Rank and directional entropy

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8. **EXTRAS**
**Cutting and Stacking**

- **Elementary** method to construct examples in ergodic theory.
- Classical version: invertible Lebesgue measure preserving transformation \( T : [0, 1) \rightarrow [0, 1) \).
- Equivalently, a measure preserving \( \mathbb{Z} \) action (MPZA).
- Easily generalizes to \( \mathbb{Z}^d \) or \( \mathbb{R}^d \) to produce \( \text{MPZ}^d A \) or \( \text{MPR}^d A \).
- More general than substitutions.
Entrophy

- **Kolmogorov-Sinai, 1959**: Entropy $h(T)$ of a measure preserving transformation $T$. Average “information” per time step.
- Straightforward generalization to $d$-dimensional entropy $h(T)$ of $\text{MP}\mathbb{Z}^d A$ $T$.
- **Adler-Konheim-McAndrew, 1965**: Topological entropy $h_{\text{top}}(T)$ of continuous map (or $\mathbb{Z}^d$ action) $T$. Exponential growth in “complexity”. $h(T) \leq h_{\text{top}}(T)$.
- **Milnor, 1986**: Directional entropy $h_n(V, T)$ of $\text{MP}\mathbb{Z}^d A, T$. Here $V \subseteq \mathbb{R}^d$ subspace, $\dim(V) = n$. 
Von Neuman’s “Adding Machine”

Step 0

Step 1

Step 3

Step n

Figure: Ergodic mpt $T$, but not aperiodic ($\ldots$ RRB RBB $\ldots$).
Illustrated as block concatenation

The tower is turned on its side, with individual levels blurred.

\[ W_1 = 0, \quad W_{n+1} = W_n W_n. \]
As $T : [0, 1) \rightarrow [0, 1)$
As Toeplitz sequence

Action together with partition equals process.
Here the combinatorial data is \( W_1 = 0 \) and \( W_{n+1} = W_n W_n 1 W_n \).
Chacon's transformation

Step 0

Step 1

Step 2
Rank 1

**Definition.** $T$ is **rank 1** if it can be constructed by cutting and stacking with one large **tower** in each step.

- Left over interval called a **spacer**.

**Theorem**

*Rank 1 implies (uniquely) ergodic. (Also minimal if number of adjacent spacers is bounded.)*

- Adding machine has **discrete spectrum**. Chacon’s transformation has **continuous spectrum** (i.e., is **weakly mixing**.)
- Any ergodic $T$ with discrete spectrum is **rank 1** (e.g., irrational rotation transformation).
(Smorodinski)-Adams (1998) version (see also Ornstein (1968)).

Recurrence relation: $W_1 = 0$, $W_{n+1} = W_n W_n 1 W_n 1^2 \ldots W_n 1^{q_n}$. Mixing provided $q_n \uparrow \infty$ sufficiently fast.
**The Morse dynamical system**

\[ W_1^0 = 0 \]
\[ W_1^1 = 1 \]

\[ \ldots W_{n+1}^0 = W_n^0 \]
\[ \ldots W_{n+1}^1 = W_n^1 \]
Morse sequences

Step 0

Step 1

Step 2

Step 3

0 1 1 0 1 0 0 1 1 0 0 1 0 1

0 1 1 0 1 0 0 1 1 0 0 1 0 1
Finite Rank

In this example, there are 2 towers at each step. We say $T$ has rank $\leq 2$.

- A. del Junco showed this $T$ is not rank 1. Thus $T$ is rank 2.
- The spectrum of $T$ is simple, and mixed (both discrete and continuous).
- Can similarly define rank $\leq r$, rank $r$, and finite rank.

**Theorem** (see Queffelec, (1987/2010))

A substitution on $r$ letters is rank $\leq r$. 
Theorem (Baxter, 1971)

Finite rank implies $h(T) = 0$.

Proof.
- Rank $n$ implies spectral multiplicity $M_T \leq n$ (Chacon, 1970).
- Positive entropy ($h(T) > 0$) implies $M_T = +\infty$ (Bernoulli factor) (Sinai’s Theorem).
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Rohlin Towers

- Let $T : X \to X$ be a MPZA on a probability space $(X, \mathcal{B}, \mu)$.
- If $B, TB, T^2 B, \ldots, T^{h-1} B$ are pairwise disjoint, we call it a Rohlin tower with height $h$ and base $B$.
- The error is $E = \left( \bigcup_{k=0}^{h-1} T^k B \right)^c$.
- Call $\xi = \{ B, TB, \ldots, T^{h-1} B, E \}$ a Rohlin partition.

**Theorem (Rohlin’s Lemma)**

If $T$ is ergodic, then for any $h \in \mathbb{N}$ and $\epsilon > 0$, there is a height $h$ Rohlin tower with $\mu(E) < \epsilon$. 
Let $\xi_n$ be a sequence of partitions. Say $\xi_n$ separates ($\xi_n \to \varepsilon$) if for any $A \in \mathcal{B}$ there is $A_n \leq \xi_n$ so that $\mu(A \Delta A_n) \to 0$.

**Definition**

$T$ is rank 1 if there is a sequence $\xi_n$ of Rohlin towers so that $\xi_n \to \varepsilon$.

Cutting and stacking definition of Rank 1 implies this one: $\xi_n \to \varepsilon$ follows from $\text{diam}(B_n) \to 0$.

**Theorem (Baxter, 1971)**

$\xi_n$ may be chosen so that $\xi_n \leq \xi_{n+1}$ and $B_{n+1} \subseteq B_n$.

Thus all these $T$ may be obtained by cutting and stacking.
“Funny” Rank 1

Call a finite $R \subseteq \mathbb{Z}$ a shape.

Suppose $\mu(B) > 0$ and $T^k B \cap T^\ell B = \emptyset$ for all $k, \ell \in R$, $k \neq \ell$.

Call $\xi = \{E, T^k B : k \in R\}$ a funny Rohlin tower.

In rank 1, $R = \{0, 1, \ldots, h-1\}$.

Define funny rank 1 analogously.

Shape matters! Rank 1 implies “loosely Bernoulli” (Katok, 1977, Ornstein-Rudolph-Weiss 1982), but funny rank 1 does not (Ferenczi, 1985).
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Actions of $\mathbb{Z}^d$

- Let $(X, \mathcal{B}, \mu)$ be a **probability space**.
- Let $T_1, T_2 : X \to X$ be MP$\mathbb{Z}$As that **commute**: $T_1 T_2 = T_2 T_1$.
- For $n = (n_1, n_2) \in \mathbb{Z}^2$, define **MP$\mathbb{Z}^2$A** $T^n = T_1^{n_1} T_2^{n_2}$.
- Similar definition for **MP$\mathbb{Z}^d$A**, (i.e., $T_1, T_2, \ldots, T_d$ commute).
- Call a finite $R \subseteq \mathbb{Z}^d$ a **shape**.

**Definition.** A shape-$R$ **Rohlin tower** consists of disjoint sets $T^n B, n \in R$. The partition $\xi = \{E, T^n B : n \in R\}$ is a Rohlin partition.
**Definition**

A MP\(\mathbb{Z}^d\) \(A\) \(\mathbb{T}\) is rank 1 if there is a sequence \(\xi_n\) of shape \(R_n\) Rohlin partitions so that \(\xi_n \to \varepsilon\).

**Proposition (R-Sahin, 2010)**

*Rank 1 (any shape) implies ergodic and simple spectrum.*

**Corollary**

*Rank 1 (any shape) implies \(h(T) = 0\).*
The $\mathbb{Z}^2$ case

$\mathbb{Z}^d$ RANK $r$

**Definition**

Suppose $T$ is an MP $\mathbb{Z}^d$-A there are shapes $R^j_n$ and positive measure sets $B^j_n$, for $j = 1, \ldots, r$ and $n \in \mathbb{N}$, so that

$$\xi_n = \{T^n B^j_n : n \in R^j_n, j = 1, \ldots, n\} \cup \{X \setminus \bigcup_{j=1}^{n} \bigcup_{n \in R^j_n} T^n B^j_n\}$$

is a partition, and $\xi_i \rightarrow \varepsilon$. We say $T$ is rank $\leq r$ for shapes $\{R^1_n, R^2_n, \ldots, R^j_n\}$.

Rank $r$ if rank $\leq r$ and not rank $\leq r - 1$.

**Proposition**

Rank $\leq r$ implies $M_T \leq r$ and $h(T) = 0$, but not necessarily ergodic.
A sequence $\mathcal{R} = \{R_k\}$ of shapes in $\mathbb{Z}^2$ is a Følner sequence (van Hove sequence) if for any $n \in \mathbb{Z}^2$

$$\lim_{k \to \infty} \frac{|R_k \triangle (R_k + n)|}{|R_k|} = 0,$$

- A natural choice is rectangles

$$R_k = [0, \ldots, w_k - 1] \times [0, \ldots, h_k - 1],$$

where $w_k, h_k \to \infty$. 
Types of rank 1

- **Rank 1**: no shape restriction.
- **Følner rank 1**: $R_n$ a Følner sequence.

**Proposition (R-Sahin, 2010)**

If Følner, can get $\xi_n \leq \xi_{n+1}$ with the same $\mathcal{R} = \{R_n\}$.

- Cutting and stacking works!
- **Rectangular rank 1**: rectangles
- **Geometric restrictions** (on rectangular Rank 1):
  - Bounded eccentricity: $1/K \leq w_k/h_k \leq K$.
  - Subexponential eccentricity: $\log(w_k)/h_k \rightarrow 0$ ($w_k \geq h_k$).
Chacon \( Z^2 \) actions

Weak mixing, not strong mixing, & “MSJ” (R-Park, 1991).

**Note.** \( w_n/h_n = 1 \): “bounded” eccentricity.
Rudolph’s example

One of \( (\Delta w_n)(\Delta h_n) \) blocks.

\( N_n \) of these blocks in a row.
Rudolph’s example (continued)

A block consisting of all possible 
\(((\Delta w_n)(\Delta h_n))^{N_n}\)
rows, in some particular order.

There are \((((\Delta w_n)(\Delta h_n))^{N_n})!\)
of these.
Rudolph’s example (continued)

- All \( \left( \left( \left( \Delta w_n \right) \left( \Delta h_n \right) \right)^n \right)! \)
  blocks (every possible order) stacked.
- \( w_{n+1} = \left( \left( \Delta w_n \right) \left( \Delta h_n \right) \right)^n \times (w_n + \Delta w_n). \)
- \( h_{n+1} = \left( \left( \left( \left( \Delta w_n \right) \left( \Delta h_n \right) \right)^n \right)! \times \left( \left( \Delta w_n \right) \left( \Delta h_n \right) \right)^n \times (h_n + \Delta h_n). \)
Properties of Rudolph’s example

- Requires appropriate choice of $\Delta w_n \to \infty$, $\Delta h_n \to \infty$ and $N_n \to \infty$.
- Side lengths
  \[ w_{n+1} = \left( (\Delta w_n)(\Delta h_n) \right)^{N_n} (w_n + \Delta w_n), \text{ and} \]
  \[ h_{n+1} = \left( (\Delta w_n)(\Delta h_n) \right)^{N_n}! \left( (\Delta w_n)(\Delta h_n) \right)^{N_n} (h_n + \Delta h_n). \]
- Sides satisfy $\log(h_n)/w_n \to \infty$. Super exponential eccentricity.

Theorem (Rudolph, 1978)

Horizontal $T_1$ is Bernoulli shift with arbitrary finite entropy $0 < h(T_1) < \infty$. 

Before defining directional entropy, we briefly review the ordinary ($d$-dimensional) entropy of a MP $\mathbb{Z}^d A T$.

- Let $\xi$ be a finite partition. The entropy of $\xi$ is
  \[ H(\xi) = -\sum_{A \in \xi} \mu(A) \log \mu(A). \]

- Define $\xi_n = \bigvee_{n \in [0, \ldots, n]^d} T^{-n} \xi$

- The $\xi$-entropy of $T$ is
  \[ h(T, \xi) = \lim_{n \to \infty} \frac{1}{n^d} H(\xi^n). \]

- The entropy of $T$ is given by
  \[ h(T) = \sup_\xi h(T, \xi). \]

This gives usual entropy of transformation $T$ when $d = 1$. 
Subspace $V \subseteq \mathbb{R}^d$, $n = \dim(V) < d$.

$Q \subseteq V$, $Q' \subseteq V^\perp$ unit cubes, and

$S(V, t, m) = (tQ + mQ')$ (we call it a window.)
Let $T$ be a MPZ^nA, with $\xi$ a finite partition, and $\dim(V) = n$.

- $\xi_{V,t,m} := \bigvee_{n \in S(V,t,m)} T^{-n} \xi$.

- $h_n(T, V, \xi, m) := \limsup_{t \to \infty} \frac{1}{t^n} H(\xi_{V,t,m})$.

- $h_n(T, V, \xi) := \sup_{m > 0} h_n(T, V, \xi, m)$

**Definition (Milnor, 1986)**

If $1 \leq n < d$, *n*-dimensional directional entropy in direction $V$ is

$$h_n(T, V) = \sup_{\xi} h_n(T, \xi, V).$$

If $n = d$, then $h_d(T, V) = h(T)$, (where $V = \mathbb{R}^d$).
Directional entropy ($\mathbb{Z}^2$ case)

- $h_1(V, T) < \infty$ for some $V$, implies $h_2(T) = 0$.
  - Ledrappier’s $\mathbb{Z}^2$ shift $T$ has $h_1(T, V) > 0$ for all $V$.
  - K. Park (unpublished, c 1987) Chacon MP$\mathbb{Z}^2$A $T$ has $h_1(T, V) = 0$ for all $V$.
- $h_1(T, V) = ||(p, q)||^{-1} h(T^{(q, p)}), V = (p, q)\mathbb{R}, p/q \in \mathbb{Q}$.
  - Rudolph rank 1 $\mathbb{Z}^2$ has $h_1(V, T) > 0$ where $V = e_1\mathbb{R}$.
- (K. Park, 1999) If $V = v\mathbb{R}, ||v|| = 1$, then $h_1(T, V) = h(F^t v)$ for the unit $\mathbb{R}^2$ suspension $F^t$ of $T$.
- (K. Park, 1999) The function $h(v) = h(T, v\mathbb{R}), ||v|| = 1$, is upper semicontinuous, and $\{v : h(v) = 0\}$ is $G_\delta$. 
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The first result has no assumptions beyond rectangular rank 1.

**Theorem 1. (R-Sahin, 2010)**

Let $T$ be a rectangular rank-1 $MP\mathbb{Z}^d A$. Then there is a 1-dimensional subspace $V \subseteq \mathbb{R}^d$ so that $h_1(T, V) = 0$.

With additional hypotheses on the eccentricity, we can say more.

**Theorem 2. (R-Sahin, 2010)**

Let $T$ be a rectangular rank-1 $MP\mathbb{Z}^d A$ with subexponential eccentricity. If $V \subseteq \mathbb{R}^d$ is an $n$-dimensional subspace, $1 \leq n \leq d$, then $h_n(T, V) = 0$. 
Two lammmas

**Lemma (Milnor, 1988)**

The formulas that define directional entropy simplify to

\[ h_n(T, V, \xi, m) = \lim_{t \to \infty} \frac{1}{t^n} H(\xi_{V,t,m}), \text{ and} \]

\[ h_n(T, V, \xi) = \lim_{m \to \infty} h_n(T, V, \xi, m). \]

**Theorem (Boyle-Lind, 1997)**

If \( \xi_k \leq \xi_{k+1} \) and \( \xi_k \to \epsilon \) then

\[ h_n(T, V) = \lim_{k \to \infty} h_n(T, V, \xi_k). \]
Suppose $\xi_k \leq \xi_{k+1}$ and $\xi_k \to \epsilon$. If $t_j \to \infty$, and

$$\lim_{j \to \infty} \frac{1}{(t_j)^n} H((\xi_k)_V, t_j, m) = 0,$$

for all $k$ and all $m > 0$, then $h_n(T, V) = 0$.

We will use this lemma in the proofs of both theorems.
Proofs (set-up)

We do the case $d = 2$.

Let $V \subseteq \mathbb{R}^2$ be a 1-dimensional subspace (to be specified later for Theorem 1), and let $\xi_k$ be a sequence of shape-$R_k$ Rohlin towers for $T$.

Assume WOLOG:

- $\xi_k \leq \xi_{k+1}$ (Baxter’s Theorem),
- $R_k$ is $w_k \times h_k$ where $h_k \leq w_k$ for all $k$.

Note. There are no eccentricity assumptions in Theorem 1.

Let $t_j \to \infty$ be a slowly increasing sequence, to be specified later.

Ultimate Goal. For fixed $m, k$, show that $H((\xi_k)_{V,t_j,m})/t_j \to 0$. 
Let $j > k$.

Call a level $T^n B_j$ in $\xi_j$ **good** if $S(V, t_j, m) \subseteq R_j - n$.

Let $G_j \subseteq \mathbb{Z}^2$ be the set of good levels.

Let $F_j = (\bigcup_{n \in G_j} T^n B_j)^c$.

And, recall $E_j = (\bigcup_{n \in R_j} T^n B_j)^c$. 
Proofs (Good Levels)
Proofs (New partitions)

- $\xi_j^* := \{T^n B_j : n \in G_j\} \cup \{F_j\}$.
- $\eta_j := (\xi_k)^{T,t_j,m} \vee \xi_j^*$.
- Note that $(\xi_k)^{T,t_j,m} \leq \eta_j$.
- Thus $H((\xi_k)^{T,t_j,m}) \leq H(\eta_j)$.
- So it suffices to show $H(\eta_j)/t_j \to 0$.
- (This will achieve our **Ultimate Goal**.)
**Proofs (Relations among partitions)**

**Key observation:** Each of the sets $T^n B_j$ for $n \in G_j$ belong to the partition $\eta_j$.

"Goodness" insures the partition $(\xi_k)_{V,t_j,m}$ is "constant" on levels $T^n B_j$, for $n \in G_j$. In other words, each $T^n B_j$ is a subset of some $A \in (\xi_k)_{V,t_j,m}$.

\[
H(\eta_j)/t_j = -\frac{1}{t_j} \sum_{A \in \eta_j} \mu(A) \log \mu(A)
\]

\[
= -\frac{1}{t_j} \left( \sum_{n \in G_j} \mu(T^n B_j) \log \mu(T^n B_j) + \sum_{A \in \eta'_j} \mu(A) \log \mu(A) \right)
\]

\[
= -\frac{1}{t_j} \left( |G_j| \mu(B_j) \log \mu(B_j) - \sum_{A \in \eta'_j} \mu(A) \log \mu(A) \right).
\]
Proofs (Left term Goal)

\[-\frac{1}{t_j}|G_j|\mu(B_j) \log \mu(B_j) \leq -\frac{1}{t_j}|R_j|\mu(B_j) \log \mu(B_j)\]

\[= -\left(\frac{w_j h_j}{t_j}\right) \left(\frac{1 - \epsilon_j}{w_j h_j}\right) \log \left(\frac{1 - \epsilon_j}{w_j h_j}\right)\]

\[\leq \frac{\log(w_j h_j) - \log(1 - \epsilon_j)}{t_j},\]

where \(\epsilon_j = \mu(E_j)\).

Left Term Goal. Show \(\log(w_j h_j)/t_j \to 0\). (Insubstantial entropy from (uniformly covered) good set)
Local entropy lemma

**Theorem (Shields, 1996)**

Suppose $\xi$ is a partition, $\xi' \subseteq \xi$ and $\beta = \mu(\bigcup_{A \in \xi'} A)$. Then

$$- \sum_{A \in \xi'} \mu(A) \log \mu(A) \leq \beta \log |\xi'| - \beta \log \beta.$$
Proofs (Right Term)

- $|\xi'_j| \leq (|R_k| + 1)|S(V, t_j, m)|.$
- $\log |\xi'_j| = |S(V, t_j, m)| \log(|R_k| + 1) \leq 2|S(V, t_j, m)| \log |R_k|.$
- $|S(V, t_j, m)| \leq 2t_j m.$
- $\log |R_k| = K.$

Thus

$$\log |\xi'_j| \leq 2Kt_j m.$$
Also

\[ \beta = \mu(F_j) = |B_j \setminus G_j| \mu(B_j) + \mu(E_j) \leq \frac{|B_j \setminus G_j|}{w_j h_j} + \epsilon_j. \]

So by the local entropy lemma

\[ -\frac{1}{t_j} \sum_{A \in \xi'} \mu(A) \log \mu(A) \leq 2Km \left( \frac{|B_j \setminus G_j|}{w_j h_j} + \epsilon_j \right) - \frac{\beta \log \beta}{t_j}. \]

\((t_j/t_j \text{ cancels in the first term}). \text{ Since } \beta < 1, (\beta \log \beta)/t_j \to 0.\)

**Right Term Goal.** \[ \frac{|B_j \setminus G_j|}{w_j h_j} \to 0. \text{ (This is essentially that measure of bad part, } \beta \to 0). \)
Proof of Theorem 1 (Left Term Goal)

- Assume $w_j \geq h_j$ for all $j$.
- Take $V = e_1 \mathbb{R}$.
- We want $t_j \to \infty$ so that $\frac{\log(w_j)}{t_j} \to 0$ and $\frac{t_j}{w_j} \to 0$.

Define $t_j = \sqrt{w_j \log w_j}$.

$$\frac{\log(w_j h_j)}{t_j} \leq 2 \frac{\log(w_j)}{t_j} \to 0.$$  **Left Term Goal Achieved.**
Proof of Theorem 1 (Right Term Goal)

We have, \(|R_j \setminus G_j| \leq h_j t_j + mw_j\).

\[
\frac{|R_j \setminus G_j|}{w_j h_j} = \frac{t_j}{w_j} + \frac{m}{h_j} \to 0,
\]

since \(\frac{t_j}{w_j} = \sqrt{w_j \log w_j} = \sqrt{\frac{\log w_j}{w_j}} \to 0\). **Right Term Goal Achieved.**
Proof of Theorem 2 (Left Term Goal)

- Take $V \subseteq \mathbb{R}^2$, $\dim(V) = 1$.
- Assume $w_j \geq h_j$ and define $t_j = \sqrt{h_j \log w_j}$.

$$\frac{\log w_j}{t_j} = \frac{\log w_j}{\sqrt{w_j \log(w_j)}} = \sqrt{\frac{\log w_j}{w_j}} \to 0$$

$$\frac{t_j}{h_j} = \frac{\sqrt{h_j \log w_j}}{h_j} = \sqrt{\frac{\log w_j}{h_j}} \to 0$$

(by subexponential eccentricity).

$$\frac{\log(w_j h_j)}{t_j} \leq \frac{2 \log(w_j)}{t_j} \to 0.$$  

Left Term Goal achieved.
Proof of Theorem 2 (Right Term Goal)

We have, \(|R_j \setminus G_j| \leq h_j(t_j + m) \cos \theta + w_j(t_j + m) \sin \theta.\)

\[
\frac{|R_j \setminus G_j|}{w_j h_j} = \frac{t_j + m}{w_j} \cos \theta + \frac{t_j + m}{h_j} \sin \theta \to 0,
\]

since \(\frac{t_j}{h_j} \to 0,\) (and \(\frac{t_j}{w_j}, \frac{m}{h_j}, \frac{m}{w_j} \to 0.\)) **Right Term Goal achieved.**
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Here is what we can prove in rank $r$. For simplicity, we discuss only the case $T$ is an ergodic rectangular rank $\leq 2$ $MP\mathbb{Z}^2 A$. Let $R^1_n$ be $w^1_n \times h^1_n$ and $R^2_n$ be $w^2_n \times h^2_n$.

**Theorem A.** If $w^1_n \geq h^1_n$ and $w^2_n \geq h^2_n$ for infinitely many $n$ then there exists $V$ so that $h_1(T, V) = 0$ (i.e., $h(T_1) = 0$).

**Theorem B.** Under the same hypotheses as above, if $\log(w^1_n)/h^1_n \to 0$, and $\log(w^2_n)/h^2_n \to 0$, then $h_1(T, V) = 0$ for all 1-dimensional $V$.

**Theorem C.** If $w^1_n \geq h^1_n$ and $w^2_n \leq h^2_n$ for all $n$, and $\log(w^1_n)/h^1_n \to 0$, and $\log(h^2_n)/w^2_n \to 0$, then $h_1(T, V) = 0$ for all 1-dimensional $V$. 


As mentioned before, a substitution on $r$ letters has rank $\leq r$. This is also true for a substitution tiling with $r$ distinct prototiles. The eccentricity is bounded. This implies a substitution tiling system has all directional entropies zero.

Another way to prove this is to note that the complexity of a substitution tiling satisfies $c(n) \leq Kn^e$ (where $e = d$ in the self similar case).

A. Julien (2009) proved $c(n) \leq Kn^e$ for a cut and project tiling where the acceptance domain is polyhedral and “almost canonical”. This implies all directional entropies zero.

More generally a model set with a topologically and measure theoretically regular acceptance domain has discrete spectrum, so is rank 1. This implies all directional entropies zero.
Ledrappier’s shift has \( c(n) = Ke^{2n} \) (exponential complexity in smaller dimension). It has positive directional entropy in every direction.

Radin showed that any uniquely ergodic \( \mathbb{Z}^2 \) SFT has \( c(n) \leq Ke^{\ell n} \). Can it have positive directional entropy.

Not for the examples that come from substitutions and model sets!
Theorem (Johnson-Sahin, 1998)

A rectangular rank 1 MP $\mathbb{Z}^2 A T$ with bounded eccentricity is loosely Bernoulli.

- This $T$ can be chosen to have $T_1$ be non LB.
- Johnson-Sahin (1998) prove that the same result holds for rank $r > 1$ provided towers have uniformly bounded eccentricity.
Loosely Bernoulli

**Theorem (R-Sahin 2011?)**

If $T$ is a loosely Bernoulli $MP_{\mathbb{Z}^d}A$ with $h_d(T) = 0$ then $h_n(T, V) = 0$ for all $V$.

Implications:
- Ledrappier’s shift is not loosely LB (a “folk theorem”).
- Rudolph’s rank 1 is not LB.
1. INTRODUCTION
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6. DIRECTIONAL ENTROPY AND RANK 1
7. MORE . . .
8. EXTRAS
Say the Rohlin lemma holds for a shape $R \subseteq \mathbb{Z}^d$ if for any ergodic $\mathbb{Z}^d$ action $T$, and $\epsilon > 0$, there exists $B \in \mathcal{B}$ so that $X$ is partitioned by $\xi = \{ E, T^n B : n \in R \}$ and $\mu(\cup_{n \in R} T^n B) > 1 - \epsilon$.

A shape $R$ tiles $\mathbb{Z}^d$ if there exists $C \subseteq \mathbb{Z}^d$ so that $\{ T^n R : n \in C \}$ is a partition of $\mathbb{Z}^d$.

**Theorem (Ornstein-Weiss, 1980)**

A Rohlin lemma holds for a shape $R$ if and only if $R$ tiles $\mathbb{Z}^d$. 