Research and Productivity Growth Across Industries: Asymmetry and Calibration

L. Rachel Ngai and Roberto M. Samaniego*

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Abstract

These notes contain two sets of results. First, we construct a class of asymmetric equilibria for the model economy and show that industry TFP growth rates and R&D intensities are the same as in the symmetric equilibrium. Second, we examine an alternative calibration, focusing on the question of how to map between prices and productivities in an environment with intermediate goods.

1 Introduction

In these notes we show that symmetry is not essential to the results of Ngai and Samaniego (2008a). We define a class of asymmetric equilibria in which the same results hold. The class of equilibria is defined by the property that, while there is a distribution of knowledge across firms in each industry, firms maintain their relative positions within this distribution over time.

We also examine an alternative calibration, focusing on the question of how to map between prices and productivities in an environment with intermediate goods.

* L. Rachel Ngai, Department of Economics, London School of Economics, Houghton Street, London WC2A 2AE, Tel: +44 (0)20 7955 7017, Fax: +44 (0)20 7831 1840, Email: L.Ngai@lse.ac.uk; Roberto M. Samaniego, Department of Economics, The George Washington University, 1922 F St NW Suite 208, Washington, DC 20052. Tel: +1 (202) 994-6153, Fax: +1 (202) 994-6147, Email: roberto@gwu.edu
2 Asymmetry

Some preliminary derivations

2.1 Step 1: reformulation of the problem in "relative terms"

Before, we had

$$T_{ih,t+1} = F_{ih} + (1 - \delta_T) T_{ih}$$

(1)

where new knowledge $F_{ih}$ is produced according to

$$F_{ih} = A_i T_{it}^{\kappa_i} T_{st}^{\sigma_i} (Q_{iht}^{1-\alpha} L_{iht}^{1-\alpha})^{\psi_i}.$$  

(2)

Let us rewrite this problem as $T_{ih} = z_{ht} T_{it}$. Now we have

$$F_{ih} = A_i (z_{ht} T_{it})^{\kappa_i} T_{it}^{\sigma_i} (Q_{iht}^{1-\alpha} L_{iht}^{1-\alpha})^{\psi_i}.$$  

or

$$F_{ih} = z_{ht}^{\kappa_i} A_i T_{it}^{\rho_i} (Q_{iht}^{1-\alpha} L_{iht}^{1-\alpha})^{\psi_i}.$$  

(3)

where $z_{ht}$ is given, and $z_{ht+1}$ is a control variable.

In addition, suppose that there is some probability of being successful $\eta$. We can rewrite the entire problem in terms of choosing $z_{ht+1}$ (your position relative to the average knowledge holdings) instead of $T_{ih+1}$ (the quantity of knowledge):

$$z_{ih+1} T_{it+1} = (\eta_t) F_{ih} + (1 - \delta_T) z_{ih} T_{it}$$

(4)

$$z_{ih+1} T_{it+1} = (\eta_t) \ z_{ht}^{\kappa_i} A_i T_{it}^{\rho_i} (Q_{iht}^{1-\alpha} L_{iht}^{1-\alpha})^{\psi_i} + (1 - \delta_T) z_{ih} T_{it}$$

(5)

Let’s look at the maximization problem for this firm. Profits are

$$\Pi_{ih} = p_{ih} T_{ih} K_{ih}^{\alpha} N_{ih}^{1-\alpha} - w_{it} (N_{ih} + L_{ih}) - R_t (K_{ih} + Q_{iht})$$

(6)

or

$$\Pi_{ih} = p_{ih} z_{ht} T_{it} K_{ih}^{\alpha} N_{ih}^{1-\alpha} - w_{it} (N_{ih} + L_{ih}) - R_t (K_{ih} + Q_{ih})$$

(7)

A symmetric equilibrium is one in which $z = 1$ is constant over time. In this case, you start with $z_{ih} = 1$, and the maximum is one in all periods. We know that this exists under the conditions discussed in the paper.
2.2 Step 2: consider one firm that deviates from the norm

Suppose we are on such a symmetric BGP, and there is ONE FIRM (only one) that has a different value of $z$ at the beginning. Since there is a continuum of firms, aggregates are not affected – it takes aggregate paths, prices, etc as given. We wish to see how this firm behaves.

The firm’s maximization problem is

$$\max_{\{N_{ih},K_{ih},Q_{ih},L_{ih}\}} \sum_{t=0}^{\infty} \lambda_t \frac{\Pi_{ih}}{p_t} \quad s.t. \quad (8)$$

$$T_{ih+1} = A_i T_{it}^{\lambda_i} (Q_{ih}^{\alpha_i} L_{ih}^{1-\alpha_i})^{\lambda_i} + (1 - \delta) T_{ih}$$
$$Y_{ih} = N_i c_{ih} \quad if \quad i = 1,..m - 1$$
$$Y_{ih} = N_i x_{ih} \quad if \quad i = m,..z$$

where demand is

$$c_{ih} = s_c \left( \frac{p_i}{p_{ih}} \right) \omega_i$$

$$\left( \frac{p_i c_{ih}}{s_c \omega_i} \right)^{\frac{1}{\mu}} p_i = p_{ih}$$

In terms of $z$,

$$\Pi_{ih} = \left( \frac{p_{ih} z_{ih} T_{ih} K_{ih}^{\alpha_i} N_{ih}^{1-\alpha_i}}{s_c \omega_i} \right)^{-\frac{1}{\mu}} p_{ih} z_{ih} T_{ih} K_{ih}^{\alpha_i} N_{ih}^{1-\alpha_i} - w_t (N_{ih} + L_{ih}) - R_t (K_{ih} + Q_{ih})$$

$$\max_{\{N_{ih},K_{ih},Q_{ih},L_{ih}\}} \sum_{t=0}^{\infty} \lambda_t \frac{\Pi_{ih}}{p_t} \quad s.t. \quad (11)$$

$$z_{ih+1} T_{ih+1} = (\eta_t) A_i z_{ih}^{\lambda_i} T_{ih}^{\mu_i} (Q_{ih}^{\alpha_i} L_{ih}^{1-\alpha_i})^{\mu_i} + (1 - \delta) z_{ih} T_{ih}$$
$$Y_{ih} = N_i c_{ih} \quad if \quad i = 1,..m - 1$$
$$Y_{ih} = N_i x_{ih} \quad if \quad i = m,..z$$

We know that if $z_{ih0} = 1$ then there is a "constant" solution to this problem. Let $s_{ih} = z_{ih}/z_{ih0}$, so $s_{ih} z_{ih0} = z_{ih}$. Rewrite again in terms of $s$:

$$\Pi_{ih} = \left( \frac{p_{ih} s_{ih} z_{ih0} T_{ih} K_{ih}^{\alpha_i} N_{ih}^{1-\alpha_i}}{s_c \omega_i} \right)^{-\frac{1}{\mu}} p_{ih} s_{ih} z_{ih0} T_{ih} K_{ih}^{\alpha_i} N_{ih}^{1-\alpha_i} - w_t (N_{ih} + L_{ih}) - R_t (K_{ih} + Q_{ih})$$

$$\max_{\{N_{ih},K_{ih},Q_{ih},L_{ih}\}} \sum_{t=0}^{\infty} \lambda_t \frac{\Pi_{ih}}{p_t} \quad s.t. \quad (12)$$
\[
\max_{\{N_{iht}, K_{iht}, Q_{iht}, L_{iht}\}} \sum_{t=0}^{\infty} \lambda_t \Pi_{iht} / p_{ct} \quad \text{s.t.}
\]
\[s_{ht+1}z_{h0}T_{it+1} = (\eta_t) A_i z_{iht}^{\psi_i} s_{ht} T_{it}^\psi_i \left( Q_{iht}^\alpha L_{iht}^{1-\alpha} \right)^\psi_i + (1 - \delta_T) s_{ht} z_{h0} T_{it}\]
\[Y_{iht} = N_{iht} c_{iht} \quad \text{if} \quad i = 1, m - 1\]
\[Y_{iht} = N_{iht} x_{jht} \quad \text{if} \quad i = m, z\]

Now define \(\tilde{T}_{it} = T_{it} z_{h0}\), and rewrite again.

\[
\Pi_{iht} = \left( \frac{p_{it} s_{ht} \tilde{T}_{it}^\psi_i N_{iht}^{1-\alpha}}{s_{ht} \omega_i} \right)^{-\frac{1}{\mu}} p_{it} s_{ht} \tilde{T}_{it}^\psi_i N_{iht}^{\alpha} - w_t (N_{iht} + L_{iht}) - R_t (K_{iht} + Q_{iht})
\]

\[
\max_{\{N_{iht}, K_{iht}, Q_{iht}, L_{iht}\}} \sum_{t=0}^{\infty} \lambda_t \Pi_{iht} / p_{ct} \quad \text{s.t.}
\]
\[s_{ht+1} \tilde{T}_{it+1} = (\eta_t) s_{ht} \tilde{A}_{ih} \tilde{T}_{it}^\psi_i \left( Q_{iht}^\alpha L_{iht}^{1-\alpha} \right)^\psi_i + (1 - \delta_T) s_{ht} \tilde{T}_{it}\]
\[Y_{iht} = N_{iht} c_{iht} \quad \text{if} \quad i = 1, m - 1\]
\[Y_{iht} = N_{iht} x_{jht} \quad \text{if} \quad i = m, z\]

where
\[
\tilde{A}_{ih} = A_i \frac{z_{iht}}{z_{h0}^\psi_i}
\]

So now this is exactly the same as problem (8). The only difference is that the firm has its own \(A_{ih}\). However, given the growth rate of \(\tilde{T}_{it}\) (which is exogenous and equal to that of \(T_{it}\)), we know that there exists a solution to this problem where \(s_{ht} = 1\) at all dates – in other words, where \(z_{ht} = z_{h0}\) at all dates.

### 2.3 Step 3: Necessary conditions for a distribution of firms to behave the same as in the symmetric equilibrium

Now suppose that it is not one firm that has \(z_{h0} \neq 1\), but a positive mass of firms (or all of them). We require the distribution of \(z_{h0}\) to be such that it generates the same aggregate behavior as in the symmetric equilibrium. We already showed that, if that is true, then there is a solution to the firm’s optimization problem whereby firms keep \(z_{ht}\) constant over time. Their R&D intensity will be the same as under symmetry, as will their TPF growth rates.

What has not been shown is whether one can aggregate these firms to yield the same aggregate behavior as under symmetry. In particular, we need to show that the firms will produce the same quantities of each good, and have the same aggregate
employment. What we now do is construct the distribution (or distributions) that satisfy that property.

We know from the consumer’s problem that

\[ c_i = \left( \int_0^1 c_{ih}^{\mu_i-1} dh' \right)^{\frac{\mu_i}{\mu_i - 1}} = c_{ih} \left[ \int (p_{ih}/p_{ih'})^{\mu_i-1} dh' \right]^{\frac{\mu_i}{\mu_i - 1}} \]  

(14)

for some arbitrary \( h \). We know that

\[ \frac{p_{iht}}{p_{jht}} = \frac{T_{iht} (1 - 1/\mu_j)}{T_{iht} (1 - 1/\mu_i)} \]  

(15)

so for two competing firms

\[ \frac{p_{iht}}{p_{ih't}} = \frac{T_{ih't}}{T_{ih't}} = \frac{z_{ih't}}{z_{ih't}} \]  

(16)

If firm \( h \) is one with \( z = 1 \), then

\[ \frac{p_{iht}}{p_{ih't}} = z_{ih't} \]  

(17)

this means that \( c_i \) and \( c_{ih} \) are the same (and the same as in the symmetric BGP by assumption). Thus, we have the necessary condition

\[ 1 = \left[ \int z_{ih't}^{\mu_i-1} dh' \right]^{\frac{\mu_i}{\mu_i - 1}} \]  

(18)

or

\[ 1 = \int z_{ih't}^{\mu_i-1} dh' \]  

(19)

so any distribution that satisfies this condition will do. So, for instance, if \( \tilde{z} \) is distributed according to \( F(\tilde{z}) \) and \( E(\tilde{z}) = 1 \), then \( z \tilde{F}(z^{\mu_i-1}) \) would work.

The next question is about whether we get the same labor allocations. Showing that \( N_{iht} = N_i \int n_{ih't}dh \) would do. We just need to know how \( n \) depends on \( z \).

In problem (13), when the solution is \( s = 1 \), \( z \) only shows up in the value of \( A_{ih} \) and in the fact that the value of \( \tilde{T}_{i0} \) will be different. How does \( A_{ih} \) affect \( n \)? How does \( \tilde{T}_{i0} \) matter?

Essentially, the difference between two firms is two productivity level constants.

If there were no R&D, and firms have \( T_i \) and \( T_j \) in a competitive environment then relative labor is relative \( T \).
Setting $s = 1,$

$$
\Pi_{iht} = \left( \frac{p_i t \tilde{T}_it n_{ih} t k^{\alpha} N}{s \omega_i} \right)^{-\frac{1}{\mu}} p_i s t \tilde{T}_it k^{\alpha} n_{ih} t N - \left[ w_t \left( 1 + \frac{L_{ih} t}{N} \right) + R_t \left( 1 + \frac{L_{ih} t}{N} \right) k \right] n_{ih} t N
$$

(20)

$$
\max_{\{N_{ih}, K_{ih}, Q_{ih}, L_{ih} \}} \sum_{t=0}^{\infty} \lambda_t \Pi_{iht} / p_{ct} \quad \text{s.t.}
$$

$$
\tilde{T}_{it+1} = \tilde{A}_{ih} \tilde{T}_{it}^\rho \left( k^{\alpha} n_{ih} t \left[ \frac{L_{ih} t}{n_{ih} t} \right] \psi_i \right) + (1 - \delta_T) \tilde{T}_it
$$

$$
Y_{ih} = N_t c_{ih} \quad \text{if} \quad i = 1, \ldots, m - 1
$$

$$
Y_{ih} = N_t x_{ih} \quad \text{if} \quad i = m, \ldots, z
$$

$$
\tilde{A}_{ih} = A_i \tilde{T}_{ih}^{\kappa_i} \tilde{T}_{it} = T_{ih} z_{ih_0}
$$

$$
\tilde{T}_{it+1} = A_i \tilde{T}_{ih}^{\kappa_i} \tilde{T}_{it}^\rho \left( k^{\alpha} n_{ih} t \left[ \frac{L_{ih} t}{n_{ih} t} \right] \psi_i \right) + (1 - \delta_T) \tilde{T}_it
$$

Write

$$
\tilde{T}_{it+1} x = A_i \tilde{T}_{ih}^{\kappa_i} \left( \tilde{T}_{it} x \right)^\rho \left( k^{\alpha} n_{ih} t \left[ \frac{L_{ih} t}{n_{ih} t} \right] \psi_i \right) + (1 - \delta_T) \tilde{T}_it x
$$

which is

$$
\tilde{T}_{it+1} = \left( \frac{A_i \tilde{T}_{ih}^{\kappa_i} \left( \tilde{T}_{it} x \right)^\rho}{x} \right) \left( k^{\alpha} n_{ih} t \left[ \frac{L_{ih} t}{n_{ih} t} \right] \psi_i \right) + (1 - \delta_T) \tilde{T}_it
$$

Let $\tilde{T}_{it} = \tilde{T}_{it} z_{ih_0}^{1 - \rho_i} = T_{ih} z_{ih_0}^{1 - \rho_i}$

WTS

$$
\tilde{T}_{it+1} = A_i \tilde{T}_{it}^{\rho_i} \left( k^{\alpha} n_{ih} t \left[ \frac{L_{ih} t}{n_{ih} t} \right] \psi_i \right) + (1 - \delta_T) \tilde{T}_it
$$

$$
\tilde{T}_{i+1} z_{ih_0}^{1 - \rho_i} = A_i \tilde{T}_{i}^{\rho_i} z_{ih_0}^{1 - \rho_i} \left( k^{\alpha} n_{ih} t \left[ \frac{L_{ih} t}{n_{ih} t} \right] \psi_i \right) + (1 - \delta_T) \tilde{T}_{ih_0}^{1 - \rho_i}
$$

$$
\tilde{T}_{i+1} = A_i \tilde{T}_{i}^{\rho_i} z_{ih_0}^{1 - \rho_i} \left( k^{\alpha} n_{ih} t \left[ \frac{L_{ih} t}{n_{ih} t} \right] \psi_i \right) + (1 - \delta_T) \tilde{T}
$$
\[
\begin{align*}
A_i z_{h0}^{-\frac{\kappa_i - \rho_i}{1-\rho_i}} &= A_i z_{h0}^{-\frac{\kappa_i - \rho_i}{1-\rho_i} + \frac{\kappa_i - \rho_i}{1-\rho_i}} \\
&= A_i z_{h0}^{-\frac{\kappa_i - \rho_i}{1-\rho_i} (\mu - 1)} \\
&= A_i z_{h0}^\kappa_i = \tilde{A}_{ih}
\end{align*}
\]

So, this is equivalent to having firms with the same \(A\) but with different levels of \(T\) given by \(z_{h0}^{1-\rho_i}\).

Labor would be allocated across these firms according to \(z_{h0}^{1-\rho_i}\), so for the same allocations to hold in equilibrium for each industry you would need

\[
\int z_{h0}^{1-\rho_i} dh = 1 \tag{21}
\]

also

\[
1 = \int z_{ih/t}^{\mu_i - 1} dh' \tag{22}
\]

from before.

As a result, in any equilibrium satisfying (21) and (22), R&D intensity and TFP growth will be the same since, in effect, industries are composed of firms that vary only in terms of \(Z_i\).

### 3 Candidate distributions

In this section we examine several candidate distributions. Finding a distribution that satisfies conditions (21) and (22) is sufficient to demonstrate that the set of asymmetric equilibria defined above is non-empty. We also examine several distributions that do not work.

The candidate distributions will have to be distributions with at least two parameters, one for each condition. Also they can’t be negative.

In what follows, I use a change of variables. Instead of (21) and (22), let \(x = z_{ih/t}^{\mu_i - 1}\). I look for distributions \(f\) such that

\[
1 = \int x f(x) \, dx \tag{23}
\]

and

\[
\int x^{\frac{1-\rho_i}{(\mu_i - 1)}} f(x) \, dx = 1. \tag{24}
\]
This is a lot easier because \( \int x f(x) \, dx \) is just the mean of \( x \).

First, if \( \frac{1-\kappa_i}{(1-\rho_i)(\mu_i-1)} = 1 \), it is clear that any distribution works. Hence, in what follows, we assume a generic parameterization, i.e. \( \frac{1-\kappa_i}{(1-\rho_i)(\mu_i-1)} \neq 1 \).

### 3.1 A Pareto distribution works if \( \alpha > 1 \)

\[ g(x) = \frac{k x^k}{x^{k+1}} : \text{mean} \frac{k x_m}{k-1} . \]
Both have to be positive. For any one, there is the other that gives the mean of one. For example, given \( k \), \( x_m = \frac{k-1}{k} \), so \( k > 1 \) is necessary.

Let \( \alpha = \frac{1-\kappa_i}{(1-\rho_i)(\mu_i-1)} \), so we want

\[
\int_{x_m}^\infty x^\alpha \frac{k x^k}{x^{k+1}} \, dx = \int_{x_m}^\infty \frac{k x_m^k}{x^{k+1-\alpha}} \, dx
\]

\[
= \int_{x_m}^\infty \frac{k x_m^k}{x^{k+1-\alpha}} \, dx
\]

\[
= \int_{x_m}^\infty x_m^a (k-a) \frac{k x_m^{k-a}}{x^{k+1-\alpha}} \, dx
\]

\[
= \frac{x_m^a k}{(k-a)} \int_{x_m}^\infty (k-a) \frac{x_m^{k-a}}{x^{k+1-\alpha}} \, dx
\]

\[
= \frac{x_m^a}{(k-a)}
\]

Since this has to be one, we need

\[
\frac{x_m^a}{(k-a)} = 1
\]

\[
\left( \frac{k-1}{k} \right)^a = (k-a)
\]

or

\[
a \log \left( \frac{k-1}{k} \right) = \log (k-a)
\]

Given \( a \), is there a \( k \) that gives this? The LHS goes from \(-\infty\) to zero for \( k > 1 \), and is increasing in \( j \). On the other hand, the RHS is increasing in \( k \) from \( \log (1-a) \). This works provided (it seems) \( \alpha > 1 \), or \( \frac{1-(1-\rho_i)(\mu_i-1)}{\rho_i} > A_i \) if \( \rho_i > 0 \), or \( \frac{1-(1-\rho_i)(\mu_i-1)}{\rho_i} < A_i \) if \( \rho_i < 0 \). So for industries with positive rho, you need markups to be below a certain point, or rho to be big enough.
3.2 A truncated lognormal distribution works if $\alpha < 1$

Suppose $g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\log(x) - \mu)^2}{2\sigma^2}} :$ mean $= e^{\mu + \sigma^2/2} - F(\bar{x})$, so for any $\mu, \bar{x}$ there is a $\sigma$ and vice versa. Let $G$ be the cdf.

Let $\alpha = \frac{1 - \kappa_i}{(1 - \rho_i)(\mu_i - 1)}$, so we want

\[
\int_0^{G(\bar{x})} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\log(x) - \mu)^2}{2\sigma^2}} \, dx
\]

Let $z = x^{1-\alpha}$, $\tilde{z} = \bar{x}^{1-\alpha}$

\[
\int_0^{\tilde{x}} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\log(z) - \mu)^2}{2\sigma^2}} \, dx = \int_0^{\tilde{x}} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(1 - \tilde{x}^{1-\alpha} - \mu)^2}{2\sigma^2}} \, dx
\]

\[
= \int_0^{\tilde{x}} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\log(z) - \mu(1-\alpha))}{2\sigma^2(1-\alpha)}} \, dx
\]

\[
= \int_0^{\sqrt{1-\alpha}} \tilde{z} \sqrt{1-\alpha} \sqrt{2\pi} e^{-\frac{(\log(z) - \mu(1-\alpha))}{2\sigma^2(1-\alpha)}} \, dx
\]

\[
= \tilde{G}(\tilde{x}) \int_0^{\sqrt{1-\alpha}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\log(z) - \mu(1-\alpha))}{2\sigma^2(1-\alpha)}} \, dx
\]

\[
= \sqrt{1-\alpha} \tilde{G}(\tilde{x})
\]

\[
\tilde{G}(\tilde{x}) = \frac{\sqrt{1-\alpha} \tilde{G}(\tilde{x})}{\tilde{G}(\tilde{x})}
\]

note that $\tilde{G}$ is not the same as $G$ because they are computed using different distributions. The mean and s.d. for $G$ is $(\mu, \sigma)$, whereas for $\tilde{G}$ they are $\left(\mu(1-\alpha), \sigma \sqrt{1-\alpha}\right)$. In fact $\tilde{G}$ is the same distribution computed for $(1 - \alpha) \log(x)$ instead of $x$. Hence, if $\alpha < 1$, the distribution for $(1 - \alpha) \log(x)$ will be less spread out than $x$, compressed to the left, and so $\tilde{G}(\tilde{x}) > G(x)$.

So, can we construct it? Any two parameters pin down the third because the mean is one. The second condition depends on all parameters because $G$ does.

FIRST CONDITION: Given $\bar{x}$ and $\sigma$, if you raise $\mu$ you raise the mean. If you lower $\bar{x}$ you lower the mean.

SECOND: Given $\mu$ and $\sigma$, you can pick $\bar{x}$ so that the end works, starting with infinity and lowering it. What does changing $\mu$ do to the second condition? Raising $\mu$ lowers $\frac{\sqrt{1-\alpha} \tilde{G}(\tilde{x})}{\tilde{G}(\tilde{x})}$:
\[ G(x) = 0.5 + 0.5 \text{erf} \left( \frac{\log x - \mu}{\sigma \sqrt{2}} \right) \]
\[ \frac{d \text{erf}}{dz} = \frac{2}{\sqrt{\pi}} e^{-z^2} \]
\[ G_\mu(x) = -\frac{1}{\sigma \sqrt{2\pi}} e^{-\left( \frac{\log x - \mu}{\sigma \sqrt{2}} \right)^2} < 0 \]

Evaluate this at \( \bar{x} \).

\[ \tilde{G}(x) = 0.5 + 0.5 \text{erf} \left( \frac{(1 - \alpha) \log(x) - \mu}{\sigma \sqrt{2}} \right) \]
\[ \frac{d \text{erf}}{dz} = \frac{2}{\sqrt{\pi}} e^{-z^2} \]
\[ \tilde{G}_\mu(x) = -\frac{1}{\sigma \sqrt{2\pi}} e^{-\left( \frac{(1-\alpha) \log(x) - \mu}{\sigma \sqrt{2}} \right)^2} < 0 \]

Suppose \((1 - \alpha) \log(\bar{x}) > \mu\).

\[ e^{-\left( \frac{(1-\alpha) \log(x) - \mu}{\sigma \sqrt{2}} \right)^2} > e^{-\left( \frac{\log x - \mu}{\sigma \sqrt{2}} \right)^2} \]
\[ (1 - \alpha) < 1 \]

So \( \tilde{G} \) is lowered more than \( G \) by \( \mu \). What of \( \bar{x} \)?

\[ \tilde{G}(x) = 0.5 + 0.5 \text{erf} \left( \frac{(1 - \alpha) \log(x) - \mu}{\sigma \sqrt{2}} \right) \]
\[ \frac{d \text{erf}}{dz} = \frac{2}{\sqrt{\pi}} e^{-z^2} \]
\[ \tilde{G}_\bar{x}(x) = \frac{(1 - \alpha)}{x \sigma \sqrt{2 \pi}} e^{-\left( \frac{(1-\alpha) \log(x) - \mu}{\sigma \sqrt{2}} \right)^2} \]

So \( \tilde{G}_\bar{x}(x) > 0 \) and \( \tilde{G}_\bar{x}(x) > G_\bar{x}(x) \). So raising \( \bar{x} \) raises \( \frac{1 - \alpha \tilde{G}(\bar{x})}{G(\bar{x})} \). So for all \( \sigma \) there should be a pair of \( \mu, \bar{x} \) that works.
4 Alternative calibrations

Now we turn to the problem of calibrating the model. There are two factors to which we wish to assess robustness. First, our calibration of $\gamma_x$ assumes that $g_q = 1.026^{-1}$. This value assumes (as in Cummins and Violante 2002) that there is no quality adjustment for goods. However, Bils and Klenow (2000) find that estimates of the quality bias in the CPI are around 0.6% per year. This suggests that $g_q = 1.02^{-1}$.

Second, the relative prices used to calibrate the model are all prices of gross output, whereas the model is a model of value added. Ngai and Samaniego (2008b) argue that, while there is a correspondence between a multisector model of value added and of gross output, this correspondence is such that gross output prices do not map into value-added TFP growth rates. Rather, these relative prices need to be raised to the power $\frac{1}{1-\alpha_m}$, where $\alpha_m$ is the intermediate goods share of gross output. This implies that $g_q = 1.02^{-1/(1-\alpha_m)}$.

We set $\alpha_m = 0.45$, as suggested by Yamano and Ahmad (2006) and by the input-output tables of the US Bureau of Economic Analysis. Then, $g_q = 1.0334$. Interestingly, with this value of $g_q$, $g_x = 1.0389$ and $g_c = 1.0054$. $\rho_x = 0.78$, and $\rho_c = -0.58$. These numbers are very close to those used in the baseline calibration.

What changes more significantly is the fact that the relative prices of capital used to calibrate $g_i$ for durables industries must also be raised to the power $\frac{1}{1-\alpha_m}$. Table A1 shows that the resulting values of $\gamma_i$ are generally higher, as are the corresponding values of $\rho_i$. 

11
Table A1 – TFP growth rates across capital goods, based on the quality-adjusted relative price of capital from Cummins and Violante (2002) ($\gamma^C V_i$).

Values of $\rho_i$ are based on $\gamma^C V_i$, assuming no cross-industry spillovers, and assuming benchmark values of parameters.

Tables A2 and A3 look at how these values affect R&D in the model, as well as the relative magnitude of $\kappa_i$ and $\sigma_i$. Model R&D changes very little, whereas the planner’s R&D increases more significantly, so that the difference between them widens. Given the low levels of appropriability in the patent data, most of the increase in $\rho_i$ over the original calibration is in terms of $\sigma_i$ rather than $\kappa_i$, and hence represents an uninternalized spillover. Consequently, optimal subsidies increase by about 10% across the board.
<table>
<thead>
<tr>
<th>Capital good sector</th>
<th>$\rho_i$</th>
<th>$A_i$</th>
<th>$\kappa_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computers and office equipment</td>
<td>0.97</td>
<td>0.16</td>
<td>0.16</td>
</tr>
<tr>
<td>Communication equipment</td>
<td>0.95</td>
<td>0.16</td>
<td>0.15</td>
</tr>
<tr>
<td>Aircraft</td>
<td>0.95</td>
<td>0.19</td>
<td>0.18</td>
</tr>
<tr>
<td>Instruments and photocopiers</td>
<td>0.93</td>
<td>0.17</td>
<td>0.16</td>
</tr>
<tr>
<td>Fabricated metal products</td>
<td>0.87</td>
<td>0.21</td>
<td>0.17</td>
</tr>
<tr>
<td>Autos and trucks</td>
<td>0.87</td>
<td>0.19</td>
<td>0.18</td>
</tr>
<tr>
<td>Electrical transm. distrib. and industrial appl.</td>
<td>0.87</td>
<td>0.16</td>
<td>0.14</td>
</tr>
<tr>
<td>Other Durables</td>
<td>0.86</td>
<td>0.19</td>
<td>0.16</td>
</tr>
<tr>
<td>Ships and boats</td>
<td>0.84</td>
<td>0.16</td>
<td>0.13</td>
</tr>
<tr>
<td>Electrical equipment, n.e.c.</td>
<td>0.83</td>
<td>0.22</td>
<td>0.15</td>
</tr>
<tr>
<td>Machinery</td>
<td>0.82</td>
<td>0.18</td>
<td>0.18</td>
</tr>
<tr>
<td>Mining and oilfield machinery</td>
<td>0.78</td>
<td>0.34</td>
<td>0.27</td>
</tr>
<tr>
<td>Furniture and fixtures</td>
<td>0.76</td>
<td>0.13</td>
<td>0.10</td>
</tr>
<tr>
<td>Structures</td>
<td>0.71</td>
<td>0.12</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Table A2 – Receptivity $\rho_i$ from Table A1 and appropriability $A_i$ based on the NBER patent citation database. Values of $\rho_i$ are higher than in the benchmark calibration but, given that $A_i$ is small, in absolute terms most of this is due to an increase in $\sigma_i$ rather than $\kappa_i$. 
<table>
<thead>
<tr>
<th>Capital good sector</th>
<th>Model</th>
<th>Planner</th>
<th>Ratio</th>
<th>Subs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computers and office equipment</td>
<td>23.7</td>
<td>67.2</td>
<td>35.3</td>
<td>84.8</td>
</tr>
<tr>
<td>Communication equipment</td>
<td>22.0</td>
<td>53.0</td>
<td>41.5</td>
<td>75.0</td>
</tr>
<tr>
<td>Aircraft</td>
<td>22.1</td>
<td>51.0</td>
<td>43.4</td>
<td>72.7</td>
</tr>
<tr>
<td>Instruments and photocopiers</td>
<td>20.6</td>
<td>43.6</td>
<td>47.3</td>
<td>66.5</td>
</tr>
<tr>
<td>Fabricated metal products</td>
<td>17.8</td>
<td>30.5</td>
<td>58.3</td>
<td>50.7</td>
</tr>
<tr>
<td>Autos and trucks</td>
<td>17.9</td>
<td>30.2</td>
<td>59.2</td>
<td>49.7</td>
</tr>
<tr>
<td>Electrical transm. distrib. and industrial appl.</td>
<td>17.4</td>
<td>30.0</td>
<td>57.9</td>
<td>50.9</td>
</tr>
<tr>
<td>Other durables</td>
<td>17.2</td>
<td>28.6</td>
<td>60.2</td>
<td>48.0</td>
</tr>
<tr>
<td>Ships and boats</td>
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<td>26.7</td>
<td>61.4</td>
<td>46.1</td>
</tr>
<tr>
<td>Electrical equipment, n.e.c.</td>
<td>16.2</td>
<td>25.6</td>
<td>63.2</td>
<td>43.9</td>
</tr>
<tr>
<td>Machinery</td>
<td>16.0</td>
<td>24.4</td>
<td>65.7</td>
<td>40.8</td>
</tr>
<tr>
<td>Mining and oilfield machinery</td>
<td>15.6</td>
<td>21.3</td>
<td>73.1</td>
<td>31.8</td>
</tr>
<tr>
<td>Furniture and fixtures</td>
<td>13.7</td>
<td>19.5</td>
<td>69.9</td>
<td>34.8</td>
</tr>
<tr>
<td>Structures</td>
<td>12.5</td>
<td>16.9</td>
<td>73.6</td>
<td>29.9</td>
</tr>
</tbody>
</table>

Table A3 – R&D intensity in the decentralized model and the planner’s solution. The third column is the ratio of model R&D to the planner’s. The fourth is the subsidy rate $h_i$. All values are percentages. Model R&D intensity is almost the same as in the benchmark calibration. However, the planner’s R&D rises and optimal subsidies increase by roughly 10% points.

5  Concluding remarks

We showed that a class of asymmetric equilibria exists such that industry TFP and R&D comparisons are the same as in a symmetric equilibrium. This demonstrates that symmetry per se is not essential for the results of the paper. We also checked the robustness of the quantitative results to an alternative calibration strategy, incorporating the concern raised by Ngai and Samaniego (2008b) regarding the appropriate calibration of TFP growth rates using prices in a multisector environment.

6  References
