Young Measures in a Nonlocal Phase Transition Problem

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Abstract

A nonlocal variational problem modelling phase transitions is studied in the framework of Young measures. The existence of global minimisers among functions with internal layers on an infinite tube is proved by combining a weak convergence result for Young measures and the principle of concentration-compactness. The regularity of such global minimisers is discussed, and the nonlocal variational problem is also considered on asymptotic tubes.

1 Introduction

We are concerned with a variational problem modelling phase transitions where a nonlocal term is involved. We assume that u is a phase-field parameter defined in an unbounded tube Ω in R^d. The values $u=-1$ and $u=1$ represent two configurations of a perfect crystal and $u \in (-1,1)$ represents a mixture. We associate with u its free energy

$$E(u) = \frac{1}{4}\int_\Omega \int_\Omega J(x-y)(u(x)-u(y))^2 dxdy + \int_\Omega W(u(x))dx. \tag{1.1}$$

The function W in the second part of (1.1) is a balanced double-well potential (see Figure 1) describing the bulk-energy density. It penalises u's that take values other than -1 or 1.

The double integral in (1.1) characterises u's self interaction energy. The kernel J is a nonnegative function with $J(-\xi)=J(\xi)$ and is normalised so that $\int_{R^d} J(\xi)d\xi = 1$. The size of $J(x-y)$ measures the intensity of interaction between sites x and y. It takes both short and long range interaction into account, and therefore the double integral

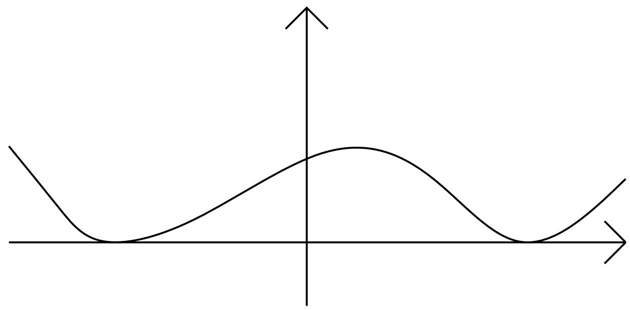

Figure 1: The graph of W.

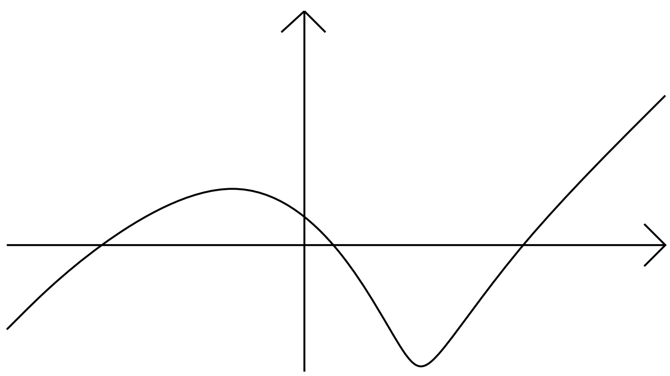

Figure 2: The graph of f.

differs from the traditional gradient model, which describes the interaction energy by $\int_\Omega |\nabla u|^2 dx$, in that the gradient model ignores interaction at a distance. The dependence of J on x and y through $x-y$ suggests that the material is homogeneous. Since we do not require that $J(\xi)$ depends only on $|\xi|$, the material is allowed to be anisotropic.

We refer to Bates, Fife, Ren and Wang **[4]** for more about this model. The Euler-Lagrange equation of (1.1) is

$$\int_\Omega J(x-y)u(y)dy - K(x)u(x) - f(u(x)) = 0 \text{ in } \Omega \tag{1.2}$$

as calculated in Fife **[14]** where $K(x) = \int_\Omega J(x-y)dy$ and $f(u) = W'(u)$ (see Figure 2). In this paper we study (1.2) by directly minimising the energy functional (1.1). Being a minimisation problem on an unbounded domain, our work involves the concentration-compactness principle of Lions **[17]**. We require that u is close to -1 at one end of the tube Ω and close to 1 at the other end so that a transition of phase occurs somewhere in Ω. The precise meaning of this boundary condition at infinity is given in (2.10). The nonlinear integral equation (1.2) differs from the standard Allen-Cahn equation

$$\begin{cases} \Delta u - f(u) = 0 \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases}$$

which is often used to model phase transitions, in that the first term of (1.2) is nonlocal. This nonlocal term together with $-K(x)u(x)$ comes from the double integral term in (1.1).

The main difficulty of our approach with regard to the existence question arises from the lack of pointwise convergence of minimising sequences. It is easy to show that a sequence $\{u_n\}$ of bounded functions with more and more rapid oscillations can have uniformly bounded energy. Therefore a minimising sequence does not have to converge pointwise and its weak$-*$ limit in $L^\infty(\Omega)$ is not necessarily an energy minimiser. To overcome this problem we study (1.1) in a much wider class of functions. We allow measure-valued functions to enter the admissible set. Such generalised functions are known as families of Young measures. In this new class the energy attains its minimum for a measure-valued function. A key fact is the following: Given a sequence of bounded measure-valued functions whose energy converges to a limit we can find another measure-valued function whose energy is this limit. We utilise this fact together with the concentration-compactness principle to prove the existence of global minimisers in "thin" periodic tubes.

Furthermore we discuss the regularity of the measure-valued solutions under various conditions on J and W. Since our solutions are measure-valued functions the first question to ask is whether they are "usual" functions. If yes, are they continuous? We show that sometimes the global minimisers are usual functions, and sometimes the global minimisers cannot be continuous, should they be usual functions. Among others, we consider the case where $W(a) + K(x)a^2/2$ as a function in a is a balanced double-well potential which achieves its minimum at exactly two points $p_c(x)$ and $q_c(x)$ with $-1 < p_c(x) < q_c(x) < 1$ and is strictly convex on $[-1, p_c(x)]$ and $[q_c(x), 1]$. Then we prove the following regularity results. The support of a Young measure minimiser has empty intersection with the open interval $(p_c(x), q_c(x))$ for a.e. $x \in \overline{\Omega}$. This implies that a minimiser which is a usual function cannot be continuous on $\overline{\Omega}$. Furthermore, for a.e. $x \in \overline{\Omega}$ a Young measure minimiser is either a Dirac mass or it satisfies

$$\mu_x = \theta(x)p_c(x) + (1 - \theta(x))q_c(x), \qquad \theta(x) \in (0, 1).$$

These results can be found in Propositions 4.2, 4.5 and Corollaries 4.3, 4.4. Previous results in this direction have been obtained only in the case where the kernel $J(x-y)$ represents the Green's function of a boundary-value problem. Two papers in this regards are **[7]** and **[15]**. Many questions concerning the regularity of global minimisers are still open.

Finally we indicate how to prove existence of global minimisers for "fat" tubes. We also discuss tubes that are only asymptotically periodic, and give a sufficient condition for existence.

Let us comment on some related work. In **[4]** travelling waves in R^1 of the gradient flow of (1.1) are studied for not necessarily balanced W, meaning that the two wells of W may have different depths, via a homotopy argument. One result there says that in the case when W is balanced (1.2) has a stable solution in R^1 connecting -1 to 1. The argument relies on the monotonicity of the solutions and cannot be generalised to treat less regular domains in R^d.

Bates *et al* approached (1.2) by truncation and formal approximation by higher order differential, hence local, equations. Though the higher order local equations can be studied via geometric singular perturbation (**[2, 3, 16]**), the concentration-compactness principle (**[5, 6]**), and the deformation method (**[19]**), the convergence to the nonlocal equation remains open.

Other nonlocal problems in the literature include **[8, 20]** of Brandon *et al* who studied ferromagnetic materials with the help of measure-valued functions, and **[9, 10, 11, 12, 18]** of De Masi *et al* who studied a phase-transition problem derived from statistical mechanics, for which stable travelling waves are found by a perturbation argument.

The paper is organised as follows: in section 2 we provide some facts about measure-valued functions and about the admissible class for our variational problem. In section 3 we prove existence in thin tubes. In section 4 we discuss the regularity of measure-valued minimisers. Finally in section 5 we indicate how to prove existence in fat or asymptotically periodic tubes.

We use C to denote a generic constant that may change from line to line. The domain Ω is always assumed smooth. We often do not explicitly mention passing to subsequences.

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2 Preliminaries

We assume throughout this paper that J and W satisfy

H-1 J is bounded, Borel measurable and nonnegative on R^d with $J(-\xi) = J(\xi)$, $\int_{R^d} J(\xi)d\xi = 1$ and $J(\xi) > c > 0$ when $|\xi| < \eta$ for some $\eta > 0$.

H-2 W is a C^2 function that has exactly two global minimisers at -1 and 1 where $W(-1) = W(1) = 0$, and $W''(-1) > 0$, $W''(1) > 0$.

Definition 2.1 A *measure-valued function* ν with range $[-1, 1]$ is a family of probability measures $\{\nu_x\}$ on $[-1, 1]$ parametrised by $x \in \overline{\Omega}$ such that for each Borel set $E \subset [-1, 1]$ $\nu_x(E)$ is Borel measurable with respect to $x \in \overline{\Omega}$.

General references for measure-valued functions include **[1, 13, 21, 22]**. A measure-valued function is a usual function if $\nu_x = \delta(u(x))$ for some Borel measurable function u. Here $\delta(u)$ denotes the Dirac delta function with its support at the point u. Denote the set of all the measure-valued functions with range $[-1, 1]$ by $\mathcal{M}(\overline{\Omega})$.

As a remark about the range $[-1,1]$ one could start with measure-valued functions with the range $(-\infty,\infty)$. For such a measure-valued function ν' define a new measure-valued function ν with range $[-1,1]$ by

$$\nu_x = \nu'_x|_{[-1,1]} + \nu'_x((-\infty,-1))\delta(-1) + \nu'_x((1,\infty))\delta(1)$$

where $\nu'_x|_{[-1,1]}$ denotes the restriction measure of ν'_x on $[-1,1]$, i.e., for each Borel set $A \subset (-\infty,\infty)$

$$\nu'_x|_{[-1,1]}(A) = \nu'_x(A \cap [-1,1]).$$

It easy to verify $E(\nu) \leq E(\nu')$, so we only consider measure-valued functions with range $[-1,1]$.

Lemma 2.2 *On every open set $\Omega \subset R^d$ if $\{\nu_n\} \subset \mathcal{M}(\overline{\Omega})$, there exists $\nu \in \mathcal{M}(\overline{\Omega})$ such that along a subsequence of $\{\nu_n\}$ again labeled by n*

$$\lim_{n\to\infty}[\int_{\overline{\Omega}^2}\int_{[-1,1]^2} F(x,y,a,b)d\nu_{n,x}d\nu_{n,y}dxdy + \int_{\overline{\Omega}}\int_{[-1,1]} G(x,a)d\nu_{n,x}dx]$$

$$= \int_{\overline{\Omega}^2}\int_{[-1,1]^2} F(x,y,a,b)d\nu_x d\nu_y dxdy + \int_{\overline{\Omega}}\int_{[-1,1]} G(x,a)d\nu_x dx$$

for all continuous F defined on $\overline{\Omega}^2 \times [-1,1]^2$ and for all continuous G defined on $\overline{\Omega} \times [-1,1]$, both with compact supports.

Note that we write $d\nu_{n,x}$ for $d\nu_{n,x}(a)$ and $d\nu_{n,y}$ for $d\nu_{n,y}(b)$ suppressing a and b.

Proof. Define measures σ_n on $\overline{\Omega} \times [-1,1]$ by

$$\sigma_n(E) = \int_{\overline{\Omega}} \nu_{n,x}(\{a : (x,a) \in E\})dx$$

for each Borel set $E \subset \overline{\Omega} \times [-1,1]$. Note that $\{a : (x,a) \in E\}$ is the x-slice of E. Clearly $\{\sigma_n\}$ is a sequence of Radon measures uniformly bounded on each compact subset of $\overline{\Omega} \times [-1,1]$. Then there exists a Radon measure σ on $\overline{\Omega} \times [-1,1]$ such that for each continuous function G on $\overline{\Omega} \times [-1,1]$ with compact support

$$\lim_{n\to\infty}\int_{\overline{\Omega}}\int_{[-1,1]} G(x,a)d\nu_{n,x}dx = \lim_{n\to\infty}\int_{\overline{\Omega}\times[-1,1]} Gd\sigma_n = \int_{\overline{\Omega}\times[-1,1]} Gd\sigma.$$

Applying Theorem 10 and the second step of the proof of Theorem 11 in **[13]**, Chapter 1, we find a measure-valued function ν such that

$$\int_{\overline{\Omega}\times[-1,1]} Gd\sigma = \int_{\overline{\Omega}}\int_{[-1,1]} G(x,a)d\nu_x dx.$$

Therefore

$$\lim_{n\to\infty}\int_{\overline{\Omega}}\int_{[-1,1]} G(x,a)d\nu_{n,x}dx = \int_{\overline{\Omega}}\int_{[-1,1]} G(x,a)d\nu_x dx. \tag{2.1}$$

Now take Σ_n to be the product measure $\sigma_n \times \sigma_n$. Identify $\overline{\Omega}^2 \times [-1,1]^2$ with $(\overline{\Omega} \times [-1,1])^2$. Then $\Sigma_n \to \Sigma = \sigma \times \sigma$ in the sense that for every continuous F on $\overline{\Omega}^2 \times [-1,1]^2$ with compact support

$$\lim_{n\to\infty} \int_{\overline{\Omega}^2 \times [-1,1]^2} F d\Sigma_n = \int_{\overline{\Omega}^2 \times [-1,1]^2} F d\Sigma.$$

Then by Fubini's theorem

$$\lim_{n\to\infty} \int_{\overline{\Omega}^2} \int_{[-1,1]^2} F(x,y,a,b) d\nu_{n,x} d\nu_{n,y} dx dy = \lim_{n\to\infty} \int_{\overline{\Omega}^2 \times [-1,1]^2} F d\Sigma_n$$

$$= \int_{\overline{\Omega}^2 \times [-1,1]^2} F d\Sigma = \int_{\overline{\Omega}^2} \int_{[-1,1]^2} F(x,y,a,b) d\nu_x d\nu_y dx dy,$$

which together with (2.1) proves the lemma. □

It is an easy exercise to show

Corollary 2.3 *In Lemma 2.2 F and G need only to be bounded and Borel measurable with compact support.*

For each $\nu \in \mathcal{M}(\overline{\Omega})$ set

$$< \nu_x > := \int_{[-1,1]} a \; d\nu_x, \tag{2.2}$$

which stands for the mean of ν_x. Define an energy functional

$$E(\nu) = \frac{1}{4} \int_{\overline{\Omega}^2} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy + \int_{\overline{\Omega}} \int_{[-1,1]} W(a) d\nu_x dx \tag{2.3}$$

for each $\nu \in \mathcal{M}(\overline{\Omega})$, which generalises (1.1). Note that the energy of ν is not necessarily finite. We set

$$\mathcal{A} = \{\nu : E(\nu) < \infty\} \tag{2.4}$$

so that E is a functional from $\mathcal{A}$ to the set of nonnegative real numbers.

Definition 2.4 A set $\Omega \subset R^d$ is a *tube* if there exists $K > 0$ so that $\Omega \subset R^1 \times [-K,K]^{d-1} \subset R^d$ and if there exists $T = (T^1, 0, ..., 0) \in R^d$ such that $x + kT \in \Omega$ if and only if $x \in \Omega$ for all integers k (see Figure 3 where the two ends are denoted by e_1 and e_2).

We single out a simple case when the tube is "thin" compared to J.

Definition 2.5 A set $\Omega \subset R^d$ is a *thin tube* if K in Definition 2.4 and $\eta > 0$ can be chosen so that $J(\xi) > c > 0$ for all $\xi \in [-2\eta, 2\eta] \times [-2K, 2K]^{d-1}$.

A segment $S(z^1, r)$ of Ω is defined by

$$S(z^1, r) = \{x \in \overline{\Omega} : |x^1 - z^1| < r\}, \tag{2.5}$$

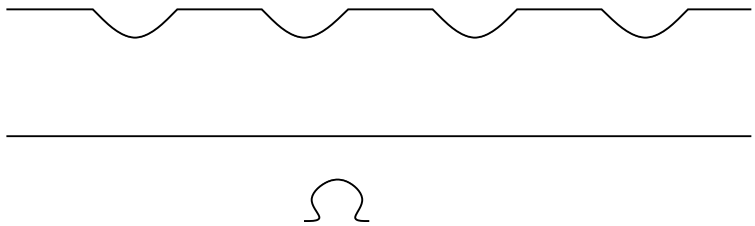

Figure 3: An example of tubes.

where $x = (x^1, x')$.

We now set

$$\widehat{\nu}(x^1) = \frac{1}{|S(x^1,\eta)|}\int_{z\in S(x^1,\eta)} <\nu_z> dz \tag{2.6}$$

for $x^1 \in R^1$. Here $\eta > 0$ is chosen such that $J(\xi) > c(\eta) > 0$ for all $|\xi| < 2\eta$, and $|S(x^1,\eta)|$ is the Lebesgue measure of $S(x^1,\eta)$. Clearly $\widehat{\nu}$ is continuous.

Lemma 2.6 *Suppose Ω is a thin tube. For every $\epsilon_1 > 0$ there is $\theta > 0$ such that for every $\nu \in \mathcal{A}$, $z^1 \in R^1$*

$$\frac{1}{4}\int_{S^2(z^1,\eta)} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_x d\nu_y dx dy + \int_{S(z^1,\eta)}\int_{[-1,1]} W(a)d\nu_x dx < \theta$$

implies that either $|\widehat{\nu}(z^1)+1| < \epsilon_1$ or $|\widehat{\nu}(z^1)-1| < \epsilon_1$.

Proof. For each $\nu \in \mathcal{A}$, $z^1 \in R^1$ we estimate $W(\widehat{\nu}(z^1))$ as follows.

$$W(\widehat{\nu}(z^1)) = \frac{1}{|S(z^1,\eta)|}\int_{S(z^1,\eta)} W(\widehat{\nu}(z^1))dx$$

$$= \frac{1}{|S(z^1,\eta)|}\int_{S(z^1,\eta)}\int_{[-1,1]} W(a+\widehat{\nu}(z^1)-a)d\nu_x dx$$

$$= \frac{1}{|S|}\int_{S(z^1,\eta)}\int_{[-1,1]}[W(a)+f(a)(\widehat{\nu}(z^1)-a)+\frac{f'(\cdot)}{2}(\widehat{\nu}(z^1)-a)^2]d\nu_x dx$$

$$\leq \frac{1}{|S|}\int_{S(z^1,\eta)}\int_{[-1,1]} W(a)d\nu_x dx + \frac{1}{|S|}[\int_{S(z^1,\eta)}\int_{[-1,1]} f^2(a)d\nu_x dx]^{1/2}\cdot$$

$$[\int_{S(z^1,\eta)}\int_{[-1,1]}(a-\widehat{\nu}(z^1))^2 d\nu_x dx]^{1/2} + C\int_{S(z^1,\eta)}\int_{[-1,1]}(a-\widehat{\nu}(z^1))^2 d\nu_x dx$$

$$\leq \frac{1}{|S|}\int_{S(z^1,\eta)}\int_{[-1,1]} W(a)d\nu_x dx + \frac{1}{|S|}[\int_{S(z^1,\eta)}\int_{[-1,1]} W(a)d\nu_x dx]^{1/2}\cdot$$

$$[\int_{S(z^1,\eta)}\int_{[-1,1]}(a-\widehat{\nu}(z^1))^2 d\nu_x dx]^{1/2} + C\int_{S(z^1,\eta)}\int_{[-1,1]}(a-\widehat{\nu}(z^1))^2 d\nu_x dx$$

$$\leq C \int_{S(z^1,\eta)} \int_{[-1,1]} W(a) d\nu_x dx + C \int_{S(z^1,\eta)} \int_{[-1,1]} (a - \widehat{\nu}(z^1))^2 d\nu_x dx \tag{2.7}$$

by Hölder's inequality and H-2. On the other hand by the definition of η and since we assume Ω to be thin

$$\int_{S^2(z^1,\eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy \geq C \int_{S^2} \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy$$

$$= C \int_{S^2} \int_{[-1,1]^2} [(a - \widehat{\nu}(z^1))^2 + (b - \widehat{\nu}(z^1))^2 - 2(a - \widehat{\nu}(z^1))(b - \widehat{\nu}(z^1))] d\nu_x d\nu_y dx dy$$

$$= C \int_{S^2(z^1,\eta)} \int_{[-1,1]^2} [(a - \widehat{\nu}(z^1))^2 + (b - \widehat{\nu}(z^1))^2] d\nu_x d\nu_y dx dy$$

$$\geq C \int_{S(z^1,\eta)} \int_{[-1,1]} (a - \widehat{\nu}(z^1))^2 d\nu_x dx.$$

In short we have

$$\int_{S(z^1,\eta)} \int_{[-1,1]} (a - \widehat{\nu}(z^1))^2 d\nu_x dx$$
$$\leq C \int_{S^2(z^1,\eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy. \tag{2.8}$$

Combing (2.8) with (2.7) we find

$$W(\widehat{\nu}(z^1)) \leq C[\frac{1}{4} \int_{S^2(z^1,\eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy$$
$$+ \int_{S(z^1,\eta)} \int_{[-1,1]} W(a) d\nu_x dx], \tag{2.9}$$

which immediately implies the lemma. $\square$

Corollary 2.7 *For each $\nu \in \mathcal{A}$ on a thin tube $\widehat{\nu}$ has limits ± 1 as $z^1 \to \pm\infty$.*

We then set

$$\mathcal{A}^1_{-1} = \{\nu \in \mathcal{A} : \lim_{x^1 \to -\infty} \widehat{\nu}(x^1) = -1 \text{ and } \lim_{x^1 \to \infty} \widehat{\nu}(x^1) = 1\}. \tag{2.10}$$

We define $\mathcal{A}^{-1}_{-1}$, $\mathcal{A}^{-1}_{1}$ and $\mathcal{A}^1_1$ in similar ways, and clearly these subclasses are mutually disjoint and

$$\mathcal{A} = \mathcal{A}^{-1}_{-1} \cup \mathcal{A}^1_{-1} \cup \mathcal{A}^{-1}_1 \cup \mathcal{A}^1_1.$$

We look for global minimisers of E in each of the four subclasses. Clearly $\nu_x = \delta(-1)$ is the only global minimum in $\mathcal{A}^{-1}_{-1}$ and $\nu_x = \delta(1)$ is the only global minimum in $\mathcal{A}^1_1$, so we only consider $\mathcal{A}^1_{-1}$ and the subclass $\mathcal{A}^{-1}_1$ can be treated in the same way.

3 Existence of Measure-valued Solutions

We now state our main theorem about the existence. We consider thin tubes (see Definition 2.5) in this section. General tubes are treated in Section 5.

Theorem 3.1 *On each thin tube Ω there exists $\mu \in \mathcal{A}^1_{-1}$ such that $E(\mu) = \inf_{\nu \in \mathcal{A}^1_{-1}} E(\nu)$.*

Let $\{\nu_n\} \subset \mathcal{A}^1_{-1}$ be a minimising sequence, i.e.,

$$\lim_{n\to\infty} E(\nu_n) = \inf_{\nu \in \mathcal{A}^1_{-1}} E(\nu). \tag{3.1}$$

Remark 3.2 *One can assume that the ν_n's are usual functions (see Theorem 1,* **[22]***).*

We will construct a minimiser from $\{\nu_n\}$. To this end we prove a few lemmas.

Lemma 3.3 *There exists $c > 0$ such that $E(\nu) > c$ for all $\nu \in \mathcal{A}^1_{-1}$.*

Proof. Suppose there exist $\nu_n \in \mathcal{A}^1_{-1}$ such that $\lim_{n\to\infty} E(\nu_n) = 0$. Then Lemma 2.6 implies that for every small $\epsilon > 0$ $|\widehat{\nu}_n(x^1) + 1| < \epsilon$ or $|\widehat{\nu}_n(x^1) - 1| < \epsilon$ for all $x^1 \in R^1$ and large n. But $\widehat{\nu}_n(x^1_n) = 0$ for some x^1_n since

$$\lim_{x^1\to-\infty} \widehat{\nu}_n(x^1) = -1 \text{ and } \lim_{x^1\to\infty} \widehat{\nu}_n(x^1) = 1.$$

This contradiction proves the lemma. □

To handle the possible loss of compactness due to the unboundedness of Ω we proceed as in **[5, 6]**. Define Lévy concentration functions Q_n associated with ν_n by

$$Q_n(t) = \sup_{z^1\in R^1} [\frac{1}{4} \int_{S^2(z^1,t)} J(x-y)$$

$$\int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy + \int_{S(z^1,t)} \int_{[-1,1]} W(a) d\nu_{n,x} dx] \tag{3.2}$$

for $t \geq 0$. Clearly $\{Q_n\}$ is a sequence of nonnegative nondecreasing functions such that

$$\lim_{n\to\infty}\lim_{t\to\infty} Q_n(t) = \lim_{n\to\infty} E(\nu_n) = \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu) > 0. \tag{3.3}$$

Note that the last inequality follows from Lemma 3.3. By a classical lemma there exists a nonnegative nondecreasing function $Q(t)$ such that along a suitable subsequence of $\{\nu_n\}$, again denoted by $\{\nu_n\}$,

$$\lim_{n\to\infty} Q_n(t) = Q(t) \tag{3.4}$$

for all $t \geq 0$. Set $\lambda = \lim_{t\to\infty} Q(t)$. Clearly $0 \leq \lambda \leq \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu)$. By (3.2), (3.3) and (3.4) we make the following three statements (see **[17]** for details).

I If $\lambda = \inf_{\nu \in \mathcal{A}^1_{-1}} E(\nu)$, there exist a subsequence of $\{\nu_n\}$ labeled by the same index and $z^1_n \in R^1$ so that for every $\epsilon > 0$ there exists $r > 0$ satisfying

$$\frac{1}{4}\int_{S^2(z^1_n,r)} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_{n,x}\nu_{n,y}dxdy$$
$$+\int_{S(z^1_n,r)}\int_{[-1,1]} W(a)d\nu_{n,x}dx > \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu) - \epsilon.$$

II If $\lambda = 0$, then for all $r > 0$

$$\lim_{n\to\infty} \sup_{z^1\in R^1} [\frac{1}{4}\int_{S^2(z^1,r)} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_{n,x}d\nu_{n,y}dxdy$$
$$+\int_{S(z^1,r)}\int_{[-1,1]} W(a)d\nu_{n,x}dx] = 0.$$

III If $0 < \lambda < \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu)$, there exist a subsequence of $\{\nu_n\}$ labeled by the same index such that for every $\epsilon > 0$ there exist $r > 0$, $r_n \to \infty$, and a positive integer n_1 such that for all $n > n_1$

$$\lambda - \epsilon < Q_n(r),\ Q_n(r_n) < \lambda + \epsilon.$$

Lemma 3.4 *The case $\lambda = 0$ cannot occur.*

Proof. The proof is the same as that of Lemma 3.3. $\square$

Lemma 3.5 *The case $0 < \lambda < \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu)$ cannot occur.*

We postpone the proof of this lemma to the end of this section.

We now deduce from Statement I and Lemma 2.6 that there exist integers k_n such that for every $\epsilon_1 > 0$ there exists $r > 0$ so that for all n

$$|\widehat{\nu}_n(x^1 - k_nT^1) + 1| < \epsilon_1 \text{ for } x^1 < -r,\ |\widehat{\nu}_n(x^1 - k_nT^1) - 1| < \epsilon_1 \text{ for } x^1 > r. \quad (3.5)$$

We introduce a shift operator τ so that

$$(\tau_k\nu)_x = \nu_{x-kT} \quad (3.6)$$

where k is an integer. Consider the shifted sequence $\{\tau_{k_n}\nu_n\}$. (3.5) becomes

$$|\widehat{\tau}_{k_n}\nu_n(x^1) + 1| < \epsilon_1 \text{ for } x^1 < -r,\ |\widehat{\tau}_{k_n}\nu_n(x^1) - 1| < \epsilon_1 \text{ for } x^1 > r. \quad (3.7)$$

Set $\mu \in \mathcal{M}(\overline{\Omega})$ to be the limit of $\{\tau_{k_n}\nu_n\}$ in the sense of Lemma 2.2. Corollary 2.3 and (3.7) imply for $x^1 < -r$

$$\widehat{\mu}(x^1) = \frac{1}{|S(x^1,\eta)|}\int_{S(x^1,\eta)}\int_{[-1,1]} a\ d\mu_y dy$$

$$= \lim_{n\to\infty} \frac{1}{|S(x^1,\eta)|} \int_{S(x^1,\eta)} \int_{[-1,1]} a \ d(\tau_{k_n}\nu_n)_y dy \in (-1-\epsilon_1, -1+\epsilon_1).$$

The same argument shows that for $x^1 > r$ $\widehat{\mu}(x^1) \in (1-\epsilon_1, 1+\epsilon_1)$. Therefore

$$\lim_{x^1\to-\infty} \widehat{\mu}(x^1) = -1 \text{ and } \lim_{x^1\to\infty} \widehat{\mu}(x^1) = 1. \tag{3.8}$$

Corollary 2.3 implies

$$\frac{1}{4}\int_{S^2(0,r)} J(x-y)\int_{[-1,1]^2} (a-b)^2 d\mu_x d\mu_y dxdy + \int_{S(0,r)}\int_{[-1,1]} W(a) d\mu_x dx$$
$$= \lim_{n\to\infty} [\frac{1}{4}\int_{S^2(0,r)} J(x-y)\int_{[-1,1]^2} (a-b)^2 d(\tau_{k_n}\nu_n)_x d(\tau_{k_n}\nu_n)_y dxdy$$
$$+ \int_{S(0,r)}\int_{[-1,1]} W(a) d(\tau_{k_n}\nu_n)_x dx] \le \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu).$$

By sending $r\to\infty$ we find $E(\mu) \le \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu)$. This and (3.8) imply $\mu \in \mathcal{A}^1_{-1}$ and $E(\mu) = \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu)$. This proves Theorem 3.1. □

We close this section by proving Lemma 3.5.

Proof of Lemma 3.5. It follows from Statement III that for every $\epsilon > 0$ we have $r > 0$ and a sequence $r_n \to \infty$, such that $Q_n(r),\ Q_n(r_n) \in (\lambda-\epsilon, \lambda+\epsilon)$ for large n. Therefore we can find z^1_n so that

$$\frac{1}{4}\int_{S^2(z^1_n,r)} J(x-y)\int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dxdy$$
$$+ \int_{S(z^1_n,r)}\int_{[-1,1]} W(a) d\nu_{n,x} dx \in (\lambda-\epsilon, \lambda+\epsilon), \tag{3.9}$$

$$\frac{1}{4}\int_{(\overline{\Omega}\times\overline{\Omega})\setminus(S(z^1_n,r_n)\times S(z^1_n,r_n))} J(x-y)\int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dxdy$$
$$+ \int_{\overline{\Omega}\setminus S(z^1_n,r_n)} W(a) d\nu_{n,x} dx \in (\inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu) - \lambda - \epsilon, \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu) - \lambda + \epsilon) \tag{3.10}$$

for large n.

Let us denote the two segments of $S(z^1_n, r_n)\setminus S(z^1_n, r)$ by D_{1n} and D_{2n} with the first one lying to the left of the second one. Note $D_{1n} = S(z^1_n - (r_n+r)/2, (r_n-r)/2)$ and $D_{2n} = S(z^1_n + (r_n+r)/2, (r_n-r)/2)$. Clearly $|D_{1n}|, |D_{2n}| \to \infty$ as $n\to\infty$. Also denote $\overline{\Omega}\setminus S(z^1_n, r)$ by M_{1n} and M_{2n} with $D_{1n} \subset M_{1n}$ and $D_{2n} \subset M_{2n}$. For large n

$$\frac{1}{4}\int_{D^2_{1n}} J(x-y)\int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dxdy + \int_{D_{1n}}\int_{[-1,1]} W d\nu_{n,x} dx < 2\epsilon$$

and a similar inequality holds for D_{2n}. Then Lemma 2.6 implies that the $\widehat{\nu}_n(x^1)$'s stay close to -1 or 1 on D_{1n} and D_{2n}. We consider three cases

Case 1: Suppose, choosing a subsequence if necessary, the $\widehat{\nu}_n(x^1)$'s stay close to -1 on D_{2n} in the sense of Lemma 2.6.

We will show that in this case the ν_n's already have energies at least not much less than $\inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu)$ in M_{2n}, so the total energy is by (3.9) is at least not much less than $\inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu)+\lambda$. This shows that ν_n cannot be a minimising sequence.

To this end we truncate ν_n at z^1_n+r. Set

$$\nu'_{n,x}=\begin{cases}\nu_{n,x} \text{ if } x^1\geq z^1_n+r\\ \delta(-1) \text{ if } x^1<z^1_n+r\end{cases} \tag{3.11}$$

where $x=(x^1,x')$. Clearly $\nu'_n\in\mathcal{A}^1_{-1}$, so

$$E(\nu'_n)\geq \inf_{\nu\in\mathcal{A}^1_{-1}} E(\nu). \tag{3.12}$$

We proceed to show that $E(\nu'_n)$ and

$$\frac{1}{4}\int_{M^2_{2n}} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_{n,x}d\nu_{n,y}dxdy+\int_{M_{2n}}\int_{[-1,1]} W(a)d\nu_{n,x}dx$$

are comparable. First we note by (3.11)

$$\int_{\overline{\Omega}}\int_{[-1,1]} W(a)d\nu'_{n,x}dx=\int_{M_{2n}}\int_{[-1,1]} W(a)d\nu_{n,x}dx. \tag{3.13}$$

The other part of $E(\nu'_n)$ is estimated as follows. Write, denoting $\{x\in\overline{\Omega}: x^1< z^1_n+r\}$ by $x^1<z^1_n+r$ where $x=(x^1,x')$,

$$\begin{aligned}&\int_{\overline{\Omega}^2} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu'_{n,x}d\nu'_{n,y}dxdy\\ &=\int_{M^2_{2n}}...+\int_{M_{2n}}\int_{y^1<z^1_n+r}...+\int_{x^1<z^1_n+r}\int_{M_{2n}}...+\int_{x^1<z^1_n+r}\int_{y^1<z^1_n+r}...\\ &=\int_{M^2_{2n}}...+\int_{M_{2n}}\int_{y^1<z^1_n+r}...+\int_{x^1<z^1_n+r}\int_{M_{2n}}...\end{aligned} \tag{3.14}$$

since

$$\int_{x^1<z^1_n+r}\int_{y^1<z^1_n+r}\int_{[-1,1]^2}(a-b)^2 d\nu'_{n,x}d\nu'_{n,y}=0$$

by (3.11). We need to show that the last two terms are small (of order ϵ) as $n\to\infty$. Since they are symmetric, we only consider

$$\int_{M_{2n}}\int_{y^1<z^1_n+r}...=\int_{x^1\geq z^1_n+r_n}\int_{y^1<z^1_n+r}...+\int_{D_{2n}}\int_{y^1<z^1_n+r}...$$

$$\leq C\int_{x^1\geq z_n^1+r_n}\int_{y^1<z_n^1+r} J(x-y)dxdy+\int_{D_{2n}}\int_{y^1<z_n^1+r}....$$

The first term approaches 0 as $n\to\infty$ since $r_n\to\infty$. The last line is therefore bounded by

$$\int_{D_{2n}}\int_{y^1<z_n^1+r} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_{n,x}d\delta(-1)dxdy+o(1)$$

$$\leq C\int_{D_{2n}}\int_{[-1,1]}(a+1)^2 d\nu_{n,x}dx+o(1)$$

since function $x\to\int_{y^1<z_n^1+r} J(x-y)dy\leq 1$ by H-2 is uniformly bounded.

Cover D_{2n} with m many mutually disjoint $S(t_{n,1},\eta)$, $S(t_{n,2},\eta)$,..., $S(t_{n,m},\eta)$ with $D_{2n}=\cup_{i=1}^m S(t_{n,i},\eta)$. This can be done if we choose r_n appropriately in the first place. Then

$$C\int_{D_{2n}}\int_{[-1,1]}(a+1)^2 d\nu_{n,x}dx+o(1)$$

$$=C\sum_{i=1}^m\int_{S(t_{n,i},\eta)}\int_{[-1,1]}(a+1)^2 d\nu_{n,x}dx+o(1)$$

$$=C\sum_{i=1}^m\int_{S(t_{n,i},\eta)}\int_{[-1,1]}(a-\widehat{\nu}(t_{n,i})+\widehat{\nu}(t_{n,i})+1)^2 d\nu_{n,x}dx+o(1)$$

$$=C\sum_{i=1}^m[\int_{S(t_{n,i},\eta)}\int_{[-1,1]}(a-\widehat{\nu}(t_{n,i}))^2 d\nu_{n,x}dx+|S(t_{n,i},\eta)|(\widehat{\nu}(t_{n,i})+1)^2]+o(1)$$

$$\leq C\sum_{i=1}^m[\int_{S(t_{n,i},\eta)}\int_{[-1,1]}(a-\widehat{\nu}(t_{n,i}))^2 d\nu_{n,x}dx+W(\widehat{\nu}(t_{n,i}))]+o(1)$$

by H-1 since $\widehat{\nu}(t_{n,i})$ is close to -1 on D_{2n}. Applying (2.8) and (2.9) we find that the last line is bounded by

$$C\sum_{i=1}^m[\frac{1}{4}\int_{S^2(t_{n,i},\eta)} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_{n,x}d\nu_{n,y}dxdy$$

$$+\int_{S(t_{n,i},\eta)}\int_{[-1,1]} W(a)d\nu_{n,x}dx]+o(1)\leq C[\frac{1}{4}\int_{D_{2n}^2}...+\int_{D_{2n}}...]+o(1).$$

Applying (3.9) and (3.10) we find

$$\int_{M_{2n}}\int_{y^1<z_n^1+r} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_{n,x}d\nu_{n,y}dxdy\leq 3\epsilon \tag{3.15}$$

for large n. We deduce from (3.13), (3.14) and (3.15) that

$$E(\nu_n')\leq\frac{1}{4}\int_{M_{2n}^2} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_{n,x}d\nu_{n,y}dxdy$$

$$+\int_{M_{2n}}\int_{[-1,1]} W(a)d\nu_{n,x}dx + 6\epsilon$$

for large n. This, (3.9) and (3.12) imply

$$E(\nu_n) \geq \lambda - \epsilon + [\frac{1}{4}\int_{\overline{\Omega}^2\setminus S^2(z_n^1,r)} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_{n,x}d\nu_{n,y}dxdy$$

$$+\int_{\overline{\Omega}\setminus S(z_n^1,r)}\int_{[-1,1]} W(a)d\nu_{n,x}dx] \geq \lambda - \epsilon + [\frac{1}{4}\int_{M_{2n}^2} ... + \int_{M_{2n}} ...]$$

$$\geq \lambda - \epsilon + E(\nu_n') - 6\epsilon \geq \lambda - 7\epsilon + \inf_{\nu\in\mathcal{A}_{-1}^1} E(\nu).$$

By choosing $7\epsilon < \lambda/2$ we find that $\{\nu_n\}$ can not be a minimising sequence in $\mathcal{A}_{-1}^1$. This rules out case 1.

Case 2: Suppose, choosing a subsequence if necessary, the $\widehat{\nu}_n(x^1)$'s stay close to 1 on D_{1n} in the sense of Lemma 2.6.

This case is handled along the same line. One shows that ν_n already has large energy on M_{1n}.

Case 3: Suppose, choosing a subsequence if necessary, the $\widehat{\nu}_n(x^1)$'s stay close to -1 on D_{1n} and close to 1 on D_{2n} in the sense of Lemma 2.6.

We will show that ν_n has large energy on $S(z_n^1, r_n)$. To this end we truncate ν_n at $z_n^1 - r_n$ and $z_n^1 + r_n$. Set

$$\nu_{n,x}' = \begin{cases} \delta(-1) \text{ if } x^1 < z_n^1 - r_n \\ \nu_{n,x} \text{ if } z_n^1 - r_n \leq x^1 \leq z_n^1 + r_n \\ \delta(1) \text{ if } x^1 > z_n^1 + r_n \end{cases} \tag{3.16}$$

where $x = (x^1, x')$. Clearly $\nu_n' \in \mathcal{A}_{-1}^1$, so

$$E(\nu_n') \geq \inf_{\nu\in\mathcal{A}_{-1}^1} E(\nu). \tag{3.17}$$

We compare $E(\nu_n')$ with

$$\frac{1}{4}\int_{S^2(z_n^1,r_n)} J(x-y)\int_{[-1,1]^2}(a-b)^2 d\nu_{n,x}d\nu_{n,y}dxdy + \int_S\int_{[-1,1]} W(a)d\nu_{n,x}dx.$$

Clearly by (3.16)

$$\int_{\overline{\Omega}}\int_{[-1,1]} W(a)d\nu_{n,x}'dx = \int_{S(z_n^1,r_n)}\int_{[-1,1]} W(a)d\nu_{n,x}dx \tag{3.18}$$

The other part of $E(\nu_n')$ is estimated as follows.

$$\int_{\overline{\Omega}^2} ... = \int_{S^2(z_n^1,r_n)} ... + \int_{x^1<z_n^1-r_n}\int_{y^1<z_n^1-r_n} ...$$

$$
\begin{aligned}
&+\int_{x^1>z_n^1+r_n}\int_{y^1>z_n^1+r_n}...+\int_{x^1<z_n^1-r_n}\int_{y^1>z_n^1+r_n}...+\int_{x^1>z_n^1+r_n}\int_{y^1<z_n^1-r_n}...\\
&+\int_S\int_{y^1<z_n^1-r_n}...+\int_S\int_{y^1>z_n^1+r_n}...+\int_{x^1<z_n^1-r_n}\int_S...+\int_{x^1>z_n^1+r_n}\int_S...\\
&=\int_{S^2}...+\int_{x^1<z_n^1-r_n}\int_{y^1>z_n^1+r_n}...+\int_{x^1>z_n^1+r_n}\int_{y^1<z_n^1-r_n}...\\
&+\int_S\int_{y^1<z_n^1-r_n}...+\int_S\int_{y^1>z_n^1+r_n}...+\int_{x^1<z_n^1-r_n}\int_S...+\int_{x^1>z_n^1+r_n}\int_S...
\end{aligned}
$$

by (3.16). The second and third terms are again symmetric, so we only consider

$$
\int_{x^1<z_n^1-r_n}\int_{y^1>z_n^1+r_n}...\leq C\int_{x^1<z_n^1-r_n}\int_{y^1>z_n^1+r_n}J(x-y)dxdy\to 0 \tag{3.19}
$$

as $n\to\infty$ since $r_n\to\infty$. The last four terms are estimated in a way similar to that leading to (3.15) and each of them is bounded by, say, 3ϵ. Together with (3.18) and (3.19) we find

$$
\begin{aligned}
E(\nu_n')\leq{}&\frac{1}{4}\int_{S^2(z_n^1,r_n)}J(x-y)\int_{[-1,1]^2}(a-b)^2d\nu_{n,x}d\nu_{n,y}dxdy\\
&+\int_{S(z_n^1,r_n)}\int_{[-1,1]}W(a)d\nu_{n,x}dx+\epsilon+12\epsilon
\end{aligned}
$$

for large n. This, (3.10) and (3.17) imply

$$
E(\nu_n)\geq(\inf_{\nu\in\mathcal{A}_{-1}^1}E(\nu)-\lambda)-\epsilon+E(\nu_n')-13\epsilon\geq(\inf_{\nu\in\mathcal{A}_{-1}^1}E(\nu)-\lambda)+\inf_{\nu\in\mathcal{A}_{-1}^1}E(\nu)-14\epsilon.
$$

By choosing $14\epsilon<(\inf_{\nu\in\mathcal{A}_{-1}^1}E(\nu)-\lambda)/2$ we deduce that $\{\nu_n\}$ cannot be a minimising sequence in $\mathcal{A}_{-1}^1$. This rules out case 3 and the proof of Lemma 3.5 is complete. $\square$

4 Regularity of Global Minimisers

In this section we assume that Ω is a thin tube and μ is a global minimiser of E in $\mathcal{A}_{-1}^1$. The "thinness" condition on Ω will be removed in the next section. We first derive an alternative formula of $E(\nu)$ for each $\nu\in\mathcal{A}$. Note that for each $x\in\overline{\Omega}$

$$
\begin{aligned}
&\int_{[-1,1]^2}(a-b)^2d\nu_xd\nu_y\\
&=\int_{[-1,1]^2}[(a-<\nu_x>)-(b-<\nu_y>)+(<\nu_x>-<\nu_y>)]^2d\nu_xd\nu_y\\
&=(<\nu_x>-<\nu_y>)^2+2\int_{[-1,1]}(a-<\nu_x>)^2d\nu_x
\end{aligned}
$$

$$= (<\nu_x> - <\nu_y>)^2 + 2\int_{[-1,1]} a^2 d\nu_x - 2 <\nu_x>^2 .$$

We deduce

$$E(\nu) = \frac{1}{4}\int_{\overline{\Omega}^2} J(x-y)(<\nu_x> - <\nu_y>)^2 dxdy$$

$$+\int_{\overline{\Omega}}[\int_{[-1,1]}(W(a)+K(x)\frac{a^2}{2})d\nu_x - K(x)\frac{<\nu_x>^2}{2}]dx, \tag{4.1}$$

where

$$K(x) = \int_\Omega J(x-y)\,dy. \tag{4.2}$$

Note that $0 < c \le K(x) \le 1$ for $x \in \overline{\Omega}$ by H-1 and the smoothness of Ω, and $K(x) \equiv 1$ if $\Omega = R^1$. An immediate consequence of (4.1) is

Proposition 4.1 *If $W(a) + K(x)a^2/2$ is convex in $a \in [-1, 1]$ for all $x \in \overline{\Omega}$, there is a usual function as global minimiser. If $W(a) + K(x)a^2/2$ is strictly convex for all $x \in \overline{\Omega}$, every global minimiser is a usual continuous function.*

Recall that for $x \in \Omega$, $W(a)+K(x)a^2/2$ is strictly convex if $W((a+b)/2)+K(x)((a+b)/2)^2/2 < [W(a) + K(x)a^2/2 + W(b) + K(x)b^2/2]/2$ for $-1 \le a < b \le 1$.

Proof. Suppose μ is a global minimiser constructed in Section 3. Set $\mu'_x = \delta(<\mu_x>)$. Clearly $\mu' \in \mathcal{A}^1_{-1}$ and by (4.1)

$$E(\mu') - E(\mu)$$

$$= \int_{\overline{\Omega}}[W(<\mu_x>) + K(x)\frac{<\mu_x>^2}{2} - \int_{[-1,1]}(W(a)+K(x)\frac{a^2}{2})d\mu_x]dx \le 0 \tag{4.3}$$

since by the convexity

$$W(<\mu_x>) + K(x)\frac{<\mu_x>^2}{2} \le \int_{[-1,1]}(W(a)+K(x)\frac{a^2}{2})d\mu_x.$$

Therefore the usual function μ' is a global minimiser.

If $W(a)+K(x)a^2/2$ is strictly convex and μ is a global minimiser, then the equality in (4.3) holds if and only if $\mu_x = \delta(<\mu_x>)$ for almost every $x \in \overline{\Omega}$, which implies that μ is a usual function. Then $<\mu>$ solves the Euler-Lagrange equation (1.2). Since $\int_\Omega J(x-y) <\mu_y dy$ is continuous in x and $f(a) + K(x)a$ is continuously invertible in a by the strict convexity assumption, $<\mu_x>$ is continuous in $x \in \overline{\Omega}$. $\square$

Next we consider a case where $W(a) + K(x)a^2/2$ is not convex in a.

H-3 Assume $W(a) + K(x)a^2/2$ achieves its minimal value exactly at $p_c(x)$ and $q_c(x)$ with $-1 < p_c(x) < q_c(x) < 1$ (see Figure 4) and $W(a) + K(x)a^2/2$ is strictly convex on $[-1, p_c(x)]$ and $[q_c(x), 1]$.

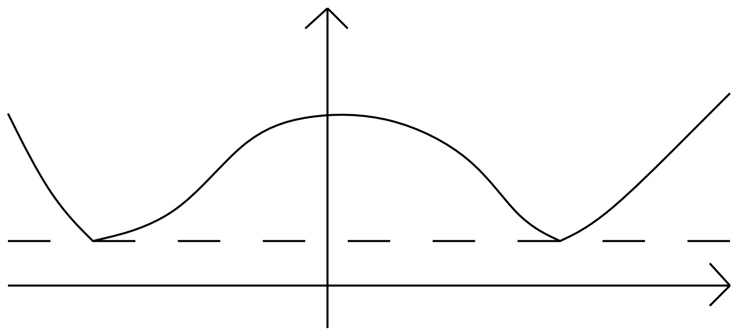

Figure 4: The graph of $W(a)+K(x)a^2/2$.

From now on we often drop the x-dependence of $p_c(x)$ and $q_c(x)$. Let μ be a global minimiser in $\mathcal{A}^1_{-1}$. Define a new measure-valued function μ' by

$$\mu'_x = \mu_x|_{[-1,p_c)} + \mu_x|_{(q_c,1]}$$
$$+\frac{\mu_x([p_c,q_c])q_c - \int_{[p_c,q_c]} a d\mu_x}{q_c-p_c}\delta(p_c) + \frac{\int_{[p_c,q_c]} a d\mu_x - \mu_x([p_c,q_c])p_c}{q_c-p_c}\delta(q_c) \qquad (4.4)$$

for each $x \in \overline{\Omega}$. Clearly $\mu' \in \mathcal{A}^1_{-1}$, $<\mu'_x>=<\mu_x>$ and $E(\mu') \leq E(\mu)$ by (4.1), where equality holds if and only if $\mu_x = \mu'_x$ for almost every $x \in \overline{\Omega}$. We have proved

Proposition 4.2 *Under H-3 for each global minimiser $\mu \in \mathcal{A}^1_{-1}$*

$$\text{supp } \mu_x \cap (p_c(x), q_c(x)) = \emptyset$$

for almost every $x \in \overline{\Omega}$ where supp μ_x *denotes the support of the measure μ_x.*

A consequence is the following irregularity result that has already appeared in a simpler case in **[4]**. Here we interpret it from an energy viewpoint.

Corollary 4.3 *Under H-3 if $u(x)$ is a usual function minimising E in $\mathcal{A}^1_{-1}$, u cannot be continuous.*

Since supp $\mu_x \subset [-1, p_c(x)] \cup [q_c(x), 1]$ for a.e. x, we further simplify μ with the help of the convexity of $W(a)+K(x)a^2/2$ on $[-1,p_c(x)]$ and $[q_c(x),1]$. Set μ' so that

$$\mu'_x = \mu_x([-1,p_c(x)])\delta(p(x)) + \mu_x([q_c(x),1])\delta(q(x)),$$

where

$$p(x) = \begin{cases} \displaystyle\int_{[-1,p_c(x)]} a d\mu_x/\mu([-1,p_c(x)]) \text{ if } \mu([-1,p_c(x)]) \neq 0 \\ \\ p_c(x) \text{ if } \mu([-1,p_c(x)]) = 0 \end{cases}$$

and

$$q(x)=\begin{cases}\displaystyle\int_{[q_c(x),1]} a d\mu_x/\mu([q_c(x),1]) \text{ if } \mu([q_c(x),1])\neq 0\\ \\ q_c(x) \text{ if } \mu([q_c(x),1])=0.\end{cases}$$

Again by (4.1) $E(\mu')<E(\mu)$ unless $\mu'_x=\mu_x$ for a.e. x. Therefore we have

Corollary 4.4 *Each global minimiser $\mu\in\mathcal{A}^1_{-1}$ satisfies*

$$\mu_x=(1-\theta(x))\delta(p(x))+\theta(x)\delta(q(x)) \tag{4.5}$$

at a.e. $x\in\overline{\Omega}$ for some $\theta(x)\in[0,1]$, $p(x)\in[-1,p_c(x)]$ and $q(x)\in[q_c(x),1]$.

Let μ written in the form (4.5) be a global minimiser in $\mathcal{A}^1_{-1}$. We derive its Euler-Lagrange equation. Fix two smooth functions ϕ and ψ on Ω with compact supports. Define a deformation μ_t of μ by

$$\mu_{t,x}=(1-\theta(x))\delta(p(x)+t\phi(x))+\theta(x)\delta(q(x)+t\psi(x)) \tag{4.6}$$

for t near 0. Clearly $\mu_0=\mu$. Since μ_t is a measure-valued function with range $(-\infty,\infty)$ (note the remark before Lemma 2.2), we have $E(\mu_0)\leq E(\mu_t)$. Therefore after some calculation

$$0=\frac{dE(\mu_t)}{dt}|_{t=0}=\int_\Omega\{(1-\theta(x))[K(x)p(x)-\int_\Omega J(x-y)((1-\theta)p+\theta q)(y)dy$$

$$+f(p(x))]\phi(x)+\theta(x)[K(x)q(x)-\int_\Omega J(x-y)((1-\theta)p+\theta q)(y)dy+f(q(x))]\psi(x)\}dx.$$

Since this is true for all ϕ and ψ, we conclude

$$\begin{cases}(1-\theta(x))[\int_\Omega(J(x-y)((1-\theta)p+\theta q)(y)dy-K(x)p(x)-f(p(x))]=0\\ \theta(x)[\int_\Omega J(x-y)((1-\theta)p+\theta q)(y)dy-K(x)q(x)-f(q(x))]=0\end{cases} \tag{4.7}$$

for a.e. $x\in\overline{\Omega}$. If $\theta(x)\in(0,1)$ then

$$\begin{cases}\int_\Omega J(x-y)((1-\theta)p+\theta q)(y)dy-K(x)p(x)-f(p(x))=0\\ \int_\Omega J(x-y)((1-\theta)p+\theta q)(y)dy-K(x)q(x)-f(q(x))=0\end{cases}$$

which implies

$$K(x)p(x)+f(p(x))=K(x)q(x)+f(q(x)).$$

According to H-3 and the fact $p(x)\in[-1,p_c(x)]$ and $q(x)\in[q_c(x),1]$, we have

$$K(x)p(x)+f(p(x))\leq 0 \text{ and } K(x)q(x)+f(q(x))\geq 0.$$

Therefore $K(x)p(x)+f(p(x))=K(x)q(x)+f(q(x))=0$ that implies

Proposition 4.5 *Under H-3 the representation (4.5) of a global minimiser μ satisfies $p(x) = p_c(x)$, $q(x) = q_c(x)$ when $\theta(x) \in (0,1)$ for a.e. $x \in \overline{\Omega}$.*

We note $((1-\theta)p + \theta q)(x) =< \mu_x >$ and

Claim $\int_\Omega J(x-y)((1-\theta)p+\theta q)(x) + K(x) \to 0$ as $x^1 \to -\infty$.

To see this we first approximate J in the L^1-norm by step functions. Then the claim holds if it holds when J is replaced by step functions and K is changed in the same way, where a step function is a linear combination of characteristic functions. For each characteristic function χ_S on a segment $S \subset \overline{\Omega}$ Corollary 2.7 implies

$$\lim_{x^1 \to -\infty} \int_\Omega \chi_S(x-y)((1-\theta)p + \theta q)(y)dy = -|S|.$$

The claim is then proved by the usual approximation argument.

The claim and the second equation of (4.7) imply that $\theta(x) = 0$ for x^1 sufficiently close to $-\infty$, i.e., $\mu_x = \delta(p(x))$ there. Now note that for such x

$$\int_\Omega J(x-y)((1-\theta)p + \theta q)(y)dy - K(x)p(x) - f(p(x)) = 0.$$

Since $p(x) \in [-1, p_c(x)]$ and $a \to K(x)a + f(a)$ is continuously invertible on $[-1, p_c(x)]$ by H-3, we find that $p(x)$ is continuous with $p(x) \to -1$ as $x^1 \to -\infty$. Applying the same argument in the case $x^1 \to \infty$ we deduce the following about the continuity of μ near infinity.

Proposition 4.6 *Under H-3 for x^1 sufficiently close to $-\infty$ (∞, respectively) $\mu_x = \delta(p(x))$ ($\mu_x = \delta(q(x))$, respectively) where $p(x)$ ($q(x)$, respectively) is continuous with $p(x) \to -1$ ($q(x) \to 1$, respectively) as $x^1 \to -\infty$ (∞, respectively).*

5 Further Results

We first extend Theorem 3.1 to general tubes. Given a tube Ω let us divide it into the union of thin tubes Ω_1, Ω_2,...,Ω_l so that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup ... \cup \overline{\Omega}_l$ and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. On each Ω_i a segment $S_i(z^1, r)$ is defined to be $\{x \in \overline{\Omega}_i : |x^1 - z^1| < r\}$. Define for each $\nu \in \mathcal{A}$

$$\widehat{\nu}_i(x^1) = \frac{1}{|S_i(x^1,\eta)|} \int_{S_i(x^1,\eta)} < \nu_z > dz.$$

Analogous to Lemma 2.6 we have

Lemma 5.1 *In a tube Ω for every given ϵ_1 there is $\theta > 0$ such that for every $\nu \in \mathcal{A}$, $z^1 \in R^1$*

$$\frac{1}{4}\int_{S^2(z^1,\eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy + \int_{S(z^1,\eta)} \int_{[-1,1]} W(a) d\nu_x dx < \theta$$

implies either $|\widehat{\nu}_i(z^1) + 1| < \epsilon_1$ for all $i = 1, 2, ..., l$, or $|\widehat{\nu}_i(z^1) - 1| < \epsilon_1$ for all $i = 1, 2, ..., l$.

Proof. Applying Lemma 2.6 on each Ω_i we find that given ϵ_1

$$|\widehat{\nu}_i(z^1)+1|<\epsilon_1 \text{ or } |\widehat{\nu}_i(z^1)-1|<\epsilon_1$$

if θ is chosen small enough. Suppose without loss of generality that $|\widehat{\nu}_1(z^1)+1|<\epsilon_1$. We need to show that $|\widehat{\nu}_i(z^1)+1|<\epsilon_1$ for $i=2,3,...,l$. To this end choose a deformation from Ω_1 to Ω_2, i.e., for each $t\in[1,2]$ find a thin tube $\Omega_{(t)}$ with $\Omega_{(1)}=\Omega_1$ and $\Omega_{(2)}=\Omega_2$. On each $\Omega_{(t)}$ set

$$S_{(t)}(z^1,\eta)=\{x\in\overline{\Omega}_{(t)}:|x^1-z^1|<\eta\} \text{ and}$$

$$\widehat{\nu}_{(t)}(z^1)=\frac{1}{|S_{(t)}(z^1,\eta)|}\int_{S_{(t)}(z^1,\eta)}<\nu_x>dx.$$

We assume that $t\rightarrow|S_{(t)}(z^1,\eta)|$ is continuous for each z^1 and η. By Lemma 2.6, $|\widehat{\nu}_{(t)}(z^1)+1|<\epsilon_1$ or $|\widehat{\nu}_{(t)}(z^1)-1|<\epsilon_1$. But since $\widehat{\nu}_{(t)}(z^1)$ is continuous in t, we find $|\widehat{\nu}_2(z^1)+1|=|\widehat{\nu}_{(2)}(z^1)+1|<\epsilon_1$. Repeating this argument we find $|\widehat{\nu}_i(z^1)+1|<\epsilon_1$ for $i=2,3,...,l$. This proves Lemma 5.1. □

Corollary 5.2 *For each* $\nu\in\mathcal{A}$ *either* $\lim\limits_{z^1\rightarrow-\infty}\widehat{\nu}_i(z^1)=-1$ *for all* $i=1,2,...,l$ *or* $\lim\limits_{z^1\rightarrow-\infty}\widehat{\nu}_i(z^1)=1$ *for all* $i=1,2,...,l$*. A similar statement holds for* $z^1\rightarrow\infty$*.*

We can now define unambiguously $\mathcal{A}^1_{-1}$ to be

$$\{\nu\in\mathcal{A}:\lim_{z^1\rightarrow-\infty}\widehat{\nu}_i(z^1)=-1,i=1,...,l \text{ and } \lim_{z^1\rightarrow\infty}\widehat{\nu}_i(z^1)=1,i=1,...,l\}. \quad (5.1)$$

Theorem 5.3 *On each tube* Ω *there is* $\mu\in\mathcal{A}^1_{-1}$ *such that* $E(\mu)=\inf\limits_{\nu\in\mathcal{A}^1_{-1}}E(\nu)$*.*

To prove this theorem one follows the argument in Section 3 using Lemma 5.1 instead of Lemma 2.6. One needs to modify the argument leading to (3.15) by modifying (2.8) and (2.9) accordingly. We leave the details to the reader.

Next we relax the periodicity condition on Ω a little to consider asymptotically periodic Ω.

Definition 5.4 Ω is an *asymptotic tube* if it can be divided into three disjoint open domains, Ω_1, Ω_2 and Ω_0, such that $\overline{\Omega}=\overline{\Omega}_1\cup\overline{\Omega}_2\cup\overline{\Omega}_0$, Ω_0 is bounded and Ω_1 and Ω_2 are "half-periodic".

We say Ω_m, $m=1,2$, is half-periodic if there exists $T_m\in R^d$, $m=1,2$, such that for every $x_m\in\Omega_m$, $x_m+T_m\in\Omega_m$, and for every $x_m\in\Omega_m$ there exist $y_m\in\Omega_0$ and a non-negative integer n_m satisfying $x_m=y_m+n_mT_m$. Ω_1 and Ω_2 are indeed the two half-periodic tails of Ω. We associate the end e_1 to Ω_1 and the other end e_2 to Ω_2 (see Figure 5). Define $\mathcal{A}$ by (2.4) and $\mathcal{A}^1_{-1}$ by a formula similar to (5.1). Set G_m, $m=1,2$, to be the periodic extension of Ω_m, i.e.,

$$G_m=\{x\in R^d:x+n_mT_m\in\Omega_m \text{ for some integer } n_m\}.$$

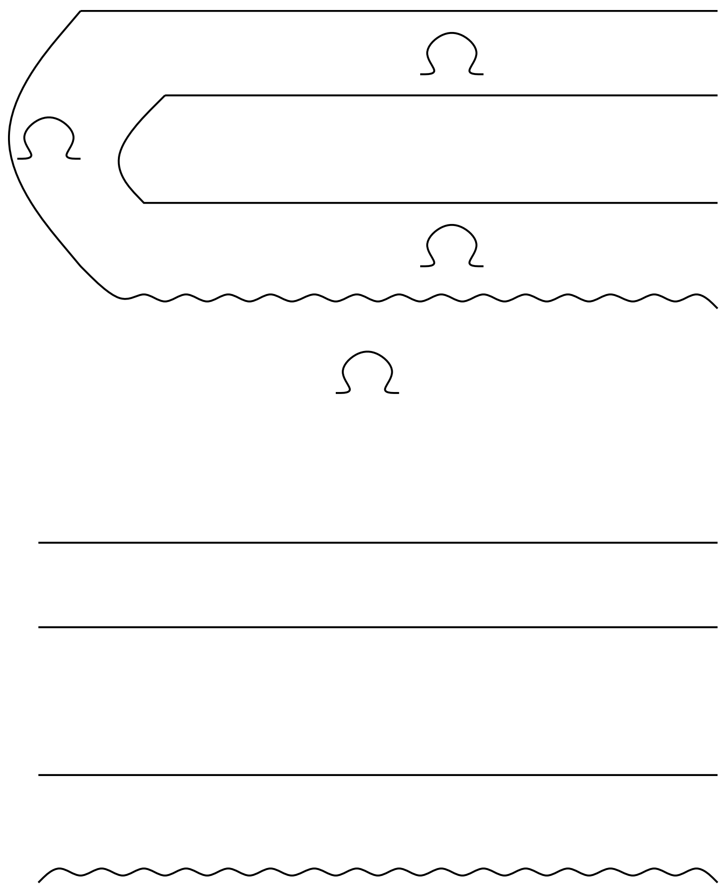

Figure 5: An example of Ω, G_1 and G_2.

Denote the other end of G_m by e'_m (see Figure 5). Define two auxiliary functionals E_m by (2.3) on G_m. Take $\mathcal{A}^1_{-1}(m)$ to be

$$\{\nu \in \mathcal{M}(\overline{G}_m) : E_m(\nu) < \infty, \lim_{x \to e_m} \nu(x) = (-1)^m, \lim_{x \to e'_m} \nu(x) = (-1)^{m+1}\}$$

where the limits are interpreted in a way similar to (5.1). The following theorem gives a sufficient condition for the existence of a global minimiser in $\mathcal{A}^1_{-1}$.

Theorem 5.5 *If Ω is an asymptotic tube, then*

$$\inf_{\nu \in \mathcal{A}^1_{-1}} E(\nu) \le \min\{\inf_{\nu \in \mathcal{A}^1_{-1}(1)} E_1(\nu), \inf_{\nu \in \mathcal{A}^1_{-1}(2)} E_2(\nu)\}.$$

If the strict inequality

$$\inf_{\nu \in \mathcal{A}^1_{-1}} E(\nu) < \min\{\inf_{\nu \in \mathcal{A}^1_{-1}(1)} E_1(\nu), \inf_{\nu \in \mathcal{A}^1_{-1}(2)} E_2(\nu)\}$$

holds, then there is $\mu \in \mathcal{A}^1_{-1}$ such that $E(\mu) = \inf_{\nu \in \mathcal{A}^1_{-1}} E(\nu)$.

We omit the proof of this theorem, which is a combination of the techniques in this paper and the ones in **[6]**. Applications of this theorem can also be found there.

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