The Dynamical Theory of Tilings
and Quasicrystallography

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ABSTRACT

A tiling \( x \) of \( \mathbb{R}^n \) is \textit{almost periodic} if a copy of any patch in \( x \) occurs within a fixed distance from an arbitrary location in \( x \). Periodic tilings are almost periodic, but aperiodic almost periodic tilings also exist; for example, the well known Penrose tilings have this property. This paper develops a generalized symmetry theory for almost periodic tilings which reduces in the periodic case to the classical theory of symmetry types. This approach to classification is based on a \textit{dynamical theory of tilings}, which can be viewed as a continuous and multidimensional generalization of symbolic dynamics.

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1. Introduction

The purpose of this paper is to describe a natural generalization of the standard theory of symmetry types for periodic tilings to a larger class of tilings called almost periodic tilings. In particular, a tiling $x$ of $\mathbb{R}^n$ is called almost periodic if a copy of any patch which occurs in $x$ re-occurs within a bounded distance from an arbitrary location in $x$. Periodic tilings are clearly almost periodic since any patch occurs periodically, but there are also many aperiodic examples of almost periodic tilings—the most famous being the Penrose tilings, discovered in around 1974 by R. Penrose [18].

Ordinary symmetry theory is based on the notion of a symmetry group—the group of all rigid motions leaving an object invariant. The symmetry groups of periodic tilings are characterized by the fact that they contains a lattice of translations as a subgroup. In contrast, for aperiodic tilings the symmetry group contains no translations, and it is typically empty. Thus a generalization of symmetry theory to almost periodic tilings must be based on different considerations. In this paper we describe a generalization of symmetry theory that uses ideas from dynamical systems theory, applied ‘tiling dynamical systems’. The simplest example of a tiling dynamical system consist of a translation invariant set of tilings, equipped with a compact metric topology (analogous to the product topology), with $\mathbb{R}^n$ acting on it by translation. In the periodic case, this shift is a transitive action. More generally, almost periodic tilings generate minimal shifts. It turns out that minimality makes it possible to develop a ‘symmetry theory’ for almost periodic tilings (2) (called the theory of quasisymmetry types) which is closely analogous to the classical theory of symmetry types for periodic tilings (see [28]). In this paper, we establish a formalism for this theory and describe its relation to symmetry theory. We also discuss some of the differences between the periodic and almost periodic cases. In particular, it turns out that there is no crystallographic restriction theorem (see [28]) for almost periodic tilings.

Part of the interest in a symmetry theory for almost periodic tilings comes from their connection with the theory of quasicrystals, a new form of solid matter discovered in 1985 by D. Schectman et al [27]. Although there is little agreement on the precise definition of a quasicrystal, roughly speaking a quasicrystal is a solid which unlike a crystal, is not made of a periodic array of atoms, but nevertheless has enough spatial order to produce sharp Bragg peaks in its diffraction pattern (see [13]). In particular, quasicrystals can have rotational ‘symmetries’ which are forbidden for ordinary crystals.

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2This idea was inspired by G. Mackey’s theory of virtual groups, [15].
by the crystallographic restriction. For example, the Schectman quasicrystal has a diffraction pattern with five-fold rotational symmetry (reminiscent of a Penrose tiling). The possibility of an ‘almost periodic crystal’ based on a 3-dimensional Penrose-like tiling was first suggested by A. Mackay [14] several years before Schectman’s discovery. Following Schectman’s discovery, almost periodic tiling models for quasicrystals quickly became popular (see for example [13], [12]), although such models have always been somewhat controversial. We avoid this controversy here, noting only that a good symmetry theory for almost periodic tilings represents an important first step in any reasonable symmetry theory for ‘not quite periodic’ structures.

Tiling dynamical systems are interesting as a subject in themselves. We view them as symbolic dynamical systems which are multidimensional and have continuous ‘time’ (i.e., $\mathbb{R}^n$ acts instead of $\mathbb{Z}^n$). In particular, they represent a promising new source of examples for dynamical systems theory. There are already several nontrivial applications of tilings to dynamical problems (see [25], [26], [16], [11]) and the concept of a tiling dynamical system provides a uniform foundation for all of these. In the first part of this paper we set up the basic framework for a topological theory of tiling dynamical systems (we briefly describe some connections to ergodic theory in later sections). However, it is not the intention of this paper to develop the general theory of tiling dynamical systems. Rather, our goal here is to exploit dynamical theory as a tool for classifying almost periodic tilings.

This paper is structured as follows: In Section 2 we set up a framework for tiling dynamical systems, and also define quasisymmetry groups, a kind of group which plays a role similar to the symmetry group in classical symmetry theory. In Section 3 we study the algebraic properties of quasisymmetry groups, proving that they satisfy an analogue of Bieberbach’s second theorem. We define almost periodic tilings in Section 4, and we discuss their connection with minimal dynamical systems. In Section 5 we show how to associate a quasisymmetry group and a ‘point group’ to an almost periodic tiling, and show that in the periodic case, the quasicrystallographic point group is the same as the classical point group. However, as we show in Section 6, quasisymmetry groups alone do not provide a very strong symmetry theory. To compensate for this we define quasisymmetry types in Section 7 in terms of certain dynamical properties of the corresponding shift and in Section 8 we prove the central result: that quasisymmetry types reduce to symmetry types in the periodic case. In section Section 9 we discuss a geometric interpretation of quasisymmetry types, showing quasisymmetry theory to be similar to symmetry theory on a geometric level. The remaining
sections discuss how quasisymmetry theory relates to various other properties of tilings and tiling dynamical systems: inflation, the spectrum and invariant measures. We conclude by discussing some additional examples.

2. Tiling Spaces and Tiling Dynamics

A tile in $\mathbb{R}^n$ is a homeomorphic image of a closed ball in $\mathbb{R}^n$. A tiling $x$ in $\mathbb{R}^n$ is a collection of tiles with disjoint interiors. The support of a tiling $x$, denoted $\text{supp}(x)$, is the union of its tiles. We will mostly be interested in tilings $x$ of $\mathbb{R}^n$ (i.e., $\text{supp}(x) = \mathbb{R}^n$). Most of the examples that we consider also satisfy the following two additional hypotheses: (i) protofiniteness: Let $p$ be a set of translationally incongruent tiles, called prototiles. A $p$-tiling is a tiling by translations of the tiles in $p$; tilings which are $p$-tilings for some $p$ will be called protofinite. The set of all $p$-tilings of $\mathbb{R}^n$ will be denoted by $X_p$. We will always assume $p$ is such that $X_p$ is nonempty (see Section 13). (ii) finite type: A patch $q$ in a tiling $x$ is a finite set of tiles with simply connected support. Given prototiles $p$, let $f = \{q_1, \ldots, q_k\}$ be a finite set of $p$-tiling patches with $\text{card}(q_i) \geq 2$. Let $X_{p,f}$ be the set of tiles $x \in X_p$ such that every pair of adjacent tiles $\tau_1, \tau_2$ in $x$ belong to a patch $q$ that is a translation of some $q_i \in f$. We call $f$ a finite type condition and refer to any $x \in X_{p,f}$ as a tiling of finite type, (the term Markov tiling is used in Rudolph [26] for a special case).

There is a natural metrizable topology on sets of tilings of $\mathbb{R}^n$ in which two tilings are close if they nearly agree on a large cube around 0 (see [19], [25], [26], [21]). To describe a metric for this topology, let $H$ denote the Hausdorff metric\(^3\), i.e., for $\omega_1, \omega_2 \subseteq \mathbb{R}^n$ compact

$$H(\omega_1, \omega_2) = \max\{\inf\{\epsilon_1 : \omega_1 \subseteq N_{\epsilon_1}(\omega_2)\}, \inf\{\epsilon_2 : \omega_2 \subseteq N_{\epsilon_2}(\omega_1)\}\},$$

where $N_{\epsilon}(\omega) = \cup_{v \in \omega} B_{\epsilon}(v)$. Then for two tilings $x$ and $y$ of $\mathbb{R}^n$ let

$$h(x, y) = \inf\{\epsilon : H(\partial(x), \partial(y)) \leq \epsilon\},$$

where $\partial(x) = \partial(C_{1/\epsilon}) \cup \bigcup_{\tau \in \mathcal{X}} (\partial(\tau) \cap C_{1/\epsilon})$ with $C_t = \{(v_1, \ldots, v_n) : |v_i| \leq t\}$. One can easily verify that $h$ is a metric.

Proposition 2.1. For any set $p$ of prototiles, the set $X_p$ is compact in the metric $h$.

This result is stated in Rudolph [26]. A proof of this is given in [21].

A compact translation invariant set $X$ of tilings will be called a shift space or ‘shift’. It is easy to see that given $p$ and $f$, both $X_p$ and $X_{p,f}$ are

\(^3\) My thanks to H. Furstenberg for suggesting the use of the Hausdorff metric in this context.
shifts. For a tiling $x$ of $\mathbb{R}^n$, let $O(x) = \{T^t x : t \in \mathbb{R}^n\}$ denote the orbit of $x$ with respect to translation, and let $\overline{O(x)}$ denote the orbit closure. We will always assume that $\overline{O(x)}$ is compact. In this case we say the tiling $x$ is protocompact. It follows that $\overline{O(x)}$ is a shift.

The group $M(n)$ of rigid motions of $\mathbb{R}^n$ is a semidirect product of the group of translations and the orthogonal group $O(n)$; for any $S \in M(n)$ there exists unique $t \in \mathbb{R}^n$ and $U \in O(n)$ with $S = T^t U$, where $T^t$ denotes translation by $t \in \mathbb{R}^n$. Similarly, the affine motions of $\mathbb{R}^n$, denoted $A(n)$, consist of (unique) products of translations and invertible linear transformations. The natural action of $M(n)$ on $\mathbb{R}^n$ induces an action of $M(n)$ on the tilings of $\mathbb{R}^n$. It is easy to see that this action is continuous.

**Definition 2.2.** A quasisymmetry group is a closed subgroup $G$ of $M(n)$ with $\mathbb{R}^n \subseteq G$.

By a dynamical system we mean a continuous left action of a locally compact group on a compact metric space. For a quasisymmetry group $G$, and a $G$-invariant shift space of tilings $X$, the natural action $L$ of $G$ on $X$ defines a dynamical system. We call such a dynamical system a tiling dynamical system. Here are two important special cases: (i) The smallest quasisymmetry group leaving $X$ invariant is $\mathbb{R}^n$. The action of $\mathbb{R}^n$ on a shift $X$ by translation will be called the shift action (usually denoted $T$). (ii) For $x \in X$, let $G_{X,x} = \{R \in M(n) : Rx \in X\}$. Then

$$G_X = \bigcap_{x \in X} G_{X,x},$$

is the largest quasisymmetry group leaving $X$ invariant. We call $G_X$ the the quasisymmetry group of $X$ and refer to the action of $G_X$ on $X$ (usually denoted $Q$) as the quasi-shift action.

### 3. The Algebraic Properties of Quasisymmetry Groups

In the previous section, a quasisymmetry group was defined to be a closed subgroup of $M(n)$ containing $\mathbb{R}^n$. In this section we show that the algebraic properties of quasisymmetry groups are similar to the properties of the symmetry groups of periodic tilings (i.e., space groups or crystallographic groups, see [28]). The results in this section are purely algebraic.

Recall that the symmetry group of a tiling $x$ of $\mathbb{R}^n$ is given by $G'_x = \{R \in M(n) : Rx = x\}$. A tiling $x$ is periodic if the translation group $\mathbb{Z}_x = G'_x \cap \mathbb{R}^n$ is isomorphic to $\mathbb{Z}^n$. In this case, $\mathbb{Z}_x$ is normal and maximal abelian in $G'_x$. The following result is known as Bieberbach’s second theorem.
Theorem 3.1. (Bieberbach [3], [4]) Let $G'_x$ and $G'_y$ be the symmetry groups for the periodic tilings $x$ and $y$ and suppose $\phi : G'_x \to G'_y$ is a group isomorphism. Then there exists $S \in A(n)$ such that $\phi(R) = SRS^{-1}$.

Our main result in this section is the following:

Theorem 3.2. Let $G$ and $G'$ be quasisymmetry groups and suppose $\phi : G \to G'$ is a topological group isomorphism. Then there exists $S \in A(n)$ such that $\phi(R) = SRS^{-1}$.

Lemma 3.3. Any quasisymmetry group $G$ is a semidirect product of $\mathbb{R}^n$ and a closed subgroup $H$ of $O(n)$. In particular, $G$ is generated by $H \cup \mathbb{R}^n$, and $\mathbb{R}^n$ is normal in $G$.

Proof: If $S \in G$ then $S = TU$ uniquely for $T \in \mathbb{R}^n$ and $U \in O(n)$. Since $\mathbb{R}^n \subseteq G$, $T \in G$ and it follows that $U \in G$, so that $G$ is a semidirect product. Now $ST^tS^{-1} = T^tU$, so that $R^n$ is normal. □

We will identify $H = G \cap O(n)$ and $G/R^n$, and refer to $H$ as the point group of $G$. Now we state our analogue of Bieberbach’s second theorem for quasisymmetry groups.

Lemma 3.4. For any $U \in O(n)$ and $t \in \mathbb{R}^n$ there exists $v \in \mathbb{R}^n$ such that $T^v(T^tU)^{-v} = T^tU$, with $Ut' = t'$.

Proof: We have $T^vT^tUT^{-v} = T^{v-Uv+t}U$, so that $t' = v - Uv + t$. Thus, to show $Ut' = t'$, it suffices to find $v$ such that $-(U - I)^2v = (U - I)t$. Let $W$ be the eigenspace for $U$ corresponding to the eigenvalue 1, and let $W^\perp$ be the ortho-complement of $W$, so that the decomposition $\mathbb{R}^n = W \oplus W^\perp$ is orthogonal and $U$–invariant. This implies $(U - I)W^\perp \subseteq W^\perp$, and since $(U - I)|_{W^\perp}$ is nonsingular, it follows that $(U - I)W^\perp = W^\perp$.

To find $v$, we first write $t = t_0 + t^\perp$, where $t_0 \in W$ and $t^\perp \in W^\perp$, and let $v = -(U - I)|_{W^\perp}^{-1}t^\perp$. Since $v \in W^\perp$, and since $(U - I)t_0 = 0$, it follows that

$$-(U - I)^2v = -(U - I)((U - I)|_{W^\perp})v = (U - I)t^\perp = (U - I)t.$$

□

Lemma 3.5. If $G$ is a quasisymmetry group, then $\mathbb{R}^n \subseteq G$ is the unique maximal normal abelian subgroup of $G$. 
Proof of Lemma 3.5 Let $K \subseteq G$ be a normal abelian subgroup and suppose $T^t U \in K$. We will show $U = I$, so that $K \subseteq \mathbb{R}^n$. Using Lemmas 3.4 and 3.3 we conjugate $T^t U$ by a translation and assume without loss of generality that $U t = t$. Since $K$ is normal, Lemma 3.3 implies $T^v T^t U T^{-v} = T^{v - U v + t} U \in K$, and $K$ abelian implies

\[ I = [T^t U, T^{v - U v + t} U] = T^{-(U - I)^2 v + U t}. \]

Since $U t = t$, (2) implies $(U - I)^2 v = 0$. Letting $W = \ker(U - I)$, write $v = v_0 + v^\perp$ where $v_0 \in W$ and $v^\perp \in W^\perp$. Then $0 = (U - I)^2 v = (U - I)^2 (v_0 + v^\perp) = (U - I)^2 v^\perp$, and since $U - I$ is nonsingular on $W^\perp$, we have $v^\perp = 0$. Hence $U v = v$ and $U = I$. \Box

Lemma 3.6. Let $H$ be a subgroup of $M(n)$ and suppose $w : H \to \mathbb{R}^n$. Then $P = \{ T^{w(U)} U : U \in H \}$ is a subgroup of $M(n)$ if and only if $w$ satisfies the cocycle equation

\[ w(UV) = w(U) + U w(V), \]

for all $U, V \in H$.

This follows from a direct computation.

Lemma 3.7. Suppose $P$ is a compact subgroup of $M(n)$. Then there exists $v \in \mathbb{R}^n$ such that for all $T^t U \in P$ with $U \in O(n)$, $T^v (T^t U) T^{-v} = U$. In particular, if $T^t U \in P$ then $t = v - U v$.

Proof: First we show that if $T^{t_1} U, T^{t_2} U \in P$ then $t_1 = t_2$. Since $(T^{t_2} U)^{-1} = T^{-U^{-1} t_2} U^{-1}$, it follows $T^{t_1 - t_2} = (T^{t_1} U)(T^{t_2} U)^{-1}$, which implies $T^{t_1 - t_2} \in P$. It follows that $t_1 = t_2$, since otherwise $T^{t_1 - t_2}$ would generate an infinite discrete subgroup of $P$.

Define a homomorphism $\pi : M(n) \to O(n)$ by $\pi(T^t U) = U$, and define $\pi' : M(n) \to \mathbb{R}^n$ by $\pi'(T^t U) = t$. Let $H = \pi(P)$. By the first step, $\pi|_P : P \to H$ is an isomorphism so that $H$ is a compact. Thus there exists $w : H \to \mathbb{R}^n$ continuous such that $T^{w(U)} U = \pi^{-1}(U)$, i.e. $w(U) = (\pi' \circ \pi^{-1})(U)$. Since $P$ is a group, $w$ satisfies (3).

Define

\[ v = -\int_H w(V) d\mu(V), \]

where $\mu$ denotes normalized Haar measure. It follows that

\[ \int_H w(UV) d\mu(V) = \int_H w(V) d\mu(V) = -v, \]

and

\[ \int_H (w(U) + U w(V)) d\mu(V) = w(U) - U v, \]
so that by (4), (5) and (6) we obtain \( -v = w(U) - Uv \). This implies
\( w(U) = Uv - v \) (i.e., \( w \) is a coboundary) and
\[
T^v(T^{w(U)}U)T^{-v} = T^{w(U)+Uv-v}U = U.
\]
\( \square \)

**Lemma 3.8.** Suppose \( \varphi : G \to G' \) is a topological group isomorphism of
quasisymmetry groups such that there exists \( \gamma : G \to \mathbb{R}^n \) with
\[
\varphi(T^sU) = T^{\gamma(T^sU)}U,
\]
and such that
\[
\varphi(T^s) = T^s.
\]
Then there exists \( v \in \mathbb{R}^n \) such that \( \gamma(T^sU) = s + v - Uv \). In particular,
\[
\varphi(T^sU) = T^v(T^sU)T^{-v}.
\]

**Proof:** Let \( H = G \cap O(n) \), and define \( P = \{ T^{\gamma(T^sU)-s}U : U \in H, s \in \mathbb{R}^n \} \). Note that \( \gamma = \pi' \circ \varphi \), and \( \pi'(T^s_1T^s_2U) = s_1 + s_2 \). Since \( \varphi(U) = \varphi(T^{-s})\varphi(T^sU) \), (8) implies \( \gamma(U) = -s + \gamma(T^sU) \), and hence
\( P = \{ T^{\gamma(U)}U : U \in H \} \). Since \( \gamma \) is continuous, \( P \) is compact, and
\[
\gamma(UV) = \pi' (\varphi(U)\varphi(V)) = \pi'(T^{\gamma(U)UT^\gamma(V)}) = \pi'(T^\gamma(U)+U\gamma(V)UV) = \gamma(U) + U\gamma(V),
\]
so that \( \gamma \) satisfies (3). It follows that \( P \) is a compact subgroup of \( M(n) \). Thus
\[
\gamma(T^sU) - s = v - Uv \text{ by Lemma 3.7, and (9) follows using (7).} \)
\( \square \)

**Proof of Theorem 3.2** Lemma 3.5 implies \( \varphi(\mathbb{R}^n) = \mathbb{R}^n \) and thus \( \varphi|_{\mathbb{R}^n} \)
is a topological group isomorphism. It follows that there exists \( G \in GL(n) \) such that
\[
\varphi(T^t) = GT^tG^{-1} = T^{Gt}. \]
For \( T \in \mathbb{R}^n \) and \( U \in O(n) \), we write \( \varphi(TU) = \varphi_1(TU)\varphi_2(TU) \), where
\( \varphi_1(TU) \in \mathbb{R}^n \) and \( \varphi_2(TU) \in O(n) \). Then \( \varphi_2(TU) = (\pi \circ \varphi)(TU) \) and
\( \varphi_1(TU) = \varphi(TU)\varphi_2(TU)^{-1} \). Since \( \pi : G \to O(n) \) is a homomorphism, (10) implies that
\[
\varphi_2(T^{t_1}U)\varphi_2(T^{t_2}U)^{-1} = (\pi \circ \varphi)((T^{t_1}U)(T^{t_2}U)^{-1})^{-1} = (\pi \circ \varphi)((T^{t_1}U)(T^{t_2}U)^{-1}) = \pi(\varphi(T^{t_1-t_2})) = \pi(T^{G(t_1-t_2)}) = I,
\]
and thus $\varphi_2(TU) = \varphi_2(U)$ for all $T \in \mathbb{R}^n$. By (10), $\varphi(T^U) = T^{GU\varphi}$. Hence if we apply $\varphi$ to the relation $(T^U)T^v(T^U)^{-1} = T^U$, we obtain

$$T^{GU\varphi} = \varphi((T^U)T^v(T^U)^{-1})$$
$$= T^{GU\varphi(U)}T^{GU\varphi(U)^{-1}T^{-1}Gt}$$
$$= T^{GU\varphi_1(U)}\varphi_2(U)T^{GU\varphi(U)^{-1}\varphi_1(U)^{-1}T^{-1}Gt}$$
$$= T^{GU\varphi_1(U)}\varphi_2(U)T^{GU\varphi_1(U)^{-1}T^{-1}Gt}$$
$$= T^{GU\varphi_1(U)}\varphi_2(U)T^{GU\varphi_1(U)^{-1}T^{-1}Gt}$$

$$= T^{GU\varphi_1(U)}\varphi_2(U).$$

Replacing $v$ with $G^{-1}v$ in (11) and applying $\pi'$, it follows that $\varphi_2(U) = GUG^{-1}$, which implies $GUG^{-1} \in \mathcal{O}(n)$. Moreover $\varphi(T^t) = \varphi_1(T^t)\varphi_2(T^t) = \varphi_1(t)$, which implies

$$\varphi_1(T^t) = T^{Gt}.$$  

Now let us define $\Phi : G \to M(n)$ by

$$\Phi(T^U) = GT^U\mathbb{G}^{-1} = T^{Gt}\mathbb{G}^{-1}.$$  

Putting $\mathbb{G}' = \Phi(G)$, we observe that $\mathbb{G}'$ is a quasisymmetry group since $\Phi$ is an isomorphism with $\Phi(T^t) = T^t$. Let $\varphi' : \mathbb{G}' \to \mathbb{G}'$ be defined by $\varphi' = \varphi \circ \Phi^{-1}$. Then $\varphi'(T^sU) = \varphi((T^{G^{-1}s}G^{-1}UG)_U) = \varphi_1((T^{G^{-1}s}G^{-1}UG))$, and since $\text{im}(\varphi_1) \subseteq \mathbb{R}^n$, it follows that $\varphi'(T^sU) = T^{\gamma(T^sU)}$, where $\gamma(T^sU) = \varphi_1((T^{G^{-1}s}G^{-1}UG))$. Thus $\gamma$ is continuous and $\varphi'$ satisfies (7). Now by (12), $\gamma(T^s) = \pi'(\varphi_1(T^{G^{-1}s})) = s$, so that (7) implies (8). By Lemma 3.8 there exists $v \in \mathbb{R}^n$ such that $\gamma(T^sU) = s + v - Uv$. This implies $\varphi(T^s) = T^vT^sUT^{-1}$, and since $\varphi = \varphi' \circ \Phi$, (13) implies

$$\varphi(T^sU) = (\varphi' \circ \Phi)(T^sU) = (T^vG)(T^sU)(T^sG)^{-1}.$$  

$\square$

4. Almost periodic tilings

Having studied the algebraic properties of symmetry groups, we now proceed to our main goal: to study the symmetry properties of individual almost periodic tilings $x$. The basic set-up will be as follows. We let $G_x = G_{\overline{O(x)}}$ and then study the dynamical properties of the quasi-shift action of $G_x$ on $\overline{O(x)}$.

Recall that a dynamical system (i.e. a group action) is called transitive if has a single orbit. A dynamical system is called minimal if every orbit is dense. It is called topologically transitive if some point has a dense orbit. The shift on $\overline{O(x)}$ is always topologically transitive. It is easy to see that a tiling $x$ is a periodic tiling if and only if the shift on $\overline{O(x)}$ is transitive, and in this case, $O(x) = \overline{O(x)}$ (i.e., $O(x)$ is closed), and $O(x)$ is isometric to an $n$–dimensional torus. Note also that any periodic tiling is of finite type.
This discussion illustrates the fundamental relation between periodicity (i.e., periodic points) and transitive actions. The following well known result of W. Gottschalk shows that almost periodicity plays a similar role in the theory of minimal actions.

For \( r > 0 \), a set \( Z \subset \mathbb{R}^n \) is called \( r \)-dense if for any \( t \in \mathbb{R}^n \), \( B_r(t) \cap Z \neq \emptyset \). A set \( Z \) is called relatively dense if it is \( r \)-dense for some \( r > 0 \). Suppose \( T \) is an action of \( \mathbb{R}^n \) on a compact metric space \( X \). A point \( x \in X \) is called an almost periodic point for \( T \) if for any \( U \subseteq X \) open, the set \( \{ t \in \mathbb{R}^n : T^t x \in U \} \) is relatively dense.

**Theorem 4.1.** (Gottschalk [7]) For an action \( T \) of \( \mathbb{R}^n \) on a compact metric space \( X \) the following are equivalent: (i) \( x \in X \) is almost periodic, (ii) the restriction of \( T \) to \( O(x) \) is minimal, (iii) every \( y \in O(x) \) is almost periodic, and (iv) \( O(y) = O(x) \) for \( y \in O(x) \).

We call a tiling \( x \) of \( \mathbb{R}^n \) almost periodic if it is an almost periodic point for the shift on \( O(x) \). A tiling \( x \) will be called rigidly almost periodic if for any patch \( q \) which occurs in \( x \), there exists \( r > 0 \) such that for any \( v \in \mathbb{R}^n \), the tiling \( x \) contains a translation of \( q \) inside \( B_r(v) \) (rigid almost periodicity is sometimes referred to as the local isomorphism property, [21]). Given a tiling \( x \), a large patch in \( x \) centered at \( 0 \in \mathbb{R}^n \) defines a neighborhood of \( x \); it consists of the tilings which have nearly the same patch around \( 0 \) that \( x \) does. If \( x \) is rigidly almost periodic, this patch repeats relatively densely throughout \( x \). It follows that rigid almost periodicity implies almost periodicity. For tilings of finite type, one can show that the converse is also true (see [21]).

Two tilings \( x \) and \( y \) are said to be of the same species\(^4\) if \( O(x) = O(y) \). By (iv) of Lemma 4.1, if \( x \) is almost periodic, this is equivalent to \( y \in O(x) \). Notice that if \( x \) and \( y \) are protofinite and made from different prototiles then they automatically belong to different species. However, tilings by the same prototiles can also belong to different species.

One can show that two finite type almost periodic tilings \( x \) and \( y \) belong to the same species if and only if every patch \( q \) occurring in \( x \) also occurs in \( y \) (such \( x \) and \( y \) are sometimes said to be locally isomorphic, see [21]). The following well known example will illustrate some of these ideas.

\(^4\)This terminology comes from [19].
Example 1: (Rhombic Penrose tilings) Consider the two rhombic tiles (see Figure 1a) marked with arrows. Let $p$ be the set of 20 marked prototiles obtained by rotating the tiles in Figure 1a by multiples of $2\pi/10$. These prototiles are laid edge-to-edge, subject to the following *matching rule*: the arrows on adjacent edges must match. This enforces a finite type condition. One can show that it is possible to tile the entire plane in this way [18] (see Figure 1b). The resulting tiles are called *rhombic Penrose tilings*.

Once the tilings are constructed we ignore the markings (so that $p$ really consists of 10 prototiles). The finite type condition $f$ can be taken to be the set of all ‘pictures’ of the vertex configurations that occur. It is well known that every rhombic Penrose tiling is aperiodic and almost periodic, and that any two rhombic Penrose tilings are of the same species (see [8]). This implies the following result.

**Corollary 4.2.** The Penrose shift $X_{p,f}$ is a minimal aperiodic shift of finite type.

This result might seem a little surprising from the point of view of 1-dimensional symbolic dynamics (where any minimal subshift of finite type consists of a single periodic orbit), but turns out to be a fairly common multidimensional phenomenon. The earliest examples of this phenomenon consist of discrete 2-dimensional subshifts of finite type in the form of aperiodic ‘Wang’ tilings of the plane (see [8]), due to M. Berger [2]. Berger was studying the ‘Tiling Problem’: *Given a set $p$ of prototiles, determine whether $X_p \neq \phi$*. He showed that in general, the tiling problem is undecidable, [2]. This fact is closely related to the existence of aperiodic minimal shifts of finite type (see [11]). Using a construction similar to Berger’s, S. Mozes recently showed that, up to an almost 1:1 extension, most 2-dimensional...
discrete substitution dynamical systems are actually subshifts of finite type. Since a typical 2-dimensional substitution system is minimal and aperiodic, this result provides an abundant source of examples of minimal aperiodic 2-dimensional discrete subshifts of finite type. Mozes also showed that weak mixing is possible for such examples [16].

Comment There is a clear qualitative similarity between discrete substitution dynamical systems and tilings which satisfy an inflation (see Section 10). An interesting open question is whether there is a tiling version of the theorem of Mozes. Specifically, is every shift of tilings satisfying an inflation rule ‘nearly’ (i.e., up to almost 1:1 extension) a shift of finite type?

5. The quasisymmetry group and point group of an almost periodic tiling

Recall that $G_x$ denotes the quasisymmetry group of the shift $O(x)$. Let $Q_x$ denote the quasi-shift action of $G_x$ on $O(x)$, and let $T_x$ denote the shift action of $\mathbf{R}^n$ on $O(x)$. The next result shows that for almost periodic tilings the group $G_x$ depends only on the species of $x$. It should be noted that the proposition is not generally true for tilings which are not almost periodic since it is essentially a corollary of Theorem 4.1.

**Proposition 5.1.** If $x$ is an almost periodic tiling of $\mathbf{R}^n$ then $G_y = G_x$ for any $y \in O(x)$.

First we note that if $y \in O(x)$ and $S \in \mathbf{A}(n)$ then $Sy \in O(Sx)$. In this case, the map $S : O(x) \rightarrow O(Sx)$ is a homeomorphism, and $O(Sx) = S(O(x))$. The proposition follows directly from the next lemma.

**Lemma 5.2.** Let $x$ be an almost periodic tiling of $\mathbf{R}^n$ and suppose that $Rx \in O(x)$ for some $R \in \mathbf{M}(n)$. If $y \in O(x)$ then $Ry \in O(x)$.

**Proof:** By Lemma 4.1 the shift $T_x$ on $O(x)$ is minimal, and since $Rx \in O(x)$, it follows that $O(Rx) = O(x)$. Since $y \in O(x)$, this implies $Ry \in O(Rx)$. $\square$

We define the point group of an almost periodic tiling $x$, by $H_x = G_x \cap O(n)$. Note that this is the point group (in the sense of Section 3) of the quasisymmetry group $G_x$. By Proposition 5.1, $H_y = H_x$ for all $y \in O(x)$.

For an (unmarked) rhombic Penrose tiling $x$ the point group is generated by the reflection $F$ through the horizontal axis and rotation $R_\theta$ by $\theta = 2\pi/10$, i.e., $H_x = D_{10}$, the dihedral group of order 20. The corresponding quasisymmetry group is generated by $D_{10}$ and $\mathbf{R}^2$. Note that by the

---

5 This definition is implicit in Niizeki [17].
crystallographic restriction $D_{10}$ cannot be the point group of any periodic tiling of $\mathbb{R}^2$. This was the observation that started the theory of quasicrystals.

**Lemma 5.3.** For a periodic tiling $x$, the classical point group $H'_x$ and the quasicrystallographic point group $H_x$ are isomorphic.

**Proof:** Define $\psi : H'_x \to H_x$ by $\psi(RZ_x) = RR^n$. Then $Z_x$ is a subgroup of $\mathbb{R}^n$ and of $G'_x$. It follows that $\psi$ is a well defined surjection, and it suffices to show $\psi$ is 1:1. If $RZ_x \in \ker(\psi)$, then $\psi(RZ_x) = RR^n$, and it follows that $RR^n = R^n$, so that $R \in \mathbb{R}^n$. But since $RZ_x \in H'_x$, it follows that $R \in G'_x$, so that $R \in G'_x \cap \mathbb{R}^n = Z_x$. Thus $\ker(\psi)$ is trivial. \hfill $\Box$

**Corollary 5.4.** If $x$ is a periodic tiling of $\mathbb{R}^n$, then $G_x$ is generated by $G'_x \cup \mathbb{R}^n$. In particular, for any $R \in G_x$ there exists $Q \in G'_x$ and $T \in \mathbb{R}^n$ such that $R = TQ$.

6. The insufficiency of a purely algebraic theory

The main invariant in the symmetry theory of periodic tilings is the isomorphism type of the symmetry group. One might expect quasisymmetry groups to play the same role for almost periodic tilings. This turns out not to be the case. We show in this section that it is not even possible to recover the symmetry type of a periodic tiling from its quasisymmetry group.

**Proposition 6.1.** There exist two periodic tilings $x$ and $y$ of $\mathbb{R}^2$ with different (non-isomorphic) symmetry groups $G'_x$ and $G'_y$ such that the quasisymmetry groups $G_x$ and $G_y$ are isomorphic.

**Proof:** Consider the two periodic tilings $x$ and $y$ of $\mathbb{R}^2$ in Figure 2, which have $G'_x = pm$ and $G'_y = pg$ (see [8] or [28]), so that in particular, the symmetry groups of $x$ and $y$ are not isomorphic.

(a) \hspace{1cm} (b)

Figure 2: Parts of two periodic tilings with symmetry groups (a) $pm$, and (b) $pg$.

On the other hand, it is easy to see that the quasisymmetry groups $G_x$ and $G_y$ are both isomorphic to the semidirect product of $\mathbb{R}^2$ and $\mathbb{Z}/2$. \hfill $\Box$

The proof of Proposition 6.1 shows that the distinction between pure reflections and glide reflections is lost in quasisymmetry groups. This is because quasisymmetry groups are always semidirect products.
7. Quasisymmetry types for almost periodic tilings

Two actions \( L_1 \) and \( L_2 \) of a locally compact group \( G \) on compact metric spaces \( X_1 \) and \( X_2 \) are said to be \textit{topologically conjugate} if there exists a homeomorphism \( \eta : X_1 \to X_2 \) such that for all \( S \in G \) and \( x \in X_1 \), \( \eta(L_1^S x) = L_2^S \eta(x) \). Topological conjugacy is the standard notion of isomorphism in topological dynamics, but it is not well suited to symmetry theory since it is not invariant under rotation or rescaling. Thus we make the following modification. Let \( L \) be a \( G \) action on \( X \) and let \( \varphi : G_1 \to G \) be a topological group homomorphism. Define a \( G_1 \) action \( L \circ \varphi \) by \( (L \circ \varphi)^S x = L^{\varphi(S)} x \). If \( L_1 \) and \( L_2 \) are continuous actions of isomorphic locally compact groups \( G_1 \) and \( G_2 \), we say \( L_1 \) and \( L_2 \circ \varphi \) are \textit{rescale topologically conjugate} if there exists a (topological group) isomorphism \( \varphi : G_1 \to G_2 \) such that \( L_1 \) and \( L_2 \circ \varphi \) are topologically conjugate \( G_1 \) actions.

\textbf{Definition 7.1.} Let \( x \) and \( y \) be almost periodic tilings of \( \mathbb{R}^n \). We say \( x \) and \( y \) have the same quasisymmetry type if (i) \( G_x \) is isomorphic to \( G_y \) and (ii) the corresponding quasi-shift actions \( Q_x \) and \( Q_y \) on \( \mathcal{O}(x) \) and \( \mathcal{O}(y) \) are rescale topologically conjugate (so that in particular, \( \mathcal{O}(x) \) and \( \mathcal{O}(y) \) are homeomorphic).

Note that a necessary condition for \( x \) and \( y \) to have the same quasisymmetry type is for \( G_x \) and \( G_y \) to be isomorphic. However, by Proposition 6.1 this is not sufficient. Clearly a sufficient condition for having the same quasisymmetry type is to have the same species. The next example shows that this is not necessary.

\textbf{Example 2:} (\textit{Kites–and–darts Penrose tilings}) Consider the two marked prototiles shown in Figure 3a, together with their rotations by multiples of \( 2\pi/10 \).

\footnote{Analogous theories of quasisymmetry types can be based \textit{almost topological conjugacy} (see [1]), and \textit{metric isomorphism} (when there is an invariant measure; see Section 12). It should be noted that for both of these alternatives, Theorem 8.1 (below) still holds.}
Figure 3: (a) Penrose kites-and-darts prototiles. (b) Part of a kites and darts Penrose tiling.

These prototiles are laid edge-to-edge, subject to the matching condition that the black and white dots on adjacent tiles match, (see Figure 3b). The resulting tilings are called *kites–and–darts Penrose tilings* ([18], see also [8]).

Like the rhombic Penrose tilings, every kites–and–darts Penrose tiling is aperiodic and almost periodic. Moreover, both kinds of Penrose tilings have the same quasisymmetry group. However, because they are made of different prototiles, they belong to different species.

There is a standard procedure for converting between the two kinds of Penrose tilings (see [8], [5]). Let $X_{p,f}$ denote the finite type shift of all rhombic Penrose tilings, and let $X_{p',f'}$ denote the finite type shift of all kites–and–darts Penrose tilings. Given $x \in X_{p,f}$, a line is drawn along the major axis of each ‘fat’ rhombic prototile, as shown in Figure 4. Then the vertices of the resulting kites and darts are colored black and white. Finally, all the lines which do not connect black and white dots (along with any remaining arrows) are erased. We denote this conversion operation by $x' = \eta(x)$. 
that periodic, it follows from Section 4 that

\[ x \eta(14) \]

Since \( \varphi \) is reversible, local in its effect, and conjugates the quasi-shift actions \( Q_x \) and \( Q_{x'} \).

8. Quasisymmetry types generalize symmetry types

**Theorem 8.1.** If \( x \) and \( y \) are periodic tilings of \( \mathbb{R}^n \) then they have the same symmetry type if and only if they have the same quasisymmetry type. In particular, for periodic tilings, symmetry types and quasisymmetry types coincide.

**Proof:** Suppose there exists a homeomorphism \( \eta : \overline{O(x)} \rightarrow \overline{O(y)} \) and a topological group isomorphism \( \varphi : G_x \rightarrow G_y \) such that

\[
\eta \circ Q^R_x = Q^R_y \circ \eta \quad \text{for all } R \in G_x.
\]

Since \( x \) is periodic, \( G'_x \) is a subgroup of \( G_x \) and so \( \varphi(G'_x) \) is a subgroup of \( G_y \). Suppose \( R \in G'_x \). By (14), \( Q^R_y \eta(x) = \eta(Q^R_x x) = \eta(Rx) = \eta(x) \), so that \( \varphi(R) \in G'_y \). Reversing the steps, it follows that for any \( R \in G'_{\eta(x)} \) that \( \varphi^{-1}(R) \in G'_x \). Thus \( G'_x \) and \( G'_{\eta(x)} \) are isomorphic. Now, since \( y \) is periodic, it follows from Section 4 that \( \overline{O(y)} \) is closed, so that \( \overline{O(y)} = O(y) = \{Ty : T \in \mathbb{R}^n\} \). Thus \( \eta(x) = Ty \) for some \( T \in \mathbb{R}^n \), and this implies that \( T^{-1}G'_y T = G'_{\eta(x)} \). Hence, \( G'_x \) and \( G'_y \) are isomorphic.

Conversely, suppose \( x \) and \( y \) are periodic tilings of \( \mathbb{R}^n \) with \( G'_x \) and \( G'_y \) isomorphic. By Theorem 3.1 there exists an affine transformation \( S \in A(n) \) such that the map \( \varphi : G'_x \rightarrow G'_y \) defined \( \varphi(R) = SRS^{-1} \) is a topological group isomorphism. Write \( S = TG \) for \( T \in \mathbb{R}^n \) and \( G \in GL(n) \). Then define \( \eta : \overline{O(x)} \rightarrow \overline{O(y)} \) by \( \eta(T^*_x x) = T^*_y y \). It follows from Lemma 3.3 that to show that \( \eta \) is a well defined homeomorphism, it suffices to show that \( Gt \in Z_y \) whenever \( t \in Z_x \). Letting \( t \in Z_x \), so that \( T^*_x x = x \), and using the fact that \( \varphi(T^*_x) \in G'_y \), we obtain \( T^*_y y = ST^*_y S^{-1} y = \varphi(T^*_y) y = y \).

The equation \( \varphi(R) = SRS^{-1} \) defines a continuous 1:1 homomorphism \( \varphi : G_x \rightarrow M(n) \). Using Corollary 5.4, we write \( R = TV \), where \( T \in \mathbb{R}^n \) and \( V \in G'_x \). Then \( \varphi(R) = \varphi(TV) = \varphi(T) \varphi(V) \), and since \( \varphi(T) \in \mathbb{R}^n \subseteq G_y \), and \( \varphi(V) \in G'_y \subseteq G_y \), we have \( \varphi(R) \in G_y \). Using the same argument

\[ \text{[15, [8].} \]
for $\varphi^{-1}$, it follows that $\varphi : G_x \to G_y$ is a topological group isomorphism. Finally, to prove (14), we let $x' = T^t x$, and compute
\[
(Q^\varphi_y)^R(x') = \varphi(R) \eta(T^t x) = SRS^{-1} T^t G_x S^{-1} x = \eta(R T^t x) = \eta(Q^\varphi_y(x'))
\]
\[
\square
\]

9. THE GEOMETRY OF QUASISYMMETRY TYPES

Let $R \in M(n)$, and define $N_R = \{S \in A(n) : SRS^{-1} \in M(n)\}$. Clearly $N_R$ is a closed subset of $A(n)$ with $M(n) \subseteq N_R$. Given $\lambda > 0$, let $M_\lambda v = \lambda v$, let $S(n) = \{M_\lambda : \lambda > 0\}$, and note that $S(n) \subseteq N_R$ for any $R$. Let $C(n)$ denote the closed subgroup of $A(n)$ generated by $S(n) \cup M(n)$.

**Lemma 9.1.** Let $x$ be an almost periodic tiling of $R^n$, let $R \in G_x$, and let $S \in N_R$. Then $SRS^{-1} \in G_S x$.

**Proof:** By definition, $SRS^{-1} \in M(n)$, and $(SRS^{-1})(Sx) = S(Rx)$. Since $R \in G_x$, it follows that $Rx \in O(x)$. Thus $S(Rx) \in O(Sx)$, and $SRS^{-1} \in G_S x$. $\square$

For a quasisymmetry group $G$ let $N_G = \cap_{R \in G} N_R$, a closed subgroup of $A(n)$ containing $C(n)$. By Lemma 9.1, $S \in N_G$ implies $SG_x S^{-1} \subseteq G_S x$.

**Theorem 9.2.** Two almost periodic tilings $x$ and $y$ of $R^n$ have the same quasisymmetry type if and only if (i) there exists $S \in N_{G_x}$ with $G_y = SG_x S^{-1}$, and (ii) the quasi-shift systems $Q_{Sx}$ and $Q_y$ are topologically conjugate.

**Proof:** By Definition 7.1 there exists an isomorphism $\varphi : G_x \to G_y$ so that the $G_x$ actions $Q^\varphi_y$ and $Q_x$ are topologically conjugate via some $\eta : O(x) \to O(y)$. By Theorem 3.2 there exists $S \in A(n)$ such that $\varphi(R) = SRS^{-1}$, so that $G_y = SG_x S^{-1}$ and $S \in N_{G_x}$. Also, $\eta(Rx) = SRS^{-1} \eta(x)$, and thus $O(Sx) = S(O(x))$. Define a homeomorphism $\omega : O(Sx) \to O(y)$ by $\omega = \eta \circ S^{-1}$ and suppose that $R \in G_y$, so that $R = SRS^{-1}$ for some
\( R' \in G_x \). Let \( z \in \overline{O(Sx)} \) and let \( x' \in \overline{O(x)} \) satisfy \( z = Sx' \). Then
\[
(\omega \circ Q^R_{Sx})(z) = \omega(Rz) = (\eta \circ S^{-1})(SR'S^{-1})(Sx') = \eta(R'x') = SR'S^{-1}\eta(x') = R(\eta \circ S^{-1})(Sx') = (R \circ \omega)(z) = (Q^R_y \circ \omega)(z).
\]

Conversely, if \( \omega : \overline{O(Sx)} \to \overline{O(y)} \) is a homeomorphism satisfying \( \omega(R(Sx')) = R(\omega(Sx')) \) for all \( x' \in \overline{O(x)} \) and some \( S \in N_{G_x} \) with \( SG_xS^{-1} = G_y \), then \( \eta = \omega \circ S : \overline{O(x)} \to \overline{O(y)} \) conjugates \( Q^x_y \) and \( Q_x \). \( \square \)

**Corollary 9.3.** If \( x \) is an almost periodic tiling of \( \mathbb{R}^n \) and \( S \in N_{G_x} \cap N_{G_{Sx}} \) then \( x \) and \( Sx \) have the same quasisymmetry type.

**Proof:** Since \( N_{G_{Sx}} \) is a group, \( S \in N_{G_{Sx}} \) implies \( S^{-1} \in N_{G_{Sx}} \). It follows that \( SN_{G_{Sx}}S^{-1} \subseteq N_{G_x} \), or equivalently, \( N_{G_{Sx}} \subseteq S^{-1}N_{G_x}S \). Since \( S \in N_{G_x} \), it follows from Lemma 9.1 that \( S^{-1}N_{G_x}S \subseteq N_{G_{Sx}} \). \( \square \)

**Corollary 9.4.** For all \( S \in C(n) \), \( x \) and \( Sx \) have the same quasisymmetry type.

**Proof:** \( C(n) \subseteq N_{M(n)} \). \( \square \)

In particular, for tilings of the plane, the quasisymmetry type is invariant under rotation, reflection, translation, and scaling. Certain quasisymmetry types are also invariant under other affine transformations, but in general, affine transformations can create or destroy quasisymmetries.

### 10. Inflation

One of the most interesting and celebrated features of Penrose tilings is the fact that they satisfy a kind of ‘self similarity’ property called an inflation. While many other almost periodic tilings satisfy inflations, there are also many examples that do not. Thus inflations (or their absence) provide important invariants for almost periodic tilings.

A tiling \( x \) is said to satisfy an inflation\(^8\) with inflation constant \( \lambda > 0 \) if there exists a homomorphism \( \psi : \overline{O(x)} \to \overline{O(x)} \), such that for all \( ST^t \in G_x \) (with \( S \in H_x \))
\[
(15) \quad \psi \circ ST^t_x = ST^\lambda_x \circ \psi.
\]

\(^8\)This usage seems natural. Unfortunately, the term deflation is also used by some authors to mean the same thing.
For protofinite tilings, inflations are usually defined in terms of a construction resembling the procedure (described above) for converting between the two kinds of Penrose tilings: First, the prototiles are marked with some ‘new edges’. Then some ‘old edges’ are removed, resulting in a new tiling on a smaller scale. Finally, the small scale tiling is ‘scaled up’ by $\lambda$ to obtain a new tiling on the original scale. The Penrose inflation, with $\lambda = (1/2)(\sqrt{5} + 1)$, is shown in Figure 5, (see [8] for many other examples).

**Figure 5:** Inflation for the rhombic Penrose prototiles.

**Proposition 10.1.** If $x$ and $y$ are tilings having the same quasisymmetry type, then $x$ satisfies an inflation with constant $\lambda$ if and only if $y$ satisfies an inflation with the same constant. In particular, the existence of an inflation with a given constant is an invariant of quasisymmetry type.

**Proof:** Let $\psi$ be the inflation with constant $\lambda$ satisfied by $x$. Define an automorphism $A_\lambda$ of $G_x$ by $A_\lambda(T^t U) = T^t G U$. By the definition of inflation, $\psi \circ Q^R_x = Q^{A_\lambda(R)}_x \circ \psi$ for all $R \in G_x$. Since $y$ has the same quasisymmetry type as $x$, Theorem 9.2 implies that there exists $S \in N_{G_y}$ such that (i) $G_x = G_y' = SG_yS^{-1}$ and, (ii) there exists a homeomorphism $\eta : \overline{O(x)} \to \overline{O(y)}$ such that $\eta \circ Q^R_x = Q^R_y \circ \eta$ for all $R \in H_x$. Let $\varphi(R) = SRS^{-1}$, and note that $S \circ Q^R_y = Q^{\varphi(R)}_y \circ S$. Note also that $\varphi$ and $A_\lambda$ commute, since if $R = T^t U$ and $S = T^s G$, then

$$
\varphi(A_\lambda R) = \varphi(T^{\lambda t} U)
= S \circ T^{\lambda t} U \circ S^{-1}
= T^{G(\lambda t)} S U S^{-1}
= A_\lambda(T^{Gt} S U S^{-1})
= A_\lambda(\varphi(R)),
$$
where $SU^{-1} S^{-1} \in O(n)$ since $S \in N_{G_y}$. Let $S' = \eta \circ S$ and let $\psi' = (S')^{-1} \psi S'$. Then
\[
(\psi')^{-1} \circ Q_y^R \circ g' = (S')^{-1} \psi^{-1} \circ \eta \circ Q_{y'}^\varphi(R) \circ \eta^{-1} \circ \psi S'
= (S')^{-1} \psi^{-1} Q_x^{\varphi(R)} \psi S'
= (S')^{-1} Q_x^{A_\lambda(R)} S'
= (S')^{-1} Q_x^{f(A_\lambda(R))} S'
= Q_y^{A_\lambda(R)}.
\]

Hence $G'$ is an inflation for $y$ with constant $\lambda$. □

**11. The spectrum**

An important technique in crystallography is to study the diffraction patterns of crystals (and other kinds of solids). For example, the icosahedral point symmetry of the Schectman quasicrystal \cite{27} was initially detected by observing five-fold rotational symmetry (not possible for crystals) in the diffraction pattern. If we assume that (at least approximately) a quasicrystal consists of atoms located at the vertices of an almost periodic tiling $x$, then one can show \cite{23} that the diffraction pattern is essentially the point spectrum of the corresponding shift action $T_x$. In this section we show how the symmetries of the point spectrum of the shift $T_x$ relate to the quasisymmetry type of $x$.

For an almost periodic tiling $x$ of $R^n$, let $\Sigma_x \subseteq R^n$ denote the (topological) point spectrum of the shift $T_x$; that is, $\Sigma_x$ consists of all $w$ such that there exists a continuous complex eigenfunction $f \in C(O(x))$, $f \neq 0$, with
\[
f(T_x^t y) = e^{2\pi i <t,w>} f(y),
\]for all $y \in O(x)$. The shift $T_x$ is said to have discrete spectrum (or be weakly mixing) if the eigenfunctions have a dense span in $C(O(x))$ (or $\Sigma_x = \{0\}$). The symmetry group of the spectrum $\Sigma_x$ is given by $H_{\Sigma_x} = \{U \in O(n) : U\Sigma_x = \Sigma_x\}$.

**Proposition 11.1.** If $x$ is an almost periodic tiling of $R^n$ and $S = TG \in A(n)$ for $T \in R^n$ and $G \in GL(n)$, then $\Sigma_{Sx} = (G^*)^{-1} \Sigma_x$. Moreover, $T_{Sx}$ has discrete spectrum (or is weakly mixing) if and only if $T_x$ has the same property.

**Proof:** Let $\chi_w : O(x) \to C$ be an eigenfunction for $w \in \Sigma_x$. Then $S^{-1} : O(Sx) \to O(x)$ is a homeomorphism, and $\chi_w \circ S^{-1} : O(Sx) \to C$
is continuous. Thus for \(z \in \overline{O(Sx)}\) and \(t \in \mathbb{R}^n\),

\[
(\chi_w \circ S^{-1})(T_x^t z) = \chi_w(S^{-1}T_x^t z) = \chi_w(T_x^{G^{-1}t}S^{-1} z)
= e^{2\pi i <G^{-1}t,w>} \chi_w(S^{-1} z) = e^{2\pi i <t,(G^*)^{-1}w>} (\chi_w \circ S^{-1})(z),
\]

and \((G^n)^{-1}w \in S_{Sx}\).

**Corollary 11.2.** If \(x\) is an almost periodic tiling of \(\mathbb{R}^n\) and \(U \in H_x\) then \(U\Sigma_x = \Sigma_x\). In particular, \(H_x \subseteq H_{\Sigma_x}\), i.e., eigenvalues provide obstructions to quasisymmetries.

**Proof:** \(U\Sigma_x = \Sigma_{Ux}\) since \(U \in O(n)\), and \(U \in H_x\) since \(\Sigma_{Ux} = \Sigma_x\).

One can show [24] that for Penrose tilings \(x\), the shift \(T_x\) has discrete spectrum with \(\Sigma_x = \mathbb{Z}[\zeta]\), where \(\zeta = e^{2\pi i/5}\) (here we view \(\mathbb{Z}[\zeta]\) as a subset of \(\mathbb{R}^2\)).

The next result is similar to Corollary 11.2. It shows how the spectrum relates to inflations.

**Proposition 11.3.** If \(x\) satisfies an inflation with constant \(\lambda\), then the spectrum \(\Sigma_x\) of \(x\) satisfies \(\lambda \cdot \Sigma_x = \Sigma_x\).

### 12. Symmetries of invariant measures

It turns out that in addition to being minimal, the shifts \(T_x\) for almost periodic tilings \(x\) are frequently also uniquely ergodic. This means that there exists a unique \(T_x\)-invariant Borel probability measure on \(\overline{O(x)}\) (see [20]). We denote this measure by \(\mu_x\). It follows from the ergodic theorem for uniquely ergodic dynamical systems that each patch in such a tiling \(x\) occurs with a uniform positive frequency. We will refer to an almost periodic finite type tiling with a uniquely ergodic shift as **strictly almost periodic**. One can easily show, for example, that Penrose tilings are strictly almost periodic (see [8]).

It has been suggested by C. Radin that one should study the symmetries of the unique invariant measure to understand the symmetries of the corresponding strictly almost periodic tiling. The results in this section show how such symmetries relate to the theory of quasisymmetry types.

**Proposition 12.1.** Let \(x\) be a strictly almost periodic tiling. If \(S \in H_x\) then \(\mu_x\) is \(S\)-invariant, i.e., \(\mu_x(S^{-1}E) = \mu_x(E)\) for all Borel sets \(E \subseteq \overline{O(x)}\).

This follows from the fact that \(\mu_x \circ S^{-1}\) is an invariant Borel measure, and so by unique ergodicity, equal to \(\mu_x\). Now let \(\overline{O_M(x)} = \{Rx : R \in M(n)\}\)
denote the orbit closure of $x$ with respect to the action of $M(n)$. We will consider the Borel probability measures on $O_M(x)$ which are invariant for the shift action. Note that $O_M(x)$ is compact in the metric $h$, and that $O_M(x) = \cup_{R \in M(n)} O(Rx)$. Thus for $R \in M(n)$ we can regard $\mu_x \circ R^{-1}$ as a measure on $O_M(x)$. In general, however, $H_x \neq O(n)$, so the shift on $O_M(x)$ need not be minimal or uniquely ergodic.

**Proposition 12.2.** Suppose $x$ is strictly almost periodic and let

$$G_{\mu_x} = \{ R \in M(n) : \mu_x \text{ is } R\text{-invariant} \},$$

(i.e., $G_{\mu_x}$ is the ‘symmetry group’ for $\mu_x$). Then $G_{\mu_x} = G_x$.

**Proof:** First note that $G_{\mu_x}$ is a quasisymmetry group. Letting $H_{\mu_x} = G_{\mu_x} \cap O(n)$, it suffices to prove $H_x = H_{\mu_x}$. By Corollary 9.4 $H_x = H_{Sx}$ for $S \in H_x$, and by Proposition 12.2 $H_{Sx} \subseteq H_{\mu_x}$, so that $H_x \subseteq H_{\mu_x}$. Now suppose $S \in H_{\mu_x}$. If $\mu$ is any ergodic $T^t$–invariant measure (see [20]), $\text{supp}(\mu) \subseteq O(Rx)$ for some $R \in O(n)$. Thus by unique ergodicity, $\mu = \mu_{Rx}$, and it follows that $S \in H_{Rx}$, which implies $S \in H_x$. $\square$

13. **Other examples**

Suppose $\overline{p}$ is a finite set of incongruent tiles. We say $x$ is a generalized $\overline{p}$-tiling if every tile in $x$ is congruent (not necessarily by translation) to a tile in $\overline{p}$. One can show that Proposition 2.1 extends to generalized $\overline{p}$-tilings (see [21]), and thus generalized $\overline{p}$-tilings are protocompact. An interesting example recently studied by Radin [22] is called pinwheel tilings. For pinwheel tilings, $\overline{p}$ consists of a single $\sqrt{5}$–right triangle, rotations and reflections of which are used to tile the plane. Without going into details, we note that pinwheel tilings are defined by an inflation rule, but Radin has shown that (up to an almost 1:1 extension) the pinwheel tiling shift is of finite type [22]. The most interesting feature of this example is its point group. In every pinwheel tiling the right–triangular ‘prototile’ occurs with infinitely many different rotational orientations. This implies that pinwheel tilings are not protofinite, and moreover that

$$H_x = O(n).$$

Radin has observed that (16) implies $T_x$ is weakly mixing (this follows, for example, from our Corollary 11.2).

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9This space was studied by Radin and Wolff, [21].

10 It should be noted that for periodic tilings, the point group is always finite. One can also show that if $p$ consists of polygons (in $\mathbb{R}^2$) then any almost periodic $x \in X_p$ must have a finite point group.
In the literature on quasicrystals and almost periodic tilings, there is a standard method for constructing examples of almost periodic tilings. There are two essentially equivalent formalisms for this method (see [6]) called the cut and project method and the dual method. The idea of this construction goes back to the work of de Bruijn [5], who showed that the Penrose tilings are this type. Tilings obtained by this construction are sometimes called quasiperiodic tilings (see [10]). Roughly speaking, the vertex set $b(x)$ of a quasiperiodic tiling $x$ of $\mathbb{R}^n$ (at least in the ‘typical’ case) is obtained by projecting part of the periodic lattice $\mathbb{Z}^r$ in $\mathbb{R}^r$, $r > n$, to an irrationally sloped $n$-dimensional plane $L$ in $\mathbb{R}^r$. Using this method, one can obtain examples of almost periodic tilings with various point groups [29]. For example, by modifying examples in [29] (to break some of the symmetry) one can obtain an almost periodic tiling $x$ of $\mathbb{R}^2$ with an arbitrary proper closed subgroup of $\text{O}(2)$ as its point group $H_x$. This illustrates a crucial fact about almost periodic tilings: almost periodic tilings are not subject to any sort of crystallographic restriction. This is one of the major differences between crystallography and quasicrystallography: there is essentially no crystallographic restriction for almost periodic tilings.

Most of the studies of the symmetry properties of quasiperiodic tilings have concentrated (at least implicitly) on studying only their point symmetries, often by studying the symmetries of the spectrum (see [10], [9]). In effect, this means looking only at the information in their quasisymmetry groups. One can also show that the shift $T_x$ for any quasiperiodic tiling $x$ has nontrivial eigenvalues. In particular, the eigenfunctions consist of the periodic functions of $\mathbb{R}^r$ restricted to $L$. It follows that quasiperiodic tilings $x$ generate shifts $T_x$ which are never weakly mixing. Since some almost periodic tilings (e.g., pinwheel tilings) have weakly mixing shifts, it follows that although quasiperiodicity implies almost periodicity, the converse is false.

References

[22] Radin, C., The pinwheel tilings of the plane, preprint, Dept. of Mathematics, University of Texas, Austin, TX 78712.