Shell structure as solution to a free boundary problem from block copolymer morphology

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Abstract

A shell like structure is sought as a solution of a free boundary problem derived from the Ohta-Kawasaki theory of diblock copolymers. The boundary of the shell satisfies an equation that involves its mean curvature and the location of the entire shell. A variant of Lyapunov-Schmidt reduction process is performed that rigorously reduces the free boundary problem to a finite dimensional problem. The finite dimensional problem is solved numerically. The problem has two parameters: a and γ. When a is small, there are a lower bound and a sequence such that if γ is greater than the lower bound and stays away from the sequence, there is a shell like solution.

Key words. diblock copolymer, shell pattern.

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Abbreviated title. Shell structure as solution.

1 Introduction

Let D be a bounded and sufficiently smooth domain in R^3. The Lebesgue measure of a subset E of D is denoted by $|E|$. The part of the boundary of E that is in D is denoted by $\partial_D E$. Let χ_E be the characteristic function of E, i.e. $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \in D \backslash E$. Given a fixed number $a \in (0, 1)$ we look for a subset E of D and a number λ such that $\partial_D E$ is a smooth curve, or a union of several smooth curves, $|E| = a|D|$, and at every point on $\partial_D E$

$$H(\partial_D E) + \gamma(-\Delta)^{-1}(\chi_E - a) = \lambda. \tag{1.1}$$

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Here $H(\partial_D E)$ is the mean curvature of $\partial_D E$ viewed from E and γ is a given positive number. The expression $(-\Delta)^{-1}(\chi_E - a)$ is the solution v of the problem

$$-\Delta v = \chi_E - a \text{ in } D,\ \partial_\nu v = 0 \text{ on the boundary of } D,\ \ \overline{v} = 0$$

where the bar over a function is the average of the function over its domain, e.g.

$$\overline{v} = \frac{1}{|D|}\int_D v(x)\,dx.$$

Because $(-\Delta)^{-1}$ is a nonlocal operator, the free boundary problem (1.1) is nonlocal.

The equation (1.1) is the Euler-Lagrange equation of the following variational problem.

$$J(E) = |D\chi_E|(D) + \frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_E - a)|^2\,dx. \tag{1.2}$$

The admissible set Σ of the functional J is the collection of all measurable subsets of D of measure $a|D|$ and of finite perimeter, i.e.

$$\Sigma = \{E \subset D : E \text{ is Lebesgue measurable},\ \ |E| = a|D|,\ \ \chi_E \in BV(D)\}. \tag{1.3}$$

Here $BV(D)$ is the space of functions of bounded variation on D. The nonlocal integral operator $(-\Delta)^{-1}$ is defined by solving

$$-\Delta v = q \text{ in } D,\ \partial_\nu v = 0 \text{ on the boundary of } D, \overline{v} = 0$$

for $q \in L^2(D)$, $\overline{q} = 0$. Then $(-\Delta)^{-1/2}$ is the positive square root of $(-\Delta)^{-1}$.

Since $\chi_E \in BV(D)$, we view $D\chi_E$, the derivative of χ_E, as a vector valued, signed measure, and let $|D\chi_E|$ be the positive total variation measure of $D\chi_E$. The first term in (1.2), $|D\chi_E|(D)$, is the $|D\chi_E|$ measure of the entire domain D, which is known as the perimeter of E. When $\partial_D E$ is a smooth surface, or a union of several smooth surfaces, $|D\chi_E|(D)$ is just the area of $\partial_D E$. For this reason $|D\chi_E|(D)$ is called the perimeter of E in D and sometimes denoted by $P_D(E)$. See [7, Section 5.7] for more information on $P_D(E)$. The constant λ in (1.1) comes as a Lagrange multiplier from the constraint $|E| = a|D|$.

The functional J in (1.2) is derived from the Ohta-Kawasaki density functional theory of diblock copolymers [15] as a Γ-limit by Ren and Wei [16].

A diblock copolymer is a soft condensed material that in contrast to crystalline solids is characterized by fluid-like disorder on the molecular scale and a high degree of order on a longer length scale. A diblock copolymer molecule is a linear subchain of A monomers grafted covalently to another subchain of B monomers [1, 10]. Because of the repulsion between the unlike monomers, the different type subchains tend to segregate, but as they are chemically bonded in chain molecules, segregation of subchains cannot lead to a macroscopic phase separation. Only a local micro-phase separation occurs: micro-domains rich in A monomers and micro-domains rich in B monomers emerge as a result. These micro-domains form patterns that are know as morphology phases.

In our model (1.2), the diblock copolymer is in a strongly segregated state. The A-monomers occupy the set E and the B-monomers occupy the set $D\backslash E$. The

number a is the block composition fraction. It is the number of the A-monomers divided by the number of all the A- and B- monomers in a polymer chain. The interface between the A-monomer regions and B-monomer regions is $\partial_D E$ whose tension is its area. The connectivity of A and B monomers in a chain molecule is described by the nonlocal term in J.

Nishiura and Ohnishi [13] formulated the Ohta-Kawasaki theory on a bounded domain as a singularly perturbed variational problem with a nonlocal term. They also formally identified the free boundary problem (1.1). Since then much work has been done to these problems. The lamellar phase was studied by Ren and Wei [16, 18, 19, 23, 24], Fife and Hilhorst [8], Chen and Oshita [2], and Choksi and Sternberg [6]. The work of Müller [12] was related to the lamellar phase in the case $a = 1/2$, as observed in [13]. Radially symmetric bubble and ring patterns were studied by Ren and Wei [17, 22, 25]. The cylindrical phase and the spherical phase were studied by Ren and Wei [27, 26, 28]. A triblock copolymer in the lamellar phase was studied by Ren and Wei [21]. Teramoto and Nishiura [29] studied the gyroid phase numerically. Mathematically strict derivations of the density functional theories for diblock copolymers, triblock copolymers and polymer blends were given by Choksi and Ren [4, 5], and Ren and Wei [20]. Also see Ohnishi and Nishiura [14], Ohnishi *et al* [14], and Choksi [3].

In this paper we consider a shell like solution in space. A shell is a region in R^3 bounded by two concentric spheres. This paper is motivated by the recent progress in the study of (1.1) by Ren and Wei [27, 26, 28] and Kang and Ren [11]. A variant of the Lyapunov-Schmidt argument was found in [27, 26, 28] that successfully reduce the infinite dimensional free boundary problem (1.1) to some finite dimensional calculus problems when a is sufficiently small. When applied to two-dimensions, a ring like solution was found [11]. A ring is a region in plane bounded by two concentric circles. When the parameter γ in (1.1) is chosen properly the ring solution is stable.

Here we carry out a similar process to study a shell solution of (1.1). We will reduce the infinite dimensional problem (1.1) to a finite dimensional problem of finding the center and the radii of a shell that satisfies (1.1). We will see that the shell solution we find is always unstable. The conditions on γ for existence in three-dimensions are also quite different from those in two dimensions.

We point out that a shell solution of (1.1) is not a perfect shell. This is because the boundaries of a shell solution are not exact spheres. They are actually hyper surfaces that are close to spheres. One difficulty here is that we do not know at first the size of a shell. The inner radius and the outer radius of the shell will have to be determined as we find the solution. This compares differently from the spherical solutions found in [28] where we know at least approximately the radius of each sphere based on the constraint $|E| = a|D|$.

Our main reduction results are presented in Section 2. Numerical calculations on the reduced problem show the existence of an unstable shell solution and the conditions needed on γ. The Lyapunov-Schmidt reduction procedure tailored to this problem is given in Sections 3 through 6. In Section 3 we construct a family of approximate solutions and see how well they solve (1.1). We also calculate the J values of these approximate solutions. In Section 4 we introduce perturbations

to the boundaries of the approximate solutions and formulate the equations for these perturbations as a system of integro-differential equations. In section 5 we obtain detailed spectral information on the linearized operator at these approximate solutions. Section 6 consists of two steps. First we prove, via a fixed point argument, that (1.1) may be reduced to a four dimensional problem of finding the center and the inner radius of the shell solution. Second we show that any solution to the reduced problem is a solution to (1.1). A few remarks are given in Section 7.

2 Theorems and observations

Let R_1, $R_2 > 0$ be such that

$$R_2^3 - R_1^3 = 1, \tag{2.1}$$

and set

$$r = \frac{R_1}{R_2}. \tag{2.2}$$

For each integer $n \geq 2$ the quadratic equation

$$\begin{aligned} &\frac{1 - r^{2n} - \frac{(1-r^3)(2n+1)}{3}}{81(2n+1)^2 R_1 R_2}\Gamma^2 \\ +[\frac{(n+2)(n-1)}{81(2n+1)R_1R_2^4} &+ \frac{(n+2)(n-1)}{81(2n+1)R_1^4R_2}(1 - \frac{(1-r^3)(2n+1)}{3})]\Gamma \\ &+\frac{(n+2)^2(n-1)^2}{81R_1^4R_2^4} \quad = \quad 0. \end{aligned} \tag{2.3}$$

of Γ has one positive root and one negative root, because the graph of the left side, as a function of Γ, is a downward parabola, and when $\Gamma = 0$ the left side is positive. Denote the positive root of (2.3) by $\tilde{\Gamma}_n(R_1)$ as a quantity that depends on R_1. Define curves W_n in the first quadrant of the R_1-Γ plane by

$$W_n = \{(R_1, \tilde{\Gamma}_n(R_1)) : R_1 > 0\}, \ n = 2, 3, ... \tag{2.4}$$

The $\tilde{\Gamma}_n$'s have the property that for each $n = 2, 3, ...,$

$$\lim_{R_1 \to 0} \tilde{\Gamma}_n(R_1) = \frac{3(n+2)(2n+1)}{2}. \tag{2.5}$$

For large R_1 there is the asymptotic formula

$$\lim_{R_1 \to \infty} \tilde{\Gamma}_n(R_1) = 6(n-1)(n+2)(2n+1) \tag{2.6}$$

for each $n = 2, 3, ...$ Moreover when R_1 is in a compact subset of $[0, \infty)$,

$$\lim_{n \to \infty} \frac{\tilde{\Gamma}_n(R_1)}{n^2} = 3. \tag{2.7}$$

uniformly. Here $\tilde{\Gamma}_n(0)$ is defined by the limit in (2.5).

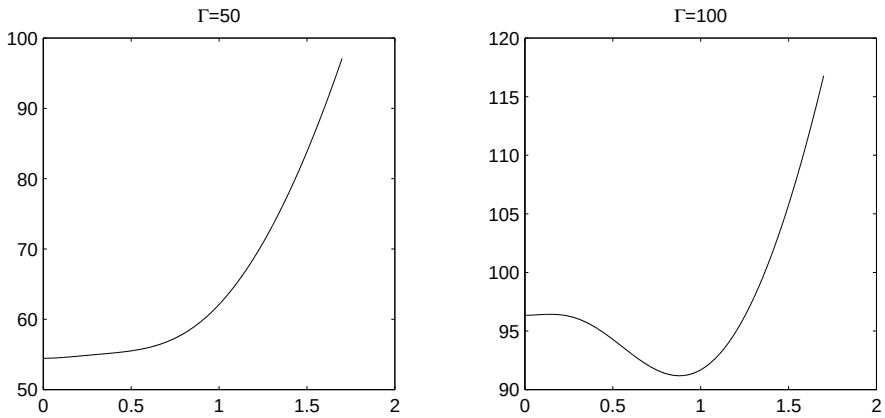

Figure 1: Two graphs of Q_Γ with $\Gamma = 50$ and $\Gamma = 100$. The first one does not have a local minimum but the second one does.

Next we define a function $Q_\Gamma(R_1)$ of $R_1 > 0$ by

$$Q_\Gamma(R_1) = 4\pi(R_1^2 + R_2^2) + \frac{\Gamma}{2}[\frac{12\pi R_1^5}{15} + \frac{8\pi R_2^5}{15} - \frac{4\pi R_1^3 R_2^2}{3}], \tag{2.8}$$

where R_2 is given by (2.1). In the function Q_Γ, Γ is a positive parameter.

Let us denote the Green's function of $-\Delta$ on D with the Neumann boundary condition by G. It is a sum of two parts:

$$G(x, y) = \frac{1}{4\pi|x - y|} + R(x, y). \tag{2.9}$$

The regular part of $G(x, y)$ is $R(x, y)$. The Green's function satisfies the equation

$$-\Delta_x G(x, y) = \delta(x - y) - \frac{1}{|D|} \text{ in } D, \quad \partial_{\nu(x)} G(x, y) = 0 \text{ on } \partial D, \quad \overline{G(\cdot, y)} = 0 \ \forall y \in D. \tag{2.10}$$

Here Δ_x is the Laplacian with respect to the x-variable of G, $\nu(x)$ is the outward normal direction at $x \in \partial D$, and $\partial_{\nu(x)}$ is the normal derivative there with respect to the x-variable.

Our first theorem addresses the existence issue.

Theorem 2.1 *If there is a point $(S_{0,1}, \Gamma)$ in the first quadrant of the R_1-Γ plane such that*

1. *$(S_{0,1}, \Gamma) \notin \cup_{n=2}^{\infty} W_n$, and*
2. *$S_{0,1}$ minimizes Q_Γ locally,*

then the Euler-Lagrange equation (1.1) with $\gamma = \Gamma(\frac{3a|D|}{4\pi})^{-1}$ admits a shell solution, when a is sufficiently small. The inner radius of the shell is approximately

$S_{0,1}(\frac{3a|D|}{4\pi})^{1/3}$ *and the outer radius of the shell is approximately* $S_{0,2}(\frac{3a|D|}{4\pi})^{1/3}$ *where* $S_{0,2}^3 - S_{0,1}^3 = 1$. *The center of the shell is close to a minimum of the function* $R(x,x)$, $x \in D$.

The next theorem addresses the stability of the shell solution.

Theorem 2.2 *If the first condition in Theorem 2.1 is satisfied because* $(S_{0,1}, \Gamma)$ *is below all the curves* W_n, *then the solution constructed in Theorem 2.1 is stable. If the first condition in Theorem 2.1 is satisfied and* $(S_{0,1}, \Gamma)$ *lies above one of the* W_n*'s, then the solution is unstable.*

These two theorems reduce the existence and the stability of a solution to a finite dimensional problem. We study this problem numerically.

The curves W_n are plotted in Figure 2. We see that these curves appear in the increasing order as n gets larger.

The function Q_Γ admits a positive local minimum only if Γ is sufficient large (see Figure 1). We have found numerically that there is a constant $\Gamma_0 > 0$ such that if $\Gamma > \Gamma_0$, Q_Γ has a positive local minimum. Approximately $\Gamma_0 = 68$, and For each $\Gamma > \Gamma_0$ we denote this positive local minimum of Q_Γ by $l(\Gamma)$. Note that $\lim_{\Gamma \to \Gamma_0+} l(\Gamma) = 0.6094$ approximately. The curve $V = \{l(\Gamma), \Gamma) : \Gamma > \Gamma_0\}$ is also plotted on Figure 2. The starting point $(\lim_{\Gamma \to \Gamma_0+} l(\Gamma), \Gamma_0) \approx (0.6094, 68)$ of V lies between W_3 and W_4. The curve V crosses W_4, W_5, W_6, etc, at $(l(\Gamma_1), \Gamma_1)$, $(l(\Gamma_2), \Gamma_2)$, $(l(\Gamma_3), \Gamma_3)$, etc.

Based on these numerical calculations, we have the following observation.

Observation 2.3 *There exists an increasing sequence* $\{\Gamma_n\}_{n=0,1,2,3,...}$ *such that for any compact subset* $\mathcal{K}$ *of* $(\Gamma_0, \infty) \backslash \{\Gamma_1, \Gamma_2, \Gamma_3, ...\}$, *there exists a constant* $a_0 > 0$ *such that if*

$$a < a_0 \quad \textit{and} \quad \frac{3a|D|}{4\pi}\gamma \in \mathcal{K},$$

there exists an unstable shell solution of (1.1).

The proofs of the two theorems start with a family of approximate solutions $F = \{x \in R^3 : r_1 < |x - \xi| < r_2\}$ which are perfect shells. These approximate solutions are parametrized by the center ξ in D and the inner radius r_1. The outer radius r_2 is determined from the inner radius via $\frac{4\pi r_2^3}{3} - \frac{4\pi r_1^3}{3} = a|D|$ since $|F| = a|D|$.

Since we look for an exact solution that deviates only a little from one F in the family of the approximate solutions, we perturb each ring F by a pair of functions $\phi = (\phi_1, \phi_2)$ defined on the unit sphere S^2, so that $r_1^3 + \phi_1$ is the perturbed inner radius cube and $r_2^3 + \phi_2$ the perturbed outer radius cube. The perturbed shell is the set

$$E_\phi = \{\xi + \alpha\theta : \ \alpha \in ((r_1^3 + \phi_1(\theta))^{1/3}, (r_2^3 + \phi_2(\theta))^{1/3}), \ \forall \theta \in S^2\}.$$

Here we perturbed the radius cubes instead of radii so that the constraint $|E_\phi| = a|D|$ becomes a simple linear constraint on ϕ_1 and ϕ_2.

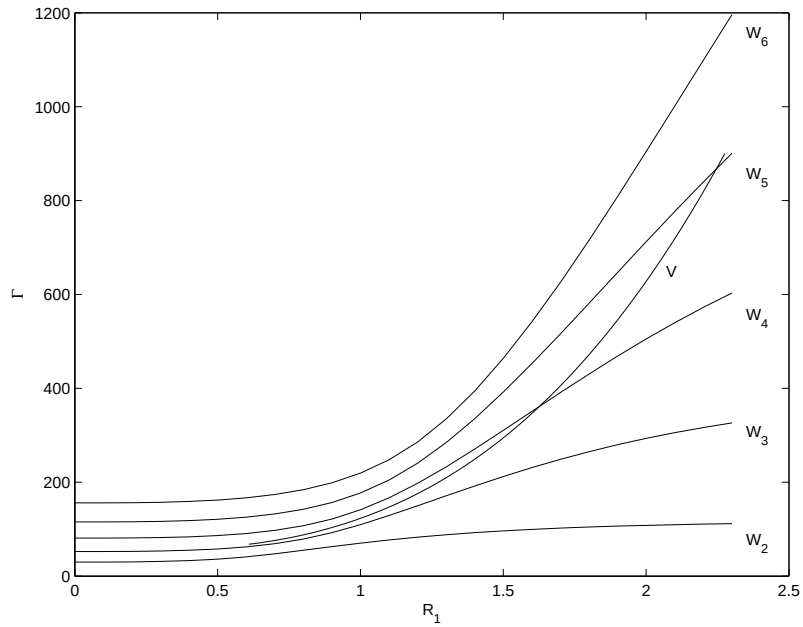

Figure 2: The illustration of Γ_0, Γ_1, and Γ_2.

In terms of ϕ the Euler-Lagrange equation (1.1) is written as $\mathcal{S}(\phi) = 0$ where $\mathcal{S}$ is a two component, nonlinear, integro-differential operator from a function space $\mathcal{X}$ to another function space $\mathcal{Y}$.

There is a subset $\mathcal{X}_*$ of $\mathcal{X}$ which, roughly speaking, ignores the effect of the translation of the center ξ and the change of the inner radius r_1. There is a also a corresponding subset $\mathcal{Y}_*$ of $\mathcal{Y}$. Given a pair (ξ, r_1), we look for $\varphi = \varphi(\theta, \xi, r_1)$ in $\mathcal{X}_*$ that solves the equation up to translation of ξ and change of r_1, i.e. $\Pi\mathcal{S}(\varphi) = 0$ where Π is the projection operator from $\mathcal{Y}$ to $\mathcal{Y}_*$.

Finally we study the dependence of $J(\varphi(\cdot, \xi, r_1))$ on (ξ, r_1). Under the conditions of Theorem 2.1 there exists (ζ, s_1) such that $J(\varphi(\cdot, \xi, r_1))$ is minimized at $(\xi, r_1) = (\zeta, s_1)$. It turns out that at this minimum, $S(\varphi(\cdot, \zeta, s_1)) = 0$.

Whether the solution $\varphi(\cdot, \xi, r_1)$ of $\Pi\mathcal{S}(\varphi) = 0$ at each (ξ, r_1) is a local minimizer of J restricted on $\mathcal{X}_*$ is the issue addressed in Theorem 2.2. If so, the solution $\varphi(\cdot, \zeta, s_1)$ of $\mathcal{S}(\varphi) = 0$ is interpreted as a stable solution. Otherwise it is considered unstable.

The L^p space on the unit sphere S^2 is $L^p(S^2)$ whose norm is denoted by $\|\cdot\|_{L^p}$. The norm of the Sobolev $W^{k,p}$ space is denoted by by $\|\cdot\|_{W^{k,p}}$. When $p = 2$, the Hilbert space $W^{k,2}(S^2)$ is denoted by $H^k(S^2)$ and its norm by $\|\cdot\|_{H^k}$

The constant C denotes a positive number which is independent of a. It can only depend on the pair $(S_{0,1}, \Gamma)$ and the domain D. The value of C usually varies from place to place.

From now on let $S_{0,1}$ and Γ be two numbers satisfying the two conditions in

Theorem 2.1. Assume that

$$\gamma = \Gamma(\frac{3a|D|}{4\pi})^{-1} \tag{2.11}$$

throughout the rest of the paper.

3 Perfect shells as approximate solutions

Let U_1 be a neighborhood of the set

$$\{z \in D: \ R(z,z) = \min_{x \in D} R(x,x)\}. \tag{3.1}$$

Since $R(x,x) \to \infty$ as $x \to \partial D$, the set defined in (3.1) is compact and we can choose U_1 so that the closure of U_1 in R^2 is compact and is contained in D.

Denote by F a perfect shell in D centered at $\xi \in U_1$ and of inner radius r_1 and outer radius r_2:

$$F = \{x \in R^3 : r_1 < |x - \xi| < r_2\}. \tag{3.2}$$

We often write $F = B_2 \backslash B_1$ (up to a set of Lebesgue measure 0) where $B_k = \{x \in R^3 : \ |x - \xi| < r_k\}$, $k = 1, 2$. Choose r_1 to be close to $S_{0,1}(\frac{3a|D|}{4\pi})^{1/3}$. Namely we let

$$r_1 \in U_2 = ((S_{0,1} - \delta_2)(\frac{3a|D|}{4\pi})^{1/3}, (S_{0,1} + \delta_2)(\frac{3a|D|}{4\pi})^{1/3}), \quad r_2^3 - r_1^3 = \frac{3a|D|}{4\pi}. \tag{3.3}$$

In (3.3) $\delta_2 > 0$ is a small number, independent of a, so that $S_{0,1}$ minimizes Q_Γ on $(S_{0,1} - \delta_2, S_{0,1} + \delta_2)$. Our constructions of U_1 and U_2 guarantee that F is inside D if a is sufficiently small.

We plug F into the left side of the equation (1.1) and see, as an approximate solution, how much error F generates. Note that when we read (1.1), the mean curvature of ∂F is viewed from the set F, so on the inner sphere the mean curvature is $-\frac{1}{r_1}$ and on the outer sphere the mean curvature is $\frac{1}{r_2}$.

Lemma 3.1 *If $E = F$, the left side of (1.1) is*

$$-\frac{1}{r_1} + \gamma[-\frac{r_1^2}{2} + \frac{r_2^2}{2} + a|D|R(\xi,\xi) + \frac{4\pi(r_2^5 - r_1^5)}{30|D|}] + O(a^{1/3})$$

on the inner sphere of F, and is

$$\frac{1}{r_2} + \gamma[\frac{r_2^2}{3} - \frac{r_1^3}{3r_2} + a|D|R(\xi,\xi) + \frac{4\pi(r_2^5 - r_1^5)}{30|D|}] + O(a^{1/3})$$

on the outer sphere of F.

Proof. Let $v = (-\Delta)^{-1}(\chi_F - a) = v_2 - v_1$, where v_k, $k = 1, 2$, satisfies

$$-\Delta v_k = \chi_{B_k} - \frac{4\pi r_k^3}{3|D|} \text{ in } D, \quad \partial_\nu v_k = 0 \text{ on } \partial D, \quad \overline{v_k} = 0 \tag{3.4}$$

Define

$$P_k(x) = \begin{cases} -\frac{|x|^2}{6} + \frac{r_k^2}{2}, & \text{if } |x| < r_k \\ \\ \frac{r_k^3}{3|x|}, & \text{if } |x| \geq r_k \end{cases}.$$

Then $-\Delta P_k(\cdot - \xi) = \chi_{B_k}$. Write $v_k = P_k(\cdot - \xi) + Q_k(\cdot, \xi)$. Clearly

$$\begin{aligned} -\Delta Q_k(x,\xi) &= -\frac{4\pi r_k^3}{3|D|} \text{ in } D, \\ \partial_\nu Q_k(x,\xi) &= -\partial_\nu \frac{4\pi r_k^3}{3} \frac{1}{4\pi|x-\xi|} \text{ on } \partial D, \\ \overline{Q_k(\cdot,\xi)} &= -\overline{P_k(|\cdot - \xi|)}. \end{aligned}$$

Here the Laplacian Δ and the outward normal derivative ∂_ν are taken with respect to x.

Note that from (2.10), $Q_k(x,\xi)$ and $\frac{4\pi r_k^3}{3} R(x,\xi)$ satisfy the same equation and the same boundary condition. Therefore they can differ only by a constant. This constant is $\overline{Q_k(\cdot,\xi)} - \frac{4\pi r_k^3}{3}\overline{R(\cdot,\xi)}$. But $\overline{v_k} = \overline{G(\cdot,\xi)} = 0$ implies that this constant is also equal to

$$-\frac{4\pi r_k^3}{3} \overline{\frac{1}{4\pi|\cdot - \xi|}} - \overline{P_k(\cdot - \xi)} = \frac{4\pi r_k^5}{3} \frac{1}{10|D|}.$$

Hence

$$Q_k(x,\xi) = \frac{4\pi r_k^3}{3} R(x,\xi) + \frac{4\pi}{3} \frac{r_k^5}{10|D|}. \tag{3.5}$$

Therefore, direct calculations show that, at each $\xi + r_1\theta$ where $\theta \in S^2$, a point on the inner sphere,

$$\begin{aligned} & -\frac{1}{r_1} + \gamma v(\xi + r_1\theta) \\ &= -\frac{1}{r_1} + \gamma[-\frac{r_1^2}{6} + \frac{r_2^2}{2} - \frac{r_1^2}{3} + \frac{4\pi(r_2^3 - r_1^3)}{3} R(\xi + r_1\theta, \xi) + \frac{4\pi(r_2^5 - r_1^5)}{30|D|}] \\ &= -\frac{1}{r_1} + \gamma[-\frac{r_1^2}{2} + \frac{r_2^2}{2} + a|D|R(\xi + r_1\theta, \xi)] + \frac{4\pi(r_2^5 - r_1^5)}{30|D|}] \\ &= -\frac{1}{r_1} + \gamma[-\frac{r_1^2}{2} + \frac{r_2^2}{2} + a|D|R(\xi,\xi) + \frac{4\pi(r_2^5 - r_1^5)}{30|D|}] + O(\gamma a^{4/3}) \\ &= -\frac{1}{r_1} + \gamma[-\frac{r_1^2}{2} + \frac{r_2^2}{2} + a|D|R(\xi,\xi) + \frac{4\pi(r_2^5 - r_1^5)}{30|D|}] + O(a^{1/3}). \end{aligned}$$

To reach the second last line we have used the fact that $r_1 = O(a^{1/3})$. At each $\xi + r_2\theta$ on the outer sphere

$$\frac{1}{r_2} + \gamma v(\xi + r_2\theta)$$

$$
\begin{aligned}
&= \frac{1}{r_2} + \gamma[\frac{r_2^2}{3} - \frac{r_1^3}{3r_2} + \frac{4\pi(r_2^3 - r_1^3)}{3} R(\xi + r_2\theta, \xi) + \frac{4\pi(r_2^5 - r_1^5)}{30|D|}] \\
&= \frac{1}{r_2} + \gamma[\frac{r_2^2}{3} - \frac{r_1^3}{3r_2} + a|D|R(\xi,\xi) + \frac{4\pi(r_2^5 - r_1^5)}{30|D|}] + O(a^{1/3}).
\end{aligned}
$$

This proves the lemma. □

Lemma 3.2 *The value of J at F is*

$$
J(F) = 4\pi(r_1^2 + r_2^2) + \frac{\gamma}{2}[\frac{12\pi r_1^5}{15} + \frac{8\pi r_2^5}{15} - \frac{4\pi r_1^3 r_2^2}{3} + a^2|D|^2 R(\xi,\xi) + \frac{4\pi a}{15}(r_2^5 - r_1^5)].
$$

Proof. Let $v = (-\Delta)^{-1}(\chi_F - a) = v_2 - v_1$ as in the proof of Lemma 3.1. The local part of $J(F)$ is just the area

$$
4\pi r_1^2 + 4\pi r_2^2. \tag{3.6}
$$

The nonlocal part of $J(F)$ is

$$
\begin{aligned}
&\frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_F - a)|^2\, dx \\
&= \frac{\gamma}{2}\int_D (\chi_F - a)v(x)\, dx \;=\; \frac{\gamma}{2}\int_D \chi_F v(x)\, dx \;=\; \frac{\gamma}{2}\int_F v(x)\, dx \\
&= \frac{\gamma}{2}\int_{B_2\setminus B_1} (v_2(x) - v_1(x))\, dx = \frac{\gamma}{2}[\int_{B_1} v_1 + \int_{B_2} v_2 - \int_{B_1} v_2 - \int_{B_2} v_1] \\
&= \frac{\gamma}{2}[I + II + III + IV]
\end{aligned} \tag{3.7}
$$

From the definition of P_k one finds that

$$
\int_{B_k} P_k(x)\, dx = \frac{8\pi r_k^5}{15}. \tag{3.8}
$$

For the integral of Q_k, note that, since $\Delta Q_k(\cdot,\xi) = \frac{4\pi r_k^3}{3|D|}$, $Q_k(x,\xi) - \frac{2\pi r_k^3}{9|D|}|x-\xi|^2$ is harmonic in x. By the Mean Value Theorem for harmonic functions

$$
\begin{aligned}
\int_{B_k} Q_k(x,\xi)\, dx &= \int_{B_k} (Q_k(x,\xi) - \frac{2\pi r_k^3}{9|D|}|x-\xi|^2)\, dx + \int_{B_k} \frac{2\pi r_k^3}{9|D|}|x-\xi|^2\, dx \\
&= \frac{4\pi r_k^3}{3} Q_k(\xi,\xi) + (\frac{4\pi}{3})^2 \frac{r_k^8}{10|D|} \\
&= (\frac{4\pi}{3})^2 r_k^6 R(\xi,\xi) + (\frac{4\pi}{3})^2 \frac{r_k^8}{5|D|}
\end{aligned} \tag{3.9}
$$

Therefore from (3.8) and (3.9)

$$
I = \frac{8\pi r_1^5}{15} + (\frac{4\pi}{3})^2 r_1^6 R(\xi,\xi) + (\frac{4\pi}{3})^2 \frac{r_1^8}{5|D|}, \tag{3.10}
$$

$$
II = \frac{8\pi r_2^5}{15} + (\frac{4\pi}{3})^2 r_2^6 R(\xi,\xi) + (\frac{4\pi}{3})^2 \frac{r_2^8}{5|D|}. \tag{3.11}
$$

Next note that

$$\begin{aligned} III &= -\int_{B_1} v_2\,dx = -\int_D \chi_{B_1} v_2 = -\int_D (\chi_{B_1} - \frac{\pi r_1^2}{|D|}) v_2\,dx \\ &= \int_D \Delta v_1 v_2 = -\int_D \nabla v_1 \cdot \nabla v_2\,dx = \int_D v_1 \Delta v_2\,dx = -\int_{B_2} v_1\,dx = IV. \end{aligned}$$

Therefore,

$$\begin{aligned} III &= IV = -\int_{B_1} v_2\,dx = -\int_{B_1} P_2\,dx - \int_{B_1} Q_2\,dx \\ &= \frac{4\pi r_1^5}{30} - \frac{4\pi r_1^3 r_2^2}{6} - \int_{B_1} (Q_2(x) - \frac{2\pi r_2^3}{9|D|}|x-\xi|^2)\,dx - \int_{B_1} \frac{2\pi r_2^3}{9|D|}|x-\xi|^2\,dx \\ &= \frac{4\pi r_1^5}{30} - \frac{4\pi r_1^3 r_2^2}{6} - \frac{4\pi r_1^3}{3} Q_2(\xi) - \frac{8\pi^2 r_1^5 r_2^3}{45|D|} \\ &= \frac{4\pi r_1^5}{30} - \frac{4\pi r_1^3 r_2^2}{6} - (\frac{4\pi}{3})^2 r_1^3 r_2^3 R(\xi,\xi) - (\frac{4\pi}{3})^2 \frac{r_1^3 r_2^5 + r_1^5 r_2^3}{10|D|}. \end{aligned}$$

Finally we sum these identities and use the fact $\frac{4\pi}{3}(r_2^3 - r_1^3) = a|D|$, to deduce the conclusion. $\square$

4 Perturbed shells

A perturbed shell E_ϕ is characterized by a pair of functions $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))$ on S^2 so that

$$E_\phi = \{\xi + \alpha\theta : \ \alpha \in ((r_1^3 + \phi_1(\theta))^{1/3}, (r_2^3 + \phi_2(\theta))^{1/3}), \ \forall \theta \in S^2\}, \tag{4.1}$$

and the boundaries of E_ϕ are two surfaces parametrized by θ: $\xi + (r_1^3 + \phi_1(\theta))^{1/3}\theta$, which is the perturbed inner sphere, and $\xi + (r_2^3 + \phi_2(\theta))^{1/3}\theta$, the perturbed outer sphere. We will restrict the size of ϕ_1, ϕ_2 so that $r_1^3 + \phi_1$, $r_2^3 + \phi_2$ are always positive. Moreover it is always assumed that $\phi \perp \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, i.e.

$$-\int_{S^2} \phi_1(\theta)\,d\theta + \int_{S^2} \phi_2(\theta)\,d\theta = 0. \tag{4.2}$$

This ensures that the volume of E_ϕ remains $a|D|$:

$$\begin{aligned} |E_\phi| &= \int_{S^2} \int_{(r_1^3+\phi_1(\theta))^{1/3}}^{(r_2^3+\phi_2(\theta))^{1/3}} r^2\,dr d\theta \ = \ \int_{S^2} \frac{r_2^3 + \phi_2(\theta) - r_1^3 - \phi_1(\theta)}{3}\,d\theta \\ &= \frac{4\pi r_2^3}{3} - \frac{4\pi r_1^4}{3} \ = \ a|D|. \end{aligned}$$

To express surface area in terms of ϕ, first define

$$L(s,p,q,\beta) = s^{-1/3}\sqrt{\frac{p^2}{9\sin^2\beta} + \frac{q^2}{9} + s^2}, \tag{4.3}$$

and then define

$$L_k(\phi_k, \frac{\partial \phi_k}{\partial \theta_1},\ \frac{\partial \phi_k}{\partial \theta_2},\ \theta_2) = r_k^2 L(1 + \frac{\phi_k}{r_k^3},\ \frac{1}{r_k^3}\frac{\partial \phi_k}{\partial \theta_1},\ \frac{1}{r_k^3}\frac{\partial \phi_k}{\partial \theta_2},\ \theta_2) \tag{4.4}$$

We have identified θ with (θ_1, θ_2) where θ_1 is the longitude and θ_2 the latitude. More precisely

$$\theta = (\cos\theta_1 \sin\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_2). \tag{4.5}$$

The surface area of $\partial_D E_\phi$ can be expressed as

$$|D\chi_{E_\phi}|(D) = \sum_{k=1}^{2} \int_{S^2} L_k(\phi_k, \frac{\partial \phi_k}{\partial \theta_1},\ \frac{\partial \phi_k}{\partial \theta_2},\ \theta_2)\, d\theta. \tag{4.6}$$

Here

$$d\theta = \sin\theta_2 d\theta_1 d\theta_2 \tag{4.7}$$

is the surface element. Calculating the variation of (4.6) we find two second order, quasi-linear, elliptic operators

$$\mathcal{H}_k(\phi_k)(\theta) = \frac{1}{\sin\theta_2}[\frac{\partial L_k}{\partial \phi_k}\sin\theta_2 - \frac{\partial}{\partial \theta_1}(\frac{\partial L_k}{\partial \phi_{k,1}}\sin\theta_2) - \frac{\partial}{\partial \theta_2}(\frac{\partial L_k}{\partial \phi_{k,2}}\sin\theta_2)]. \tag{4.8}$$

Here we have used short hand notations $\phi_{k,1} = \frac{\partial \phi_k}{\partial \theta_1}$ and $\phi_{k,2} = \frac{\partial \phi_k}{\partial \theta_2}$. Note that $\mathcal{H}_2$ gives one third of the curvature of the perturbed outer boundary viewed from E. However $\mathcal{H}_1$ is *negative* one third of the curvature of the perturbed inner boundary viewed from E.

The nonlocal part of J in (1.2) may be written in terms of ϕ as

$$\begin{aligned} \frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_{E_\phi} - a)|^2\, dx &= \frac{\gamma}{2}\int_{E_\phi}\int_{E_\phi} G(x,y)\, dxdy \\ &= \frac{\gamma}{2}\int_{S^2} d\theta \int_{(r_1^3+\phi_1(\theta))^{1/3}}^{(r_2^3+\phi_2(\theta))^{1/3}} dr \int_{S^2} d\omega \int_{(r_1^3+\phi_1(\omega))^{1/3}}^{(r_2^3+\phi_2(\omega))^{1/3}} dt \\ &\quad G(\xi + r\theta, \xi + t\omega) rt. \end{aligned} \tag{4.9}$$

The variation of (4.9) with respect to ϕ_1 is

$$-\frac{\gamma}{3}(-\Delta)^{-1}(\chi_{E_\phi} - a)(\xi + (r_1^3 + \phi_1(\theta))^{1/3}\theta) = -\frac{\gamma}{3}\int_{E_\phi} G(\xi + (r_1^3 + \phi_1(\theta))^{1/3}\theta, y)\, dy; \tag{4.10}$$

and the variation of (4.9) with respect to ϕ_2 is

$$\frac{\gamma}{3}(-\Delta)^{-1}(\chi_{E_\phi} - a)(\xi + (r_2^3 + \phi_2(\theta))^{1/3}\theta) = \frac{\gamma}{3}\int_{E_\phi} G(\xi + (r_2^3 + \phi_2(\theta))^{1/3}\theta, y)\, dy. \tag{4.11}$$

Under the constraint (4.2) the Euler-Lagrange equations of J are

$$\mathcal{H}_1(\phi_1)(\theta) - \frac{\gamma}{3}(-\Delta)^{-1}(\chi_{E_\phi} - a)(\xi + (r_1^3 + \phi_1(\theta))^{1/3}\theta) = \lambda \tag{4.12}$$

$$\mathcal{H}_2(\phi_2)(\theta) + \frac{\gamma}{3}(-\Delta)^{-1}(\chi_{E_\phi} - a)(\xi + (r_2^3 + \phi_2(\theta))^{1/3}\theta) = -\lambda \tag{4.13}$$

in terms of ϕ_1 and ϕ_2.

Remark 4.1 *Note that (4.13) differs from (1.1) by one third while (4.12) differs from (1.1) by* negative *one third.*

Let us define

$$\begin{aligned}
&\mathcal{A}_1(\phi)(\theta) \\
&= -\frac{\gamma}{3}\int_{S^2}\int_{(r_1^3+\phi_1(\omega))^{1/3}}^{(r_2^3+\phi_2(\omega))^{1/3}} \frac{1}{4\pi|(r_1^3+\phi_1(\theta))^{1/3}\theta - t\omega|} t\,dtd\omega \qquad (4.14)\\
&\mathcal{B}_1(\phi)(\theta) \\
&= -\frac{\gamma}{3}\int_{S^2}\int_{(r_1^3+\phi_1(\omega))^{1/3}}^{(r_2^3+\phi_2(\omega))^{1/3}} R(\xi+(r_1^3+\phi_1(\theta))^{1/3}\theta, \xi+t\omega)t\,dtd\omega \qquad (4.15)\\
&\mathcal{A}_2(\phi)(\theta) \\
&= \frac{\gamma}{3}\int_{S^2}\int_{(r_1^3+\phi_1(\omega))^{1/3}}^{(r_2^3+\phi_2(\omega))^{1/3}} \frac{1}{4\pi|(r_2^3+\phi_2(\theta))^{1/3}\theta - t\omega|} t\,dtd\omega \qquad (4.16)\\
&\mathcal{B}_2(\phi)(\theta) \\
&= \frac{\gamma}{3}\int_{S^2}\int_{(r_1^3+\phi_1(\omega))^{1/3}}^{(r_2^3+\phi_2(\omega))^{1/3}} R(\xi+(r_2^3+\phi_2(\theta))^{1/3}\theta, \xi+t\omega)t\,dtd\omega, \qquad (4.17)
\end{aligned}$$

so that (4.12) and (4.13) become

$$\mathcal{H}_1(\phi_1)+\mathcal{A}_1(\phi)+\mathcal{B}_1(\phi)=\lambda;\ \mathcal{H}_2(\phi_2)+\mathcal{A}_2(\phi)+\mathcal{B}_2(\phi)=-\lambda. \qquad (4.18)$$

Note that the operators $\mathcal{H}_k$ and $\mathcal{A}_k$ are independent of ξ while the operators $\mathcal{B}_k$ do depend on ξ.

Let $\mathcal{S}=(\mathcal{S}_1,\mathcal{S}_2)$ be the operator that appears on the left side of (4.18) projected to $\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}^{\perp}$, i.e.

$$\mathcal{S}_k(\phi)=\mathcal{H}_k(\phi_k)+\mathcal{A}_k(\phi)+\mathcal{B}_k(\phi)+(-1)^k\lambda(\phi) \qquad (4.19)$$

for $k=1,2$. Here $\lambda(\phi)$ is a number so chosen that $\mathcal{S}(\phi)\perp\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, i.e.

$$\int_{S^2}(-\mathcal{S}_1(\phi)+\mathcal{S}_2(\phi))\,d\theta=0. \qquad (4.20)$$

Now E_ϕ is a solution of (1.1) (and of course (4.18)) if and only if

$$\mathcal{S}(\phi)=0. \qquad (4.21)$$

The operator $\mathcal{S}=(\mathcal{S}_1,\mathcal{S}_2)$ maps from

$$\mathcal{X}=\left\{\phi=\begin{bmatrix} \phi_1(\theta) \\ \phi_2(\theta) \end{bmatrix}: \phi_k\in W^{2,p}(S^2),\ k=1,2,\ \phi\perp\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \qquad (4.22)$$

to

$$\mathcal{Y} = \left\{ q = \begin{bmatrix} q_1(\theta) \\ q_2(\theta) \end{bmatrix} : q_k \in L^p(S^2),\ k = 1, 2,\ q \perp \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}. \tag{4.23}$$

Here $\phi \perp \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ means that (4.2) holds. For technical reasons we assume that

$$p \in (2, \infty). \tag{4.24}$$

The first Fréchet derivative of $\mathcal{S}$ is given by

$$\mathcal{H}_k'(\phi_k)(u_k) = \frac{\partial \mathcal{H}_k}{\partial \phi_k} u_k + \sum_{i=1}^{2} \frac{\mathcal{H}_k}{\partial \phi_{k,i}} \frac{\partial u_k}{\partial \theta_i} + \sum_{i,j=1}^{2} \frac{\partial \mathcal{H}_k}{\partial \phi_{k,ij}} \frac{\partial^2 u_k}{\partial \theta_i \partial \theta_j} \tag{4.25}$$

$$\begin{aligned} \mathcal{A}_1'(\phi)(u)(\theta) = {} & \frac{\gamma}{9} \int_{S^2} \frac{u_1(\omega)}{4\pi |(r_1^3 + \phi_1(\theta))^{1/3}\theta - (r_1^3 + \phi_1(\omega))^{1/3}\omega|}\, d\omega \\ & + \frac{\gamma u_1(\theta)}{9(r_1^3 + \phi_1(\theta))^{2/3}} \int_{\tilde{E}_\phi} \frac{((r_1^3 + \phi_1(\theta))^{1/3}\theta - y) \cdot \theta}{4\pi |(r_1^3 + \phi_1(\theta))^{1/3}\theta - y|^3} dy \\ & - \frac{\gamma}{9} \int_{S^2} \frac{u_2(\omega)}{4\pi |(r_1^3 + \phi_1(\theta))^{1/3}\theta - (r_2^3 + \phi_2(\omega))^{1/3}\omega|}\, d\omega. \end{aligned} \tag{4.26}$$

$$\begin{aligned} \mathcal{B}_1'(\phi)(u)(\theta) = {} & \frac{\gamma}{9} \int_{S^2} u_1(\omega) R(\xi + (r_1^3 + \phi_1(\theta))^{1/3}\theta, \xi + (r_1^3 + \phi_1(\omega))^{1/3}\omega)\, d\omega \\ & - \frac{\gamma u_1(\theta)}{9(r_1^3 + \phi_1(\theta))^{2/3}} \int_{E_\phi} \nabla R(\xi + (r_1^3 + \phi_1(\theta))^{1/3}\theta, y) \cdot \theta\, dy. \\ & - \frac{\gamma}{9} \int_{S^2} u_2(\omega) R(\xi + (r_1^3 + \phi_1(\theta))^{1/3}\theta, \xi + (r_2^3 + \phi_2(\omega))^{1/3}\omega)\, d\omega \end{aligned} \tag{4.27}$$

$$\begin{aligned} \mathcal{A}_2'(\phi)(u)(\theta) = {} & -\frac{\gamma}{9} \int_{S^2} \frac{u_1(\omega)}{4\pi |(r_2^3 + \phi_2(\theta))^{1/3}\theta - (r_1^3 + \phi_1(\omega))^{1/3}\omega|}\, d\omega \\ & + \frac{\gamma}{9} \int_{S^2} \frac{u_2(\omega)}{4\pi |(r_2^3 + \phi_2(\theta))^{1/3}\theta - (r_2^3 + \phi_2(\omega))^{1/3}\omega|}\, d\omega \\ & - \frac{\gamma u_2(\theta)}{9(r_2^3 + \phi_2(\theta))^{2/3}} \int_{\tilde{E}_\phi} \frac{((r_2^3 + \phi_2(\theta))^{1/3}\theta - y) \cdot \theta}{4\pi |(r_2^3 + \phi_2(\theta))^{1/3}\theta - y|^3} dy \end{aligned} \tag{4.28}$$

$$\begin{aligned} \mathcal{B}_2'(\phi)(u)(\theta) = {} & -\frac{\gamma}{9} \int_{S^2} u_1(\omega) R(\xi + (r_2^3 + \phi_2(\theta))^{1/3}\theta, \xi + (r_1^3 + \phi_1(\omega))^{1/3}\omega)\, d\omega \\ & + \frac{\gamma}{9} \int_{S^2} u_2(\omega) R(\xi + (r_2^3 + \phi_2(\theta))^{1/3}\theta, \xi + (r_2^3 + \phi_2(\omega))^{1/3}\omega)\, d\omega \\ & + \frac{\gamma u_2(\theta)}{9(r_2^3 + \phi_2(\theta))^{2/3}} \int_{E_\phi} \nabla R(\xi + (r_2^3 + \phi_2(\theta))^{1/3}\theta, y) \cdot \theta\, dy. \end{aligned} \tag{4.29}$$

Here $\tilde{E}_\phi$ in $\mathcal{A}_k'$, $k = 1, 2$, is a shift of E_ϕ so that $\tilde{E}_\phi$ is centered at 0, i.e. $\tilde{E}_\phi = E_\phi - \xi$. The derivative of the operator λ is so chosen that

$$\mathcal{S}_k'(\phi) = \mathcal{H}_k'(\phi) + \mathcal{A}_k'(\phi) + \mathcal{B}_k'(\phi) + (-1)^k \lambda'(\phi),\ \mathcal{S}'(\phi)(u) \perp \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{4.30}$$

In $\mathcal{B}_k'$, ∇R is the gradient of R with respect to its first argument.

5 Linear analysis

Let $\mathcal{L}$ be the linearized operator of $\mathcal{S}$ at $\phi = 0$, i.e, at $E = F = B_2 \backslash B_1$:

$$\mathcal{L} = \mathcal{S}'(0). \tag{5.1}$$

Going back to (4.25), (4.27),(4.28), (4.29) and (4.29) we find that

$$\mathcal{H}_k'(0)(u_k) = -\frac{1}{9r_k^4}(\Delta_{S^2} u_k + 2u_k), \tag{5.2}$$

$$\mathcal{A}_1'(0)(u)(\theta) = \frac{\gamma}{9}\int_{S^2} \frac{u_1(\omega)}{4\pi|r_1\theta - r_1\omega|}\, d\omega - \frac{\gamma}{9}\int_{S^2} \frac{u_2(\omega)}{4\pi|r_1\theta - r_2\omega|}\, d\omega, \tag{5.3}$$

$$\begin{aligned} \mathcal{B}_1'(0)(u)(\theta) = & \frac{\gamma}{9}\int_{S^2} u_1(\omega) R(\xi + r_1\theta, \xi + r_1\omega)\, d\omega \\ & -\frac{\gamma u_1(\theta)}{9r_1^2}\int_F \nabla R(\xi + r_1\theta, y)\cdot\theta\, dy \\ & -\frac{\gamma}{9}\int_{S^2} u_2(\omega) R(\xi + r_1\theta, \xi + r_2\omega)\, d\omega, \end{aligned} \tag{5.4}$$

$$\begin{aligned} \mathcal{A}_2'(0)(u)(\theta) = & -\frac{\gamma}{9}\int_{S^2} \frac{u_1(\omega)}{4\pi|r_2\theta - r_1\omega|}\, d\omega + \frac{\gamma}{9}\int_{S^2} \frac{u_2(\omega)}{4\pi|r_2\theta - r_2\omega|}\, d\omega \\ & -\frac{\gamma u_2(\theta)}{9r_2}\left(\frac{1}{3} - \frac{r_1^3}{3r_2^3}\right), \end{aligned} \tag{5.5}$$

$$\begin{aligned} \mathcal{B}_2'(0)(u)(\theta) = & -\frac{\gamma}{9}\int_{S^2} u_1(\omega) R(\xi + r_2\theta, \xi + r_1\omega)\, d\omega \\ & +\frac{\gamma}{9}\int_{S^2} u_2(\omega) R(\xi + r_2\theta, \xi + r_2\omega)\, d\omega \\ & +\frac{\gamma u_2(\theta)}{9r_2^2}\int_F \nabla R(\xi + r_2\theta, y)\cdot\theta\, dy. \end{aligned} \tag{5.6}$$

In $\mathcal{H}_k'(0)$, Δ_{S^2} is the Laplace-Beltrami operator on S^2, given by

$$\Delta_{S^2} = \frac{1}{\sin^2\theta_2}\frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2} + \cot\theta_2 \frac{\partial}{\partial\theta_2}. \tag{5.7}$$

The derivation of $\mathcal{A}_k'(0)$ is explained in more detail in Appendix A.

Let us separate $\mathcal{L}$ into a dominant part $\mathcal{L}_1$ and a minor part $\mathcal{L}_2$. We define $\mathcal{L}_{1,k}$, the k-th component of $\mathcal{L}_1$, to be

$$\begin{aligned} \mathcal{L}_{1,1}(u)(\theta) = & -\frac{1}{9r_1^4}(\Delta_{S^2} u_1(\theta) + 2u_1(\theta)) \\ & +\frac{\gamma}{9}\int_{S^2} \frac{u_1(\omega)}{4\pi|r_1\theta - r_1\omega|}\, d\omega \\ & -\frac{\gamma}{9}\int_{S^2} \frac{u_2(\omega)}{4\pi|r_1\theta - r_2\omega|}\, d\omega \\ & -l_1(u) \end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_{1,2}(u)(\theta) &= -\frac{1}{9r_2^4}(\Delta_{S^2}u_2(\theta)+2u_2(\theta)) \\
&\quad -\frac{\gamma}{9}\int_{S^2}\frac{u_1(\omega)}{4\pi|r_2\theta-r_1\omega|}\,d\omega \\
&\quad +\frac{\gamma}{9}\int_{S^2}\frac{u_2(\omega)}{4\pi|r_2\theta-r_2\omega|}\,d\omega \\
&\quad -\frac{\gamma u_2(\theta)}{9r_2}(\frac{1}{3}-\frac{r_1^3}{3r_2^3}) \\
&\quad +l_1(u).
\end{aligned}
$$

The real valued linear operator l_1 is independent of k. It is so chosen that $\mathcal{L}_1$ maps from $\mathcal{X}$ to $\mathcal{Y}$. The rest of $\mathcal{L}$ is denoted by $\mathcal{L}_2$.

Recall $r=\frac{r_1}{r_2}<1$ first introduced in (2.2). In our Lyapunov-Schmidt procedure we are more interested in the operator $\Pi\mathcal{L}$ and $\Pi\mathcal{L}_1$ where Π is the projection operator from $\mathcal{Y}$ to

$$
\mathcal{Y}_*=\left\{q=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\in\mathcal{Y}: q\perp h\begin{bmatrix} r^2 \\ 1 \end{bmatrix}, \forall h\in H_1,\ q\perp\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}. \tag{5.8}
$$

More precisely Π is the orthogonal projection operator on the Hilbert space $L^2(S^2)\oplus L^2(S^2)$ restricted to $\mathcal{Y}$. Here H_1 is the space of spherical harmonics on S^2 of degree 1. The operator $\Pi\mathcal{L}$ is defined on

$$
\mathcal{X}_*=\left\{u=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\in\mathcal{X}: q\perp h\begin{bmatrix} r^2 \\ 1 \end{bmatrix}, \forall h\in H_1,\ q\perp\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}\subset\mathcal{Y}_*. \tag{5.9}
$$

Since every element of $\mathcal{X}$ (or $\mathcal{Y}$) is perpendicular to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, if $u=(u_1,u_2)\in\mathcal{X}_*$ (or $\mathcal{Y}_*$), it must be perpendicular to any constant vector, i.e.

$$
\int_{S^2}u_k(\theta)\,d\theta=0,\quad k=1,2. \tag{5.10}
$$

Consequently

$$
l_1(u)=0 \text{ if } u\in\mathcal{X}_*. \tag{5.11}
$$

Lemma 5.1 *1. $\|u\|_{H^2}\le Ca^{4/3}\|\mathcal{L}_1(u)\|_{L^2}$ for all $u\in\mathcal{X}_*$.*

2. Under the condition of Theorem 2.2 that $(S_{0,1},\Gamma)$ lies below all the W_n's, we have $\|u\|_{H^1}^2\le Ca^{4/3}\langle\mathcal{L}_1(u),u\rangle$ for all $u\in\mathcal{X}_*$.

Proof. The spectrum of $\Pi\mathcal{L}_1$ can be computed explicitly using spherical harmonics. Let $h=(h_1,h_2)$ with $h_1,h_2\in H_n$ where H_n is the space of spherical harmonics of degree n. In view of (5.10), we assume that $n\neq 0$. Consider $\mathcal{L}_1(h)$. Clearly

$$
-\Delta_{S^2}h_k=n(n+1)h_k. \tag{5.12}
$$

In Appendix B we show that

$$\int_{S^2} \frac{h_k(\omega)\, d\omega}{4\pi|\theta-\omega|} = \frac{h_k(\theta)}{2n+1}. \tag{5.13}$$

Also shown in the appendix are

$$\int_{S^2} \frac{h_2(\omega)\, d\omega}{4\pi|r\theta-\omega|} = \frac{r^n h_2(\theta)}{2n+1}, \quad \int_{S^2} \frac{h_1(\omega)\, d\omega}{4\pi|\theta-r\omega|} = \frac{r^n h_1(\theta)}{2n+1}. \tag{5.14}$$

With the help of (5.11) we can now write $\mathcal{L}_1(h)$ as

$$\mathcal{L}_1(h) = M_n h \tag{5.15}$$

where M_n is the 2 by 2 matrix

$$M_n = \begin{bmatrix} \frac{(n+2)(n-1)}{9r_1^4} + \frac{\gamma}{9(2n+1)r_1} & -\frac{\gamma r^n}{9(2n+1)r_2} \\ -\frac{\gamma r^n}{9(2n+1)r_2} & \frac{(n+2)(n-1)}{9r_2^4} + \frac{\gamma}{9r_2}[\frac{1}{2n+1} - \frac{1}{3} + \frac{r^3}{3}] \end{bmatrix}. \tag{5.16}$$

This means that $\mathcal{L}_1$ is invariant on $H_n \oplus H_n$.

When $n = 1$,

$$M_1 = \begin{bmatrix} \frac{\gamma}{27r_1} & -\frac{\gamma r_1}{27r_2^2} \\ -\frac{\gamma r_1}{27r_2^2} & \frac{\gamma r_1^3}{27r_2^4} \end{bmatrix}$$

has two eigenvalues. One of them is $\lambda_{1,1} = 0$, with eigenvector $\begin{bmatrix} r^2 \\ 1 \end{bmatrix}$. Hence $\mathcal{L}_1$ on $H_1 \oplus H_1$ has one eigenvalue $\lambda_{1,1} = 0$ and the corresponding eigenspace is $\{g \begin{bmatrix} r^2 \\ 1 \end{bmatrix} : g \in H_1\}$. However this subspace of $\mathcal{X}$ is perpendicular to $\mathcal{X}_*$. Therefore we ignore this eigenpair. Another consequence of this is that on $\mathcal{X}_*$, $\Pi\mathcal{L}_1 = \mathcal{L}_1$.

The second eigenvalue of M_1 is $\lambda_{1,2} = \frac{\gamma(1+r^4)}{r_1} > Ca^{-4/3}$ for some $C > 0$ independent of a, with eigenvector $\begin{bmatrix} -1 \\ r^2 \end{bmatrix}$. The corresponding eigenspace of $\mathcal{L}_1$ is $\{g \begin{bmatrix} -1 \\ r^2 \end{bmatrix} : g \in H_1\}$ which is a subspace of $\mathcal{X}_*$.

For $n \geq 2$, denote the $(1,1)$ entry of M_n by c_1, $(2,2)$ entry by c_2, and $(1,2)$ and $(2,1)$ entries by d. Then

$$\det(\lambda I - M_n) = \lambda^2 - (c_1 + c_2)\lambda + c_1 c_2 - d^2. \tag{5.17}$$

Let $\lambda_{n,1}, \lambda_{n,2}$ be the two eigenvalues of M_n, then we find that

$$\lambda_{n,1} = \frac{c_1 + c_2 + \sqrt{(c_1 - c_2)^2 + 4d^2}}{2}, \tag{5.18}$$

$$\lambda_{n,2} = \frac{c_1 + c_2 - \sqrt{(c_1 - c_2)^2 + 4d^2}}{2}. \tag{5.19}$$

It is obvious that $c_1 > c_2$, therefore

$$\lambda_{n,1} > \frac{c_1 + c_2 + c_1 - c_2}{2} = c_1 > Ca^{-4/3} > 0$$

where $C > 0$ is independent of a.

It remains to study $\lambda_{n,2}$. Let us introduce scaled variables R_j and Γ where

$$R_j = (\frac{3a|D|}{4\pi})^{-1/3} r_j, \ j = 1, 2; \quad \Gamma = (\frac{3a|D|}{4\pi})\gamma.$$

The constraint on r_j now becomes

$$R_2^3 - R_1^3 = 1.$$

The range (3.3) for r_1 and r_2 implies that

$$S_{0,1} - \delta_2 < R_1 < S_{0,1} + \delta_2. \tag{5.20}$$

The matrices M_n can be written as

$$M_n = (\frac{3a|D|}{4\pi})^{-4/3} \begin{bmatrix} \frac{(n+2)(n-1)}{9R_1^4} + \frac{\Gamma}{9(2n+1)R_1} & -\frac{\Gamma r^n}{9(2n+1)R_2} \\ -\frac{\Gamma r^n}{9(2n+1)R_2} & \frac{(n+2)(n-1)}{9R_2^4} + \frac{\Gamma}{9R_2}[\frac{1}{2n+1} - \frac{1}{3} + \frac{r^3}{3}] \end{bmatrix}$$

for $n \geq 2$. It is easy to see that asymptotically for fixed R_1 and Γ

$$\lim_{n\to\infty} \frac{\lambda_{n,1}}{(\frac{3a|D|}{4\pi})^{-4/3}(\frac{(n+2)(n-1)}{9R_1^4})} = 1, \quad \lim_{n\to\infty} \frac{\lambda_{n,2}}{(\frac{3a|D|}{4\pi})^{-4/3}(\frac{(n+2)(n-1)}{9R_2^4})} = 1. \tag{5.21}$$

Note that the second eigenvalue $\lambda_{n,2}$ is not zero if $\det M_n \neq 0$, and it is positive if $\det M_n > 0$. The equation $\det M_n = 0$ is quadratic in Γ:

$$\frac{1 - r^{2n} - \frac{(1-r^3)(2n+1)}{3}}{81(2n+1)^2 R_1 R_2}\Gamma^2$$

$$+[\frac{(n+2)(n-1)}{81(2n+1)R_1R_2^4} + \frac{(n+2)(n-1)}{81(2n+1)R_1^4R_2}(1 - \frac{(1-r^3)(2n+1)}{3})]\Gamma$$

$$+\frac{(n+2)^2(n-1)^2}{81R_1^4R_2^4} = 0.$$

This is the same as (2.3). The graph of the left side, as a function of Γ, is a downward parabola. Its intersection with the vertical axis is $(0, \frac{(n+2)^2(n-1)^2}{81R_1^4R_2^4})$. Therefore one root for Γ is negative, and the other root is positive.

We focus on the positive root which in Section 2 is denoted by $\tilde{\Gamma}_n(R_1)$. The first condition in Theorem 2.1 on $S_{0,1}$ and Γ ensures that if δ_2 is small, $\det M_n \neq 0$ and

hence the second eigenvalue $\lambda_{n,2}$ is not 0. With the help of the asymptotic formulae (5.21) we find $C > 0$, independent of a, such that

$$\lambda_{1,2} > Ca^{-4/3}, \quad \frac{|\lambda_{n,k}|}{n^2} > Ca^{-4/3}, k = 1, 2, \ n = 2, 3, ... \tag{5.22}$$

This implies that

$$\|u\|_{H^2} \leq Ca^{4/3} \|\mathcal{L}_1(u)\|_{L^2}, \tag{5.23}$$

for all $u \in \mathcal{X}_*$.

If we further assume that $(S_{0,1}, \Gamma)$ lies below all the W_n's, then (R_1, Γ) also lies below all the W_n's, if we let δ_2 be small enough. In this case $\det M_n > 0$ and there exists $C > 0$ such that

$$\lambda_{1,2} > Ca^{-4/3}, \quad \frac{\lambda_{n,k}}{n^2} > Ca^{-4/3}, k = 1, 2, \ n = 2, 3, ... \tag{5.24}$$

This implies that

$$\|u\|_{H^1}^2 \leq Ca^{4/3} \langle \mathcal{L}_1(u), u \rangle. \tag{5.25}$$

This proves the lemma. □

The second part $\mathcal{L}_2$ in $\mathcal{L}$ is a minor part.

Lemma 5.2 *There exists $C > 0$ independent of ξ, r_1, r_2 such that $\|\mathcal{L}_2(u)\|_{L^2} \leq Ca^{-2/3}\|u\|_{L^2}$ for all $u \in \mathcal{X}_*$.*

Proof. Recall $\mathcal{L}_2$:

$$\begin{aligned}
\mathcal{L}_{2,1}(u)(\theta) &= \frac{\gamma}{9} \int_{S^2} u_1(\omega) R(\xi + r_1\theta, \xi + r_1\omega)\, d\omega \\
&\quad - \frac{\gamma u_1(\theta)}{9r_1^2} \int_F \nabla R(\xi + r_1\theta, y) \cdot \theta\, dy \\
&\quad - \frac{\gamma}{9} \int_{S^2} u_2(\omega) R(\xi + r_1 e^{i\theta}, \xi + r_2 e^{i\omega})\, d\omega \\
&\quad - l_2(u) \\
\mathcal{L}_{2,2}(u)(\theta) &= -\frac{\gamma}{9} \int_{S^2} u_1(\omega) R(\xi + r_2\theta, \xi + r_1\omega)\, d\omega \\
&\quad + \frac{\gamma}{9} \int_{S^2} u_2(\omega) R(\xi + r_2\theta, \xi + r_2\omega)\, d\omega \\
&\quad + \frac{\gamma u_2(\theta)}{9r_2^2} \int_F \nabla R(\xi + r_2\theta, y) \cdot \theta\, dy \\
&\quad + l_2(u),
\end{aligned}$$

where $l_2(u)$ is real valued and is included so that $\mathcal{L}_2(u)$ is in $\mathcal{Y}$.

Because

$$R(\xi + r_k\theta, \xi + r_l\theta) - R(\xi, \xi) = O(a^{1/3})$$

and $\int_{S^2} u_k(\omega)\, d\omega = 0$, we obtain that

$$\begin{aligned}
&\|\frac{\gamma}{9}\int_{S^2} u_j(\omega) R(\xi + r_k\theta, \xi + r_l\omega)\, d\omega\|_{L^2} \\
&= \|\frac{\gamma}{9}\int_{S^2} u_j(\omega)(R(\xi + r_k\theta, \xi + r_l\omega) - R(\xi,\xi))\, d\omega\|_{L^2} \\
&\leq C\gamma a^{1/3}\|u_j\|_{L^2} \\
&\leq Ca^{-2/3}\|u_j\|_{L^2}.
\end{aligned}$$

Since the volume of F is of order $O(a)$,

$$\|\frac{\gamma u_k(\theta)}{9r_k^2}\int_F \nabla R(\xi + r_k\theta, y)\cdot\theta\, dy\|_{L^2} \leq Ca^{-2/3}\|u_k\|_{L^2}.$$

The condition

$$\overline{-\mathcal{L}_{2,1}(u) + \mathcal{L}_{2,2}(u)} = 0$$

in the definition of $\mathcal{Y}$ implies that

$$|l_2(u)| \leq Ca^{-2/3}\|u\|_{L^2}.$$

The lemma then follows. □

Lemma 5.3 *1. For $u \in \mathcal{X}_*$, $\|u\|_{H^2} \leq Ca^{4/3}\|\Pi\mathcal{L}(u)\|_{L^2}$.*

2. If $(S_{0,1}, \Gamma)$ lies below all the W_n's, then $\|u\|_{H^1}^2 \leq Ca^{4/3}\langle \Pi\mathcal{L}(u), u\rangle$.

Proof. When a is small, by Lemma 5.1 Part 1 and Lemma 5.2,

$$\begin{aligned}
\|\Pi\mathcal{L}(u)\|_{L^2} &\geq \|\Pi\mathcal{L}_1(u)\|_{L^2} - \|\Pi\mathcal{L}_2(u)\|_{L^2} \\
&\geq Ca^{-4/3}\|u\|_{L^2} - Ca^{-2/3}\|u\|_{L^2} \;\geq\; Ca^{-4/3}\|u\|_{L^2},
\end{aligned}$$

proving Part 1 of the lemma.

If $(S_{0,1}, \Gamma)$ lies below all the W_n's, it follows from Lemma 5.1 Part 2 and Lemma 5.2 that

$$\begin{aligned}
\langle \Pi\mathcal{L}(u), u\rangle &= \langle \Pi\mathcal{L}_1(u), u\rangle + \langle \Pi\mathcal{L}_2(u), u\rangle \\
&\geq Ca^{-4/3}\|u\|_{H^1}^2 - Ca^{-2/3}\|u\|_{L^2}^2 \;\geq\; Ca^{-4/3}\|u\|_{H^1}^2
\end{aligned}$$

when a is sufficiently small. □

As in [28, Lemma 5.2] we extend these estimates to the L^p setting. We skip the details of the proof.

Lemma 5.4 *1. For $u \in \mathcal{X}_*$, $\|u\|_{W^{2,p}} \leq Ca^{4/3}\|\Pi\mathcal{L}(u)\|_{L^p}$.*

2. $\Pi\mathcal{L} : \mathcal{X}_* \to \mathcal{Y}_*$ is one-to-one and onto.

Finally in this section we state a bound on the second Fréchet derivative of $\mathcal{S} = \mathcal{H} + \mathcal{A} + \mathcal{B} + \lambda$.

Lemma 5.5 *Assume that $\|\phi\|_{H^2} \le ca$ where c is sufficiently small. The following estimates hold for $u = (u_1, u_2) \in \mathcal{X}$, $v = (v_1, v_2) \in \mathcal{X}$.*

1. $\|\mathcal{H}_k''(\phi_k)(u_k, v_k)\|_{L^p} \le Ca^{-7/3}\|u_k\|_{W^{2,p}}\|v_k\|_{W^{2,p}}$.
2. $\|\mathcal{A}''(\phi)(u, v)\|_{L^p} \le Ca^{-7/3}\|u\|_{W^{1,p}}\|v\|_{W^{1,p}}$.
3. $\|\mathcal{B}''(\phi)(u, v)\|_{L^p} \le Ca^{-5/3}\|u\|_{W^{1,p}}\|v\|_{W^{1,p}}$.
4. $|\lambda''(\phi)(u, v)| \le Ca^{-7/3}\|u\|_{W^{2,p}}\|v\|_{W^{2,p}}$.

In summary $\|\mathcal{S}''(\phi)(u, v)\|_{L^p} \le Ca^{-7/3}\|u\|_{W^{2,p}}\|v\|_{W^{2,p}}$.

Note that by taking c small, we keep $r_k^3 + \phi_k$ positive, so E_ϕ is a perturbed shell. The proof of this lemma is similar to that of [28, Lemma 6.1]. We omit the details.

6 Existence and stability

In this section we will first prove that, for each ξ and r_1, there exists a pair of functions $\varphi(\cdot, \xi, r_1) = (\varphi_1(\cdot, \xi, r_1), \varphi_2(\cdot, \xi, r_1)) \in \mathcal{X}_*$ such that

$$\mathcal{S}(\varphi)(\theta) = A_1 \cos\theta_1 \sin\theta_2 \begin{bmatrix} r^2 \\ 1 \end{bmatrix} + A_2 \sin\theta_1 \sin\theta_2 \begin{bmatrix} r^2 \\ 1 \end{bmatrix} + A_3 \cos\theta_2 \begin{bmatrix} r^2 \\ 1 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{6.1}$$

for some real numbers A_1, A_2, A_3, B_1, B_2. Note that $\varphi(\cdot, \xi, r_1)$ is sought in $\mathcal{X}_*$. Later we will find a particular pair of ξ and r_1, say ζ and s_1 such that $\mathcal{S}(\varphi(\cdot, \zeta, s_1)) = 0$.

The equation (6.1) is written as

$$\Pi\mathcal{S}(\varphi(\cdot, \xi, r_1)) = 0 \tag{6.2}$$

where Π is the projection operator from $\mathcal{Y}$ to $\mathcal{Y}_*$. In the next section we will find a particular ξ and r_1, say ζ and s_1, such that at $\xi = \zeta$ and $r_1 = s_1$, $A_1 = A_2 = A_3 = B_1 = B_2 = 0$, i.e. $\mathcal{S}(\varphi(\cdot, \zeta, s_1)) = 0$. This means that by finding φ one reduces the original problem (1.1) to a problem of finding ζ and s_1 in a four dimensional set of (ξ, r_1).

Recall $\mathcal{L}$, the linearized operator of $\mathcal{S}$ at $\phi = 0$, i.e. $\mathcal{L}(u) = \mathcal{S}'(0)(u)$. Expand $\mathcal{S}(\phi)$ as

$$\mathcal{S}(\phi) = \mathcal{S}(0) + \mathcal{L}(\phi) + \mathcal{N}(\phi) \tag{6.3}$$

where $\mathcal{N}$ is a higher order term defined by (6.3). Rewrite (6.2) in a fixed point form:

$$\phi = -(\Pi\mathcal{L})^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{N}(\phi)) \tag{6.4}$$

Lemma 6.1 *There is $\varphi = \varphi(\cdot, \xi, r_1)$ such that for every $\xi \in U_1$ and $r_1 \in U_2$, $\varphi(\cdot, \xi, r_1) \in \mathcal{X}_*$ solves (6.4) and $\|\varphi(\cdot, \xi, r_1, r_2)\|_{W^{2,p}} \le ca^{5/3}$ where c is a sufficiently large constant independent of a, ξ, and r_1.*

The proof of this lemma, which we omit, is similar to that of [28, Lemma 7.1]. We state a result regarding the linearization of $\mathcal{S}$ at $\varphi(\cdot,\xi,r_1)$. Denote this linearized operator by $\tilde{\mathcal{L}}$, i.e.

$$\tilde{\mathcal{L}} = \mathcal{S}'(\varphi). \tag{6.5}$$

We have the following analogy of Lemma 5.4.

Lemma 6.2 *1. There exists $C > 0$ such that for all $u \in \mathcal{X}_*$*

$$\|u\|_{W^{2,p}} \leq Ca^{4/3}\|\Pi\tilde{\mathcal{L}}(u)\|_{L^p}$$

2. If $(S_{0,1},\Gamma)$ is below all the W_n's, then

$$\|u\|_{H^1}^2 \leq Ca^{4/3}\langle \Pi\tilde{\mathcal{L}}(u), u\rangle.$$

One consequence of this lemma is an estimate of $\frac{\partial\varphi}{\partial\xi_j}$.

Lemma 6.3 *φ satisfies $\|\dfrac{\partial\varphi}{\partial\xi_j}\|_{W^{2,p}} = O(a^{4/3})$, $j = 1,2,3$.*

The proofs of the last two lemmas can be found in the proofs of [28, Lemmas 7.2 and 7.3].

We now prove Theorems 2.1 and 2.2. From Lemma 6.1 we know that for every $\xi \in U_1$ and $r_1 \in U_2$ there exists $\varphi(\cdot,\xi,r_1) \in \mathcal{X}_*$ such that $\Pi\mathcal{S}(\varphi(\cdot,\xi,r_1)) = 0$, i.e. (6.1) holds. We now find particular ξ and r_1 denoted by ζ and s_1 such that $\mathcal{S}(\varphi(\cdot,\zeta,s_1)) = 0$.

Lemma 6.4 *$J(E_{\varphi(\cdot,\xi,r_1)}) = J(F) + O(a^2)$.*

Proof. On the one hand expanding $J(E_\varphi)$ yields

$$J(E_\varphi) = J(F) + \sum_{k=1}^{2}\int_{S^2}\mathcal{S}_k(0)\varphi_k\,d\theta + \frac{1}{2}\sum_{k=1}^{2}\int_{S^2}\mathcal{L}_k(\varphi)\varphi_k\,d\theta + O(a^{8/3}). \tag{6.6}$$

The error term in (6.6) is obtained by Lemma 5.5. and the fact $\|\varphi\|_{H^2} = O(a^{5/3})$.

On the other hand $\Pi\mathcal{S}(\varphi) = 0$ implies that

$$\Pi(\mathcal{S}(0) + \mathcal{L}(\varphi) + \mathcal{N}(\varphi)) = 0$$

where $\mathcal{N}$ is given in (6.3). We multiply the last equation by φ and integrate to derive, again with the help of Lemma 5.5,

$$\sum_{k=1}^{2}\int_{S^2}\mathcal{S}_k(0)\varphi_k\,d\theta + \sum_{k=1}^{2}\int_{S^2}\mathcal{L}_k(\varphi)\varphi_k\,d\theta = O(a^{8/3}).$$

We can now rewrite (6.6) as

$$J(E_\varphi) = J(F) + \frac{1}{2}\sum_{k=1}^{2}\int_{S^2} \mathcal{S}_k(0)\varphi_k \, d\theta + O(a^{8/3}).$$

Lemma 3.1 and the fact $\|\varphi\|_{W^{2,p}} = O(a^{5/3})$ implies that

$$J(E_\varphi) = J(F) + O(a^2) + O(a^{8/3}) = J(F) + O(a^2).$$

When we use Lemma 3.1, note that $\mathcal{S}(0)$ is a sum of a θ independent part and a quantity of order $O(a^{1/3})$, and that $\varphi_k \perp 1$. This proves the lemma. $\square$

Lemma 6.5 *As a function of (ξ, r_1), $J(E_{\varphi(\cdot,\xi,r_1)})$ is locally minimized at some (ζ, s_1), when a is small. As $a \to 0$,*

$$\zeta \to \zeta_0, \quad (\frac{3a|D|}{4\pi})^{-1/3} s_1 \to S_{0,1},$$

possibly along a subsequence, where $R(\zeta_0, \zeta_0) = \min_{x\in D} R(x,x)$.

Proof. If we consider $J(\varphi(\cdot, \xi, r_1))$ as a function of ξ and r_1, then Lemmas 3.2 and 6.4 imply that

$$\begin{aligned} J(E_\varphi) &= 4\pi(r_1^2 + r_2^2) \\ &\quad + \frac{\gamma}{2}[\frac{12\pi r_1^5}{15} + \frac{8\pi r_2^5}{15} - \frac{4\pi r_1^3 r_2^2}{3} + a^2|D|^2 R(\xi,\xi) + \frac{4\pi a(r_2^5 - r_1^5)}{15}] + O(a^2) \\ &= 4\pi(r_1^2 + r_2^2) + \frac{\gamma}{2}[\frac{12\pi r_1^5}{15} + \frac{8\pi r_2^5}{15} - \frac{4\pi r_1^3 r_2^2}{3} + a^2|D|^2 R(\xi,\xi)] + O(a^{5/3}). \end{aligned}$$

Introduce the scaled variables R_j and Γ where

$$R_j = (\frac{3a|D|}{4\pi})^{-1/3} r_j, \ j = 1, 2; \quad \Gamma = (\frac{3a|D|}{4\pi})\gamma,$$

with the new constraint $R_2^3 - R_1^3 = 1$. Recall the function Q_Γ given in (2.8). We find

$$J(E_{\varphi(\cdot,\xi,r_1)}) = (\frac{3a|D|}{4\pi})^{2/3}[Q_\Gamma(R_1) + (\frac{3}{4\pi})^{-5/3}(a|D|)^{1/3}(\frac{\Gamma}{2})R(\xi,\xi)] + O(a^{5/3}). \quad (6.7)$$

By our assumption that $S_{0,1}$ locally minimizes Q_Γ, $J(E_{\varphi(\cdot,\xi,r_1)})$ is minimized at some ζ and s_1 and $\zeta \to \zeta_0$ and $(\frac{3a|D|}{4\pi})^{-1/3} s_1 \to S_{0,1}$ as $a \to 0$. See [28, Lemma 8.2] for more detail. $\square$

We conclude that $\varphi(\cdot, \zeta, s_1)$ is an exact solution of (1.1).

Lemma 6.6 *At $\xi = \zeta$ and $r_1 = s_1$, $\mathcal{S}(\varphi) = 0$.*

This lemma is intuitively clear, but its proof is quite tricky. The details of the proof can be found in [28, Lemmas 8.3 and 8.4], and [11, Lemmas 7.3 and 7.4].

We know now that $\varphi(\cdot,\zeta,s_1)$ found in Lemma 6.5 solves $\mathcal{S}(\phi)=0$ and hence the equation (1.1). The center of the perturbed ring solution $E_{\varphi(\cdot,\zeta,s_1)}$ is ζ, the inner radius is s_1 and the outer radius is $s_2=(\frac{3a|D|}{4\pi}+s_1^3)^{1/3}$. As $a\to 0$, $\zeta\to\zeta_0$ and $(\frac{a|D|}{\pi})^{-1/2}s_1\to S_{0,1}$ possibly along a subsequence, where ζ_0 minimizes the function $R(x,x)$ in D. This proves Theorem 2.1.

In Theorem 2.2, a solution is termed stable if it is a local minimizer of J in the space

$$(U_1\times U_2)\times\{\phi=(\phi_1,\phi_2):\ \phi_k\in H^1(S^1),\ k=1,2,\ \phi\in\mathcal{Y}_*\}. \tag{6.8}$$

In Theorem 2.2 if $(S_{0,1},\Gamma)$ lies below all the W_n's, Lemma 6.2, Part 2, shows that each $\varphi(\cdot,\xi,r)$ we found in Lemma 6.1 locally minimizes J, with fixed $(\xi,r)\in U_1\times U_2$, in $\{\phi:\phi_k\in H^1(S^1),\ \phi\in\mathcal{Y}_*\}$. On the other hand $\varphi(\cdot,\zeta,s_1)$ minimizes $J(E_{\varphi(\cdot,\xi,r_1)})$ with respect to (ξ,r_1) in $U_1\times U_2$. Hence $\varphi(\cdot,\zeta,s_1)$ is a local minimizer of J in (6.8).

If $(S_{0,1},\Gamma)$ lies between two curves, there is $n\in\{2,3,...\}$ such that $(S_{0,1},\Gamma)$ is above the curve W_n. Then the eigenvalue $\lambda_{n,2}$ of $\mathcal{L}_1$ is negative. There exists $C>0$ such that

$$\lambda_{n,2}<-Ca^{-4/3},\quad \langle\mathcal{L}_1(e_{n,2}),e_{n,2}\rangle<-Ca^{-4/3}\|e_{n,2}\|_{L^2}^2$$

where $e_{n,2}$ is an eigenvector of $\mathcal{L}_1$ corresponding to $\lambda_{n,2}$. By Lemma 5.2, the last inequality implies that

$$\langle\mathcal{L}(e_{n,2}),e_{n,2}\rangle<-Ca^{-4/3}\|e_{n,2}\|_{L^2}^2.$$

Then by Lemma 5.5

$$\langle\tilde{\mathcal{L}}(e_{n,2}),e_{n,2}\rangle<-Ca^{-4/3}\|e_{n,2}\|_{L^2}^2.$$

Therefore the solution is unstable. This proves Theorem 2.2.

7 Discussion

In Figure 1 the graph of Q_Γ shows that when Γ is large, Q_Γ also has a local maximum in addition to the local minimum $S_{0,1}$. This local maximum indicates the existence of another unstable shell solution whose inner radius corresponds to the local maximum on the graph of Q_Γ.

To prove this assertion one uses the same argument and reduces the problem to $J(E_{\varphi(\cdot,\xi,r_1)})$. However instead of Lemma 6.5 where a local minimum of $J(E_{\varphi(\cdot,\xi,r_1)})$ is found, we have to prove the existence of a saddle point for $J(E_{\varphi(\cdot,\xi,r_1)})$. Roughly speaking we would like to minimize $J(E_{\varphi(\cdot,\xi,r_1)})$ with respect to ξ and *maximize* $J(E_{\varphi(\cdot,\xi,r_1)})$ with respect to r_1. This intuitive idea may be made rigorous by a type of mini-max argument.

Our numerical calculations suggest that the shell solution found in this paper is always unstable. The analogy in two-dimensions, where a ring shaped solution is sought, is quite different. In [11] the analogous graph of Figure 2 is Figure 3, and the observation is the following.

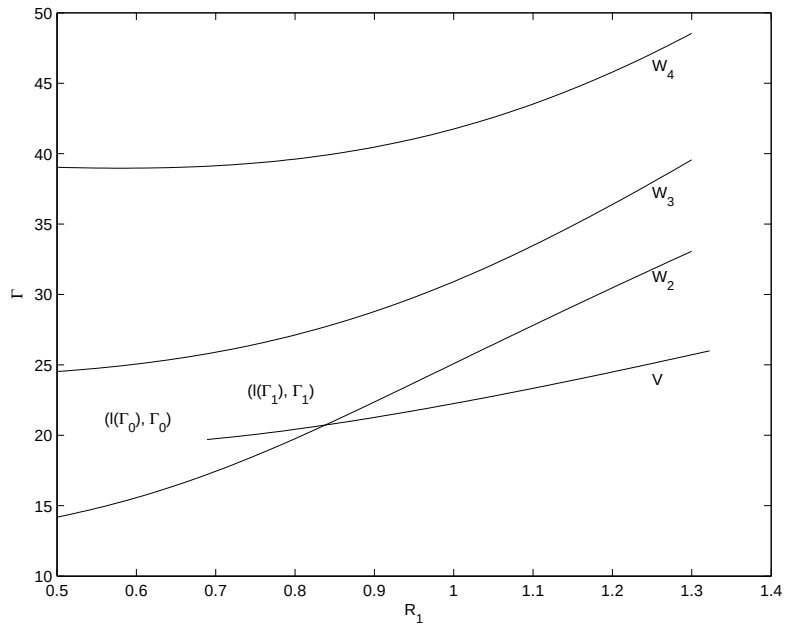

Figure 3: The illustration of Γ_0 and Γ_1 in two dimensions.

Observation 7.1 ([16]) *Let D be a bounded two dimensional domain. There exist two universal constants Γ_0 and Γ_1, with $0 < \Gamma_0 < \Gamma_1$, such that for any compact subset $\mathcal{K}$ of $(\Gamma_0, \Gamma_1) \cup (\Gamma_1, \infty)$ there is a constant $a_0 > 0$ such that if*

$$a < a_0 \quad \textit{and} \quad \gamma(\frac{a|D|}{\pi})^{3/2} \in \mathcal{K},$$

there exists a ring pattern solution of (1.1).

If $\gamma(\frac{a|D|}{\pi})^{3/2}$ is in $\mathcal{K} \cap (\Gamma_0, \Gamma_1)$, then the solution is unstable; if $\gamma(\frac{a|D|}{\pi})^{3/2}$ is in $\mathcal{K} \cap (\Gamma_1, \infty)$, then the solution is stable.

Hence in two dimensions as long as $\Gamma > \Gamma_1$, the ring solution is stable.

A Appendix

In this appendix we show that

$$\int_{\tilde{F}} \frac{(r_1\theta - y)\cdot\theta}{4\pi|r_1\theta - y|^3} dy = 0 \tag{A.1}$$

$$\int_{\tilde{F}} \frac{(r_2\theta - y)\cdot\theta}{4\pi|r_2\theta - y|^3} dy = \frac{1}{3}(r_2 - \frac{r_1^3}{r_2^2}), \tag{A.2}$$

where $\tilde{F} = B_{r_2}(0) \backslash B_{r_1}(0)$, $B_{r_1}(0)$ is the ball centered at 0 with radius r_1, and $B_{r_2}(0)$ is the ball centered at 0 with radius r_2.

We first calculate the integral

$$\int_{B_{r_1}(0)} \frac{(r_1\theta - y)\cdot\theta}{|r_1\theta - y|^3}\, dy.$$

This integral is independent of $\theta \in S^2$ so we take $\theta = (0,0,1)$. Scale $B_{r_1}(0)$ to $B_1(0)$, the disc centered at 0 of radius 1, so that

$$\int_{B_{r_1}(0)} \frac{(r_1\theta - y)\cdot\theta}{|r_1\theta - y|^3}\, dy = r_1 \int_{B_1(0)} \frac{(\theta - y)\cdot\theta}{|\theta - y|^3}\, dy$$

Use the cylindrical coordinates $y = (r\cos p, r\sin p, y_3)$ to deduce

$$\int_{B_1(0)} \frac{(\theta - y)\cdot\theta}{|\theta - y|^3}\, dy = \int_{B_1(0)} \frac{1 - y_3}{|(0,0,1) - y|^3}\, dz$$

$$= 2\pi \int_{-1}^{1}\int_0^{\sqrt{(1/r)^2 - y_3^2}} \frac{(1-y_3)\rho\, d\rho dy_3}{[(1-y_3)^2 + \rho^2]^{3/2}} = \frac{4\pi}{3}.$$

We have our first formula

$$\int_{B_{r_1}(0)} \frac{(r_1\theta - y)\cdot\theta}{|r_1\theta - y|^3}\, dy = \frac{4\pi r_1}{3}. \tag{A.3}$$

Next we calculate

$$\int_{B_{r_2}(0)} \frac{(r_1\theta - y)\cdot\theta}{|r_1\theta - y|^3}\, dy = r_1 \int_{B_{1/r}(0)} \frac{(\theta - y)\cdot\theta}{|\theta - y|^2}\, dy.$$

The integral is independent of $\theta \in S^2$. Using the cylindrical coordinates again, we find

$$\begin{aligned}
\int_{B_{1/r}(0)} \frac{(\theta - y)\cdot\theta}{|\theta - y|^3}\, dy &= 2\pi \int_{-1/r}^{1/r}\int_0^{\sqrt{(1/r)^2 - y_3^2}} \frac{(1-y_3)\rho\, d\rho dy_3}{[(1-y_3)^2 + \rho^2]^{3/2}} \\
&= 2\pi \int_{-1/r}^{1/r} [\frac{y_3 - 1}{(1 + \frac{1}{r^2} - 2y_3)^{1/2}} - \frac{y_3 - 1}{|1 - y_3|}]\, dy_3 \\
&= \frac{4\pi r_1}{3}.
\end{aligned}$$

We have our second formula

$$\int_{B_{r_2}(0)} \frac{(r_1\theta - y)\cdot\theta}{|r_1\theta - y|^3}\, dy = \frac{4\pi r_1}{3}. \tag{A.4}$$

(A.1) follows from (A.3) and (A.4).

Similarly to (A.3) we have

$$\int_{B_{r_2}(0)} \frac{(r_2\theta - y)\cdot\theta}{|r_2\theta - y|^3}\, dy = \frac{4\pi r_2}{3}. \tag{A.5}$$

But different from (A.4) we see that

$$\int_{B_{r_1}(0)} \frac{(r_2\theta - y)\cdot\theta}{|r_2\theta - y|^3} dy = r_2 \int_{B_r(0)} \frac{(\theta - y)\cdot\theta}{|\theta - y|^3}\, dy$$

and the function

$$y \to \frac{(\theta - y)\cdot\theta}{|\theta - y|^3}$$

is harmonic (without singularity) for $y \in B_r(0)$. The Mean Value Theorem for harmonic functions implies that

$$\int_{B_r(0)} \frac{(\theta - y)\cdot\theta}{|\theta - y|^3}\, dy = \frac{4\pi r^3}{3}.$$

We now have our last formula

$$\int_{B_{r_1}(0)} \frac{(r_2\theta - y)\cdot\theta}{|r_2\theta - y|^3} dy = \frac{4\pi r_1^3}{3r_2^2}. \tag{A.6}$$

(A.2) follows from (A.5) and (A.6).

B Appendix

The integral operator

$$h(\theta) \to \int_{S^2} \frac{h(\omega)\, d\omega}{|\theta - \omega|} \tag{B.1}$$

acts on spherical harmonics $h \in H_n$ in a simple way. Here H_n is the space of spherical harmonics of degree n on S^2. In general one has

$$\int_{S^2} \Phi(\theta\cdot\omega) h(\omega)\, d\omega = \alpha_n(\Phi) h(\theta) \tag{B.2}$$

where

$$\alpha_n(\Phi) = 2\pi \int_{-1}^{1} \Phi(t) P_n(t)\, dt. \tag{B.3}$$

See for instance [9, Theorem 3.4.1]. Here P_n is the n-th Legendre polynomial. In our case

$$\frac{1}{|\theta - \omega|} = \frac{1}{\sqrt{2 - 2\theta\cdot\omega}},$$

so we take

$$\Phi(t) = \frac{1}{\sqrt{2 - 2t}}. \tag{B.4}$$

The classical representation of Legendre polynomials in terms of generating functions ([9, Formula 3.3.39])

$$\frac{1}{(1 + r^2 - 2rt)^{1/2}} = \sum_{n=0}^{\infty} P_n(t) r^n, \quad r, t \in (-1, 1) \tag{B.5}$$

shows that

$$\int_{-1}^{1} \frac{P_n(t)\,dt}{(1+r^2-2rt)^{1/2}} = r^n \int_{-1}^{1} P_n^2(t)\,dt = \frac{2r^n}{2n+1}$$

where the orthogonality of the Legendre polynomials is used ([9, Formula 3.3.16]):

$$\int_{-1}^{1} P_n(t)P_m(t)\,dt = \frac{2\delta_{nm}}{2n+1}.$$

By sending $r \to 1$ we find that

$$\alpha_n(\Phi) = \frac{4\pi}{2n+1}. \tag{B.6}$$

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