any countable subgroup of \( \mathbb{R}^d \) as spectrum RR, 2004

**Model Sets-2 (Geometric part)**

**Definition:**

\[ W \subseteq H \] window

**Regular:**
\[ W = \overline{W^\circ} \]
\[ \mu_H(\partial W) = 0 \]

**Examples:**
- Period doubling
- Fibonacci
- Chair tilings
- Penrose tilings
- Octagonal tilings

**Theorem.** The model set shift \( T \) is a strictly ergodic almost 1:1 extension of \( \mathbb{R} \) with dynamical & diffraction spectrum \( \Sigma \). Any rotational “quaissymmetry” can be achieved.

**An uncountable class of examples, mostly not substitutions and no matching rules (but conditions in some cases given by T. Le (1997), Julien (2010)).**
Mixing Properties

- **Weak mixing** means $\Sigma = \{0\}$ (no discrete spectrum). False for model sets and projection tilings. Sometimes for substitutions and matching rules.

- **Mixing** means $\gamma_f(v) \to 0$ as $|v| \to \infty$. False for model/projections and FLC substitutions. Unknown in other cases.

- **Topological mixing**. False for model/projection tilings. Sometimes for substitutions.

- **Mixed spectrum** (discrete & continuous or Lebesgue). Lebesgue by itself?

  Should these count as quasicrystals?
Weak mixing

- Substitution tilings (self-similar) with expansion $\lambda \in \mathbb{C}$.

- Say algebraic integer $\lambda$ is complex-Pisot if $|\lambda| > 1$, and all conjugates $\theta \neq \lambda, \bar{\lambda}$ satisfy $|\theta| < 1$.

- Theorem. (Solomyak, 1997) $T$ is weakly mixing iff $\lambda$ is complex-Pisot.

- Complete characterization of discrete spectrum still open (the Pisot Conjecture in dimension $d$).

- Basically, this would say that all substitutions with discrete spectrum are model sets.
Visual comparison of discrete spectrum & weak mixing.

Binary: weakly mixing.  

Penrose: discrete spectrum.
**Entropy & Complexity**

**Complexity:** $C_n = \#n \times n$ patches, up to small translation.

**Entropy:** $h = \lim_{n \to \infty} \frac{\log C_n}{n^d}$.

- For substitution tiling with expansion $L$,

  $C_n \sim n^\alpha$ for $\alpha = \frac{\log |\det L|}{\log |\lambda_{\min}|}$ (Hansen, RR 2004).

- For aperiodic model sets in $\mathbb{R}^d$ with $H=\mathbb{R}^e$ and an “almost canonical” window: $C_n \sim n^\alpha$ for $d \leq \alpha \leq de$ (Julien, 2010).

In both cases $h = 0$. 
1-dimensional SFTs, if not finite sets, have $h>0$.

Tiling spaces $X$ with matching rules can have $h=0$ or $h>0$. Even if $X$ is aperiodic, $h>0$ is possible.

If $X$ is a uniquely ergodic tiling space $C_n \leq Ke^{\beta(d-1)}$.

Again $h = o$. 
# Tiling dynamical systems as symbolic dynamics

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<td>Periodic orbit</td>
<td>Location independence</td>
<td>Mixing</td>
</tr>
</tbody>
</table>
VII. Spectral Measures

- $\mathbb{R}^d$ action $T$ on tiling space $X$.
- Eigenvalue-eigenfunction $f(T^v x) = e^{v \cdot \omega} f(x)$.
- Spectral measure $\gamma_f(v) = \langle f \circ T^v, f \rangle$ (positive definite). Choose $f$ so $\gamma_f$ is maximal. Dynamical spectrum.
- Lebesgue decomposition: $\gamma_f = \delta_0 + \gamma_{\text{dis}} + \gamma_{\text{sc}} + \gamma_{\text{ac}}$
- How is this related to diffraction spectrum $\sigma$?

\textbf{Theorem.} (S. Dworkin, 1993) $\sigma \ll \gamma_f$. Diffraction is only possible if $T$ has nontrivial discrete spectrum.
### Some Examples

<table>
<thead>
<tr>
<th>Tiling</th>
<th>$d$</th>
<th>$Discrete$</th>
<th>$Singular$</th>
<th>$Lebesgue$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period doubling</td>
<td>1</td>
<td>$\mathbb{Z}[1/2]$</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Morse</td>
<td>1</td>
<td>$\mathbb{Z}[1/2]$</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Rudin-Shapiro</td>
<td>1</td>
<td>$\mathbb{Z}[1/2]$</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Fibonacci</td>
<td>1</td>
<td>$\mathbb{Z}[\lambda] \cong \mathbb{Z}^2$</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Penrose</td>
<td>2</td>
<td>$\mathbb{Z}[\xi_5] \cong \mathbb{Z}^5$</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Binary</td>
<td>2</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Chair</td>
<td>2</td>
<td>$\mathbb{Z}[1/2]^2$</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Table</td>
<td>2</td>
<td>$\mathbb{Z}[1/2]^2$</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Pinwheel</td>
<td>2</td>
<td>No</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
FIBONACCI (1-DIM. PENROSE)

Numerical work from, Dissertation of Joe Herning, 2013

O→01, 1→0
**Period Doubling**

0 → 01, 1 → 00
Morse Sequence

Diffraction spectrum: only singular.

Strictly increasing $g(t)$ with $g'(t)=0$.

$0 \rightarrow 01$, $1 \rightarrow 10$
Morse Sequence & Period Doubling

Morse: \[0 \rightarrow 01, \ 1 \rightarrow 10\]

\[u = 01101001100101101001011001101001\ldots\]

2:1 factor map: \[v_i = -(u_i + u_{i+1}) \mod 1\]

\[v = 01000101010001000100010101000101\ldots\]

Period doubling: \[0 \rightarrow 01, \ 1 \rightarrow 00\]
Mixing

\[ 0 \rightarrow 0111, \; 1 \rightarrow 0 \]
Rudin-Shapiro

Lebesgue diffraction spectrum.
**Conjecture.** (Michael Baake, 2010) Every substitution has a substitution factor isomorphic to its discrete spectrum part.

**Theorem.** (Joe Herning, 2013 Dissertation) The substitution $0 \rightarrow 00120$, $1 \rightarrow 21011$, $2 \rightarrow 12202$ has no discrete spectrum substitution factor.

**Note.** The discrete part of the spectrum is $\mathbb{Z}[1/5]$. It cannot be seen in any factors.
Visible Points

Claim (4) about the density is a standard example of the use of Möbius inversion, see [Apo76, Thm. 3.9], where the average is taken over squares. It is not difficult to see that the average over centred disks gives the same result; see [BMP00, Appendix] for details.

Remark 10.6 (Holes in $V$).

Square-shaped holes of increasing size are exponentially rare, but repeat lattice periodically. The $2 \times 2$ holes closest to the origin are those with lower left corner $(20, 14)$ and its symmetry related counterparts, eight altogether. Since the primes involved in the congruences are 2, 3, 5 and 7, each of these eight holes repeats periodically, with lattice of periods $210 \mathbb{Z}_2$.

A $3 \times 3$ hole can be found at $(1308, 1274)$, with lattice of periods $39270 \mathbb{Z}_2$, or at $(3794, 1000)$, with periods $30030 \mathbb{Z}_2$. A $4 \times 4$ example begins at the corner $(13458288, 13449225)$.

Moreover, one explicitly sees that the differentiation image is fully translation invariant, with $\mathbb{Z}_2$ as lattice of periods, in accordance with Proposition 10.3. Finally, as $GL(2, \mathbb{Z})$-transformations do not change the denominator of a rational point, we inherit complete $GL(2, \mathbb{Z})$-invariance of the differentiation measure from that of the set $V$ itself. Our summary reads as follows.

Corollary 10.3. The differentiation measure $\gamma$ of the visible lattice points $V$ of $\mathbb{Z}_2$ is a pure point measure with positive intensities precisely on the rational points with squarefree denominator. Moreover, the measure $\gamma$ has the symmetry group $\mathbb{Z}_2 \rtimes GL(2, \mathbb{Z})$.

The above results, with minor modifications, also hold for visible points in arbitrary planar lattices, and similarly for lattices in higher dimensions; Baake & Grimm, 2013.
Figure 11.7. Typical patch of a rhombus (or lozenge) random tiling, with periodic boundary conditions. The frequencies of the three orientations differ (the vertical rhombus is less frequent than the other two types), so that this example breaks (statistical) three- or sixfold symmetry.

The corresponding ensemble is in one-to-one correspondence to the ensemble of fully packed dimer configurations on the hexagonal or honeycomb packing (which is the periodic repetition of a regular hexagon). This ensemble is known to have positive entropy density, compare [Hen99, RHHB98], with the maximal contribution from the realisations with sixfold (and hence maximal) symmetry, the latter to be understood in the statistical (or ensemble) sense.

The dimer model on the honeycomb packing is an exactly solved and much studied model of statistical mechanics; see [Kas63, DG72, Ken97, H öf01] for background and applications in our context. The ensemble of all configurations is equivalent to the random tiling ensemble built from rhombuses with opening angle $\pi/3$ (lozenges) in three orientations. The ensemble is equipped with a unique Gibbs measure, which is parametrised by the frequencies of the tiles in the three orientations; compare [Ken97] for an alternative formulation. This measure is thus an ergodic measure, and its correlations are rather well understood. With similar methods as used

Figure 11.8. Dissection image of the lozenge random tiling of Figure 11.7. The point measures are represented by the three types of large spots, while the smaller spots represent the absolutely continuous background, the latter once again calculated numerically by fast Fourier transform of the periodic approximant. A fundamental domain of $\Gamma$ is indicated (shaded). The (lack of) symmetry corresponds to that of the tiling.

Note: The discrete spectrum comes from the underlying triangular lattice (a classical crystal).
A tiling space (of FLC tilings) is a \textit{lamination}: neighborhoods are homeomorphic to $\mathbb{R}^d \times C$. The space is a $d$-dimensional, connected compact metric space.

The Čech cohomology $\check{H}^*(X)$ of $X$ is well defined. FLC tiling spaces are all inverse limits of CW-complexes (easy to see in case of inflation), and are all $T^d$ fibered spaces (Sadun and Williams, 2005).

Cohomology first computed by Anderson & Putnam, Gahler

Non FLC. Cohomology not yet well understood (but see Frank & Sadun)
### Čech Cohomology

<table>
<thead>
<tr>
<th>Tiling</th>
<th>$d$</th>
<th>$\check{H}^0(X)$</th>
<th>$\check{H}^1(X)$</th>
<th>$\check{H}^2(X)$</th>
<th>$\check{H}^3(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Morse</td>
<td>1</td>
<td>$\mathbb{Z}^1$</td>
<td>$\mathbb{Z}[1/2] \oplus \mathbb{Z}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Fibonacci</td>
<td>1</td>
<td>$\mathbb{Z}^1$</td>
<td>$\mathbb{Z}^2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Penrose</td>
<td>2</td>
<td>$\mathbb{Z}^1$</td>
<td>$\mathbb{Z}^5$</td>
<td>$\mathbb{Z}^8$</td>
<td>0</td>
</tr>
<tr>
<td>Octagonal (remarked)</td>
<td>2</td>
<td>$\mathbb{Z}^1$</td>
<td>$\mathbb{Z}^8$</td>
<td>$\mathbb{Z}^{23}$</td>
<td>0</td>
</tr>
<tr>
<td>Chair</td>
<td>2</td>
<td>$\mathbb{Z}^1$</td>
<td>$\mathbb{Z}[1/2]^2$</td>
<td>$\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2$</td>
<td>0</td>
</tr>
</tbody>
</table>
Gap Labeling

- $\hat{H}^i(X)$ is related to point spectrum (Andress & RR 2012).

- $\hat{H}^i(X)$ is related to $K_0(C(X)\rtimes_T R^d)$. This has a (unique) trace $\tau$ (unique ergodicity).

- The group $\Gamma=\tau(K_0(C(X)\rtimes_T R^d)) \subseteq R$ is called the gap labeling group. It is related to Schrödinger equation with $x$ as potential.

- Define frequency module $\Phi$ the group generated by the patch frequencies.

Gap Labeling theorem. $\Gamma=\Phi$. (Conjectured Bellisard 1992, proved independently by three groups 2004).
Schrödinger Equations

Schrödinger operator: self-adjoint

\[(Hu)_n := u_{n+1} + u_{n-1} - v_n u_n\]

\[u_n \in \ell^2(\mathbb{Z}) \quad Hu = \lambda u\]

For what \(\lambda \in \mathbb{R}\) is there solution?

More generally study the spectrum \(\Sigma\) of \(H\).

In this case, \(\Sigma \subseteq \mathbb{R}\) is a Cantor set, and the lengths of the gaps (removed intervals) come from the frequency module \(\Phi\).

**Note:** Previously, only discrete spectrum and AC spectrum were observed.

(Bellisard, Damanik, Simon).
SUMMARY

Substitution
Mixing
Table

Matching rule
Mixing (remarked)
Table (remarked)

Kari/Culic?
Chair (remarked)
Penrose
Octagonal (remarked)

Special-1 (remarked?)
Special-2 (remarked?)

Model Set
Chair
Octagonal

Diffract
Generic
Visible points
Random

Special-1=Special-2?

Thursday, March 21, 13
Dan Schectman in Stockholm 2011
APPENDIX

- Parts 2 & 3 of Hilbert’s 18th problem.
- Algorithm for 2-dimensional space groups.
- Penrose’s derivation of Penrose tilings
- Architectural quasicrystals
- Quasicrystals in industry & nature
Hilbert’s 18th, Part 2:

No. Reinhardt 1928 ($n=3$), Heesch ($n=2$) 1935

Heesch: $S(X)$ action not transitive on tiles.

Ceiling of Gottingen City Hall.

**Hilbert’s 18th, Part 3.**

Kepler conjecture,
Johannes Kepler, 1611.

$$D = \frac{\pi}{18} \approx 0.74048$$

Proved by Thomas Hales 1998.

http://en.wikipedia.org/wiki/Kepler_conjecture
Brian Sanderson's Pattern Recognition Algorithm

**Origin in pentagonal symmetry**

- **Penrose:** even though plane cannot be tiled by pentagons, how close can you get?

![Diagram of Penrose tilings](image-url)
"Pentaplexy" version of Penrose tilings

Four different prototiles.
Triangular, hexagonal, pentatope and Bell number.

Open meandric number. These are the number of

\[ f_i \]

...
“**Kites & Darts**” to “**Rhombic**”

MLD to kites & darts.

Matching rule (decorated two ways)
Penrose tilings as architecture

Kites & Darts kitchen.

Rhombs bathroom

http://www.liefies.com/?p=366

More architecture

Federation Square
Melbourne, Australia
Putting Quasicrystals to Work

At a conference held August 19-23 in Ames, Iowa, “New Horizons in Quasicrystals: Research and Applications,” Jan-Olof Nilsson, manager of physical metallurgy for the R&D Centre of Sandvic Steel, Sandviken, Sweden, said: “Three years ago I thought quasicrystals were something that existed only in the academic world, but I have been forced to change my mind. As a matter of fact, we can’t avoid quasicrystalline precipitates in our commercial materials, and furthermore, we can make use of them.”

In the past several years many people have changed their minds about quasicrystals, the new form of matter discovered in 1982. (Quasicrystals differ from glasses in that their atoms are arranged in orderly patterns, and from crystals in that these patterns do not repeat periodically; see Figure 3, page 29.) At first quasicrystals were considered, for example, are extremely poor electrical and thermal conductors. The thermal conductivity of quasicrystals containing more than 70 atomic percent aluminum is two orders of magnitude below that of aluminum and roughly equivalent to that of zirconia, which is used as a refractory material. Quasicrystals are also exceptionally hard, and their surfaces have very low coefficients of friction, good wear resistance, and good oxidation and corrosion resistance. Depending on how they are prepared, quasicrystals can have coefficients of friction so low they are comparable to the coefficient of a diamond gliding over a diamond.

At first, there was an apparent obstacle to quasicrystal applications. Dan Shechtman of Technion-Israel Institute of Technology (Haifa, Israel), who discovered quasicrystals, remarked in passing that 1890—an aluminum–lithium alloy, strengthened by the application of heat. This alloy responds well to tempering, or aging, because there were some early “stealth” applications. Dan Shechtman of Technion-Israel Institute of Technology (Haifa, Israel), who discovered quasicrystals, remarked in passing that 1890—an aluminum–lithium alloy, developed by Alcan Aluminium Limitée in Montreal, that is now one of three standard aluminum–lithium alloys used in the aerospace industry—sometimes contains qua-

Figure 1. This aluminum–copper–iron quasicrystal, shown in a secondary electron microscope image, belongs to the icosahedral family of quasicrystals, and has 12 pentagonal faces.

Figure 2. The first bona fide application of quasicrystals is a steel alloy produced by Sandvic Steel that is made into medical instruments. These must be easy to form, yet very strong, to avoid breakage during use.

[Conference proceedings will be published by World Scientific Publishing, which can be reached by telephone (800–227–7562) or on the Web (http://www.wspc.com).] It was chaired by Alan Goldman, Daniel Sordelet, and Patricia Thiel of the Department of Energy’s Ames Laboratory, in Ames, Iowa, and by Jean-Marie Dubois of the École des Mines in Nancy, France.

Pinpointing the first instance of the commercial use of quasicrystals is a bit tricky, because there were some early “stealth” applications. Dan Shechtman of Technion-Israel Institute of Technology (Haifa, Israel), who discovered quasicrystals, remarked in passing that 1890—an aluminum–lithium alloy, developed by Alcan Aluminium Limitée in Montreal, that is now one of three standard aluminum–lithium alloys used in the aerospace industry—sometimes contains qua-
Naturally occurring quasicrystal


Luca Bindi, Museum of Natural History, Florence, Italy. Found quasicrystal in meteor from Koryak Mountains in Russia.