A Variational Approach to Multiple Layers of the Bistable Equation in Long Tubes

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Abstract

Multiple layer solutions of the balanced bistable equation in infinite tubes are constructed via a variational method. I start with a characterization of Palais-Smale sequences which easily gives some global minima in the desired function classes as single layers. Assuming these minima are isolated as critical points, I paste them together to serve as an approximate multiple layer solution. If there were no exact solutions near the approximate one, the negative gradient flow of the energy functional would significantly lower the energy. On the other hand, if the building minima are kept far from each other, the energy of a function near the approximate solution is not much less than that of the approximate solution. This contradiction proves the existence of a solution.

1 Introduction

I study the elliptic equation

$$\begin{cases} \Delta u - f(u) = 0 \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

where f is a balanced bistable function (see Figure 1), and Ω is an unbounded tube-shaped domain in R^d. It is the Euler-Lagrange equation of the energy functional

$$E(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 + W(u)]dx \tag{1.2}$$

where $W(u) = \displaystyle\int_{-1}^{u} f(w)dw$, and W is a balanced double well potential function (see Figure 2).

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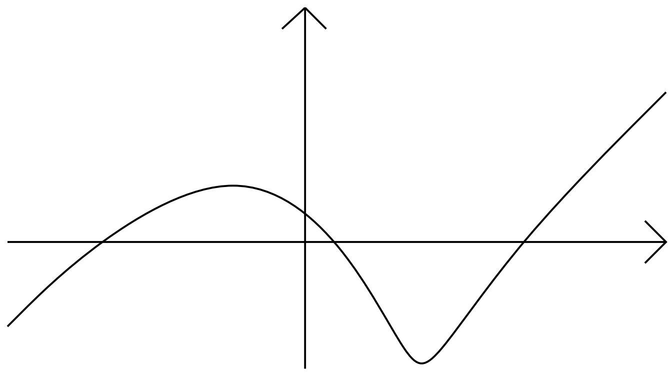

Figure 1: The graph of f.

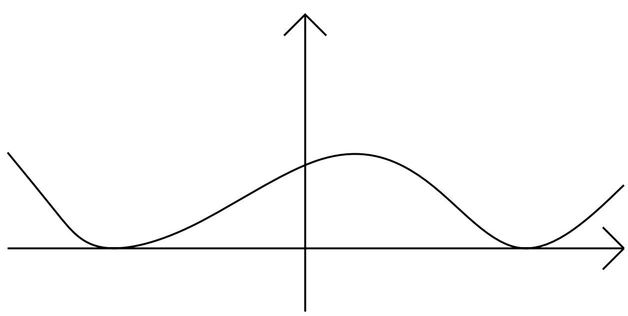

Figure 2: The graph of W.

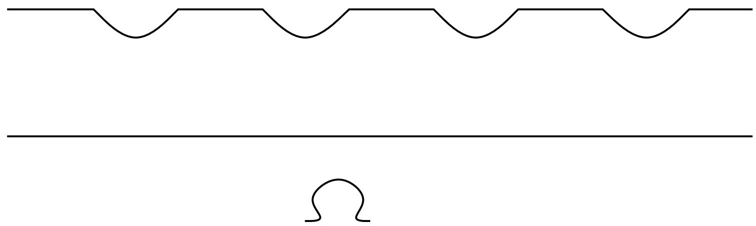

Figure 3: An example of periodic Ω.

(1.1) and (1.2) are often used to study phase transitions. The function u is an order parameter representing the state of a material. $u = -1$ and $u = 1$ represent two different orientations of a perfect crystal. Intermediate values of u then represent disordered states intermediate between the pure states. The first term in the integral of (1.2) is the interfacial energy density, and the second term is the bulk energy density. Note that the first term penalizes spacially inhomogeneous materials while the second term penalizes states that take values other than -1 or 1. In this paper I assume

H-1 W is a C^2 function that has exactly two global minima at -1 and 1 where $W(-1) = W(1) = 0$, and $W''(-1) > 0$, $W''(1) > 0$, and

H-2 There exists a positive number Θ such that $W(u) = \Theta u^2$ for all $|u| > 2$.

The second hypothesis is more of technical nature. Indeed for each W satisfying H-1 and $W''(u) > 0$ for $|u| > 1$, the maximum principle implies that bounded solutions of (1.1) lie between -1 and 1, so one can always modify W to satisfy H-2 without affecting bounded solutions. Except in Section 5 I assume throughout the paper

H-3 Ω is a smooth infinite tube T-periodic in x^1-direction.

To be more specific, I assume that there exists $T = (T^1, 0, ..., 0) \in R^d$ such that $x = (x^1, x') \in R^1 \times R^{d-1} = R^d$ is in Ω if and only if $x + T$ is in Ω. Furthermore, for every $(x^1, x') \in \Omega$, x' lies in a bounded subset of R^{d-1}. The two ends of Ω are denoted by e_1 and e_2 (See Figure 3).

The domain of the functional E is taken to be

$$\mathcal{A} = \{u \in L^1_{loc}(\Omega) : \ \nabla u \in (L^2(\Omega))^d, \ W(u) \in L^1(\Omega)\}. \tag{1.3}$$

Here $L^1_{loc}(\Omega)$ is the space of measurable functions that belong to $L^1(K)$ for every compact subset K of Ω. It turns out that $\mathcal{A}$ can be naturally divided into four mutually disjoint subclasses $\mathcal{A}^{-1}_{-1}$, $\mathcal{A}^{1}_{-1}$, $\mathcal{A}^{-1}_{1}$ and $\mathcal{A}^{1}_{1}$ where

$$\mathcal{A}^{\eta}_{\zeta} = \{u \in \mathcal{A} : \lim_{x^1 \to -\infty} u(x^1, x') = \zeta, \ \lim_{x^1 \to \infty} u(x^1, x') = \eta\}, \ \zeta, \eta \in \{-1, 1\}.$$

The limits above are understood in an appropriate sense (see (2.5)).

Obviously $u \equiv -1$ is the global minimum of E in $\mathcal{A}_{-1}^{-1}$, and $u \equiv 1$ is the global minimum in $\mathcal{A}_1^1$. I prove in Section 3 that E has global minima in $\mathcal{A}_{-1}^1$ and $\mathcal{A}_1^{-1}$ as well (see Theorem 3.2). Then in Section 4 I prove that given some isolated global minima in $\mathcal{A}_{-1}^1$ and $\mathcal{A}_1^{-1}$, I can always "paste" them together to obtain multiple layer solutions of (1.1) (see Theorem 4.4). I believe these solutions are local minima of E. The isolation condition excludes perfect tubes. A perfect tube Ω is $R^1 \times \Omega'$ for some bounded Ω' in R^{d-1}. In a perfect tube, every translation of a minimum is again a minimum. For more information about (1.1) in a perfect tube, I refer the reader to [5] of Berestycki and Nirenberg.

The main difficulty (or advantage) of this paper is the unboundedness of the domain Ω. The usual compactness argument in variational problems is no longer available. For instance, a Palais-Smale sequence does not necessarily converge along a subsequence. A characterization of Palais-Smale sequences is given in Section 3.

The key references of this work are [7] and [8] of Coti Zelati and Rabinowitz where multi-bump homoclinic type solutions of some Hamiltonian systems in R^1 and semilinear elliptic equations in R^d are studied. Another important reference is [13] of Rabinowitz. It treats a second order Hamiltonian system of the form

$$\ddot{q} + V_q(t, q) = 0,$$

with $q : R^1 \to R^n$, which can be viewed as a one-dimensional version of (1.1). There is also a paper [16] of Strobel along this line. In this ODE setting each function q in its admissible set is Holder continuous. On each interval the modulus of continuity of q is controlled by its energy there. In the PDE setting (1.1) this property is lost. To perform an estimate on a function $u \in \mathcal{A}$, I often have to use a local average $\widehat{u} : R^1 \to R^1$ (see (2.2)) of u, whose modulus of continuity can be controlled. Then I consider two cases: (1) u is close to $\widehat{u}$ and (2) u is far from $\widehat{u}$. The argument for the first case is usually simple, but the one for the second case is sometimes tedious (see the proof of Lemma 2.3 and the comment after the proof).

A somewhat related variational approach to (1.1) takes place in Kohn and Sternberg [10] that deals with the singularly perturbed equation

$$\begin{cases} \epsilon^2 \Delta u - f(u) = 0 \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

where Ω is a bounded domain. As ϵ approaches 0, the limiting problem can be identified, and local minima can be found if the limiting problem has nondegenerate local minima. I believe that the smallness condition of ϵ in [10] is related to the unboundedness and periodicity conditions of Ω in this paper.

Work for unbounded domains in the literature includes Bronsard, Gui and Schatzman [6] that treats three layered solutions $u : R^2 \to R^2$ for a three well potential, and Alama, Bronsard and Gui [1] that treats heteroclinic type solutions $u : R^2 \to R^2$ for a two well potential.

The existence of global minima in $\mathcal{A}_\zeta^\eta$ has appeared in Bates and Ren [3], [4] where a higher order variational problem is considered. In those papers the concentration-compactness principle (see [11]) is applied. Such an idea can also be used here to yield global minima, but I choose to present a more elegant approach to Theorem 3.2.

I include a section to discuss further generalizations of my results. In particular, I discuss tubes that are only periodic near infinity. I sketch an existence result of global minima. It is proved with the help of the concentration-compactness principle.

I would like to thank Professor Peter Bates for many stimulating conversations. I am grateful to Professor Xinfu Chen who showed me the proof of Lemma 2.3. Professor Paul Rabinowitz kindly shared his insight with me and brought [13] and [16] to my attention. I thank the referee who pointed out more references and corrected several errors in an earlier version of this paper.

2 Preliminary Results

I use C, C_1, C_2,.... to denote generic constants in this paper. They may vary from line to line. Note, by H-1 and H-2, there exists $C > 0$ such that $f^2(u) \le CW(u)$ for all $u \in R^1$. Define a segment $S(y^1, R)$ of Ω to be

$$S(y^1, R) := \{(x^1, x') \in \Omega : \ |x^1 - y^1| < R\}. \tag{2.1}$$

For every $u \in \mathcal{A}$ (defined in (1.3)), define

$$\widehat{u}(x^1) = \frac{1}{|S(x^1, T^1/2)|} \int_{S(x^1,T^1/2)} u(y)dy, \ x^1 \in R^1 \tag{2.2}$$

where $|S(x^1, T^1/2)|$ denotes the Lebesgue measure of $S(x^1, T^1/2)$ in R^d. Note that $\widehat{u}$ is continuous. $\widehat{u}$ can be used to describe phase transitions. The first lemma of this paper says that no phase transition occurs on a segment if the energy there is small.

Lemma 2.1 *For every $\epsilon > 0$, there exists $\delta > 0$ such that for every $u \in \mathcal{A}$, every $y^1 \in R^1$ and every $R > 0$,*

$$\int_{S(y^1,R+T^1/2)} [\frac{1}{2}|\nabla u|^2 + W(u)]dx < \delta$$

implies $|\widehat{u}(x^1) - 1| < \epsilon$ for all $x^1 \in (y^1 - R, y^1 + R)$ or $|\widehat{u}(x^1) + 1| < \epsilon$ for all $x^1 \in (y^1 - R, y^1 + R)$.

Proof. Suppose, on the contrary, there exist $\epsilon > 0$, and a sequence of $u_n \in \mathcal{A}$, a sequence of $S(y_n^1, R_n)$, such that

$$\int_{S(y_n^1,R_n+T^1/2)} [\frac{1}{2}|\nabla u_n|^2 + W(u_n)]dx \to 0 \text{ as } n \to \infty, \text{ and} \tag{2.3}$$

$$||\widehat{u}_n(x_n^1)| - 1| \ge \epsilon \text{ for some } x_n^1 \in S(y_n^1 - R_n, y_n^1 + R_n). \tag{2.4}$$

Estimate, with the help of Holder's inequality and Poincare's inequality,

$$\int_{S(x_n^1,T^1/2)} W(u_n)dx = \int_{S(x_n^1,T^1/2)} W(\widehat{u}_n(x_n^1) + u_n - \widehat{u}_n(x_n^1))dx$$

$$= \int_{S(x_n^1,T^1/2)} \{W(\widehat{u}_n(x_n^1)) + f(\widehat{u}_n(x_n^1))[u_n - \widehat{u}_n(x_n^1)] + \frac{f'(\cdot)}{2}[u_n - \widehat{u}_n(x_n^1)]^2\}dx$$

$$\geq \int_{S(x_n^1,T^1/2)} W(\widehat{u}_n(x_n^1))dx - |f(\widehat{u}_n(x_n^1))| \cdot |S(x_n^1, T^1/2)|^{1/2}$$

$$\cdot \|u_n - \widehat{u}_n(x_n^1)\|_{L^2(S(x_n^1,T^1/2))} - C_1\|u_n - \widehat{u}_n(x_n^1)\|^2_{L^2(S(x_n^1,T^1/2))}$$

$$\geq W(\widehat{u}_n(x_n^1))|S(x_n^1, T^1/2)| - C_2|f(\widehat{u}_n(x_n^1))| \cdot \|\nabla u_n\|_{L^2(S(x_n^1,T^1/2))}$$

$$-C_3\|\nabla u_n\|^2_{L^2(S(x_n^1,T^1/2))}$$

$$\geq W(\widehat{u}_n(x_n^1))|S(x_n^1, T^1/2)| - C_2[\epsilon_1|f(\widehat{u}_n(x_n^1))|^2$$

$$+\frac{1}{\epsilon_1}\|\nabla u_n\|^2_{L^2(S(x_n^1,T^1/2))}] - C_3\|\nabla u_n\|^2_{L^2(S(x_n^1,T^1/2))}$$

$$\geq W(\widehat{u}_n(x_n^1))|S(x_n^1, T^1/2)| - C_4\epsilon_1 W(\widehat{u}_n(x_n^1))$$

$$-(\frac{1}{\epsilon_1} + C_3)\|\nabla u_n\|^2_{L^2(S(x_n^1,T^1/2))}$$

$$= (|S(x_n^1, T^1/2)| - C_4\epsilon_1)W(\widehat{u}_n(x_n^1)) - (\frac{1}{\epsilon_1} + C_3)\|\nabla u_n\|^2_{L^2(S(x_n^1,T^1/2))}.$$

One chooses ϵ_1 so small that $|S(x_n^1, T^1/2)| - C_4\epsilon_1 > 0$. Then (2.3) implies that the second term in the last line approaches 0, and (2.4) implies that the first term is bounded from below by a positive number. Then

$$0 = \lim_{n\to\infty} \int_{S(y_n^1,R_n+T^1/2)} [\frac{1}{2}|\nabla u_n|^2 + W(u_n)] \geq \liminf_{n\to\infty} \int_{S(x_n^1,T^1/2)} W(u_n) > 0,$$

which is impossible. □

Corollary 2.2 *For every $u \in \mathcal{A}$, $\lim_{x^1\to-\infty} \widehat{u}(x^1)$ and $\lim_{x^1\to\infty} \widehat{u}(x^1)$ exist and equal -1 or 1.*

With the help of Corollary 2.2, define

$$\mathcal{A}_\zeta^\eta = \{u \in \mathcal{A} : \lim_{x^1\to-\infty} \widehat{u}(x^1) = \zeta, \lim_{x^1\to\infty} \widehat{u}(x^1) = \eta\}, \ \zeta, \eta \in \{-1, 1\}. \tag{2.5}$$

Then $\mathcal{A} = \mathcal{A}_{-1}^{-1} \cup \mathcal{A}_{-1}^1 \cup \mathcal{A}_1^{-1} \cup \mathcal{A}_1^1$. About the function classes $\mathcal{A}_\zeta^\eta$, I have

Lemma 2.3 *Let $u \in \mathcal{A}_\zeta^\eta$, $\zeta, \eta \in \{-1, 1\}$. Then for every $v \in L^1_{loc}(\Omega)$, $v \in \mathcal{A}_\zeta^\eta$ if and only if $v - u \in W^{1,2}(\Omega)$.*

$W^{1,2}(\Omega)$ is the Hilbert space of measurable $L^2(\Omega)$ functions whose gradients are in $(L^2(\Omega))^d$. The norm in $W^{1,2}(\Omega)$ is

$$\|w\|_{W^{1,2}(\Omega)} = \{\int_\Omega [|\nabla w|^2 + w^2]dx\}^{1/2}.$$

Proof. Let $u \in \mathcal{A}_\zeta^\eta$ and $w \in W^{1,2}(\Omega)$. Then $\nabla(u+w) \in (L^2(\Omega))^d$ since ∇u, $\nabla w \in (L^2(\Omega))^d$. To show $\int_\Omega W(u+w)dx < \infty$, estimate

$$\int_\Omega W(u+w)dx = \int_\Omega [W(u) + wf(u) + \frac{f'(\cdot)}{2}w^2]dx$$

$$\leq \int_\Omega W(u)dx + \|w\|_{L^2(\Omega)}\|f(u)\|_{L^2(\Omega)} + C_1\|w\|^2_{L^2(\Omega)}$$

$$\leq \int_\Omega W(u)dx + C_2[\int_\Omega W(u)dx]^{1/2}\|w\|_{L^2(\Omega)} + C_1\|w\|^2_{L^2(\Omega)} < \infty.$$

Now let $u, v \in \mathcal{A}_\zeta^\eta$. Clearly $\nabla(u-v) \in (L^2(\Omega))^d$. To see $u - v \in L^2(\Omega)$, set $\Omega^+ = \{x \in \Omega : x^1 > 0\}$ and $\Omega^- = \{x \in \Omega : x^1 < 0\}$. Without the loss of generality, I only show $\int_{\Omega^+}(u-v)^2 < \infty$. Note $|u-v| \leq |u-\eta| + |v-\eta|$, so I proceed to show $|u-\eta| \in L^2(\Omega^+)$. Let ϵ be a small number such that for all $u \in [\eta-\epsilon, \eta+\epsilon]$, $(u-\eta)^2 < CW(u)$ for some $C > 0$. This is guaranteed by H-1. I further divide Ω^+ into Ω_1 and Ω_2 where $\Omega_1 = \{x \in \Omega^+ : u(x) \in [\eta-\epsilon, \eta+\epsilon]\}$ and $\Omega_2 = \Omega^+ \backslash \Omega_1$. Then

$$\int_{\Omega_1}(u-\eta)^2 dx \leq C_1 \int_{\Omega_1} W(u)dx < \infty.$$

Note $u - \eta \in W^{1,2}(G)$ for every bounded subdomain G of Ω, since on every bounded subset of Ω, $(u-\eta)^2$ can be bounded by $W(u) + C_2$ where $C_2 > 0$ is some constant. I therefore write

$$\Omega_2 = (\Omega_2 \cap S(0, T^1/2)) \cup (\cup_{i=1}^\infty(\Omega_2 \cap S(iT^1, T^1/2))),$$

and I need to bound

$$\int_{\cup_{i=k}^\infty(\Omega_2 \cap S(iT^1,T^1/2))}(u-\eta)^2 dx$$

for large k. Since $\widehat{u}(x^1) \to \eta$ as $x^1 \to \infty$ by Corollary 2.2, choose k so large that $\widehat{u}(x^1) \in [\eta-\epsilon/2, \eta+\epsilon/2]$ for all $x^1 > kT^1 - T^1/2$. For every $x \in \Omega_2 \cap S(iT^1.T^1/2)$ with $i \geq k$,

$$|u(x)-\eta| \leq |u(x) - \widehat{u}(iT^1)| + |\widehat{u}(iT^1) - \eta| \leq |u(x) - \widehat{u}(iT^1)| + \epsilon/2$$

$$\leq |u(x) - \widehat{u}(iT^1)| + |u(x) - \widehat{u}(iT^1)| = 2|u(x) - \widehat{u}(iT^1)|.$$

Now estimate

$$\int_{\Omega_2\cap S(iT^1,T^{1/2})}(u(x)-\eta)^2dx \le \int_{\Omega_2\cap S(iT^1,T^{1/2})}4(u(x)-\widehat{u}(iT^1))^2dx$$

$$\le C\int_{S(iT^1,T^{1/2})}|\nabla u|^2dx,$$

by Poincare's inequality. Summing over i from k to infinity, I deduce

$$\int_{\cup_{i=k}^{\infty}(\Omega_2\cap S(iT^1,T^1/2))}(u-\eta)^2dx \le \int_{\Omega^+}|\nabla u|^2dx \le \int_\Omega |\nabla u|^2dx < \infty. \ \square$$

In the proof of this lemma the decomposition Ω_1 and Ω_2 is crucial. Ω_1 is the good set on which quantities are estimated directly by W. Ω_2 is the bad set on which quantities are estimated with the help of $\widehat{u}$ and Poincare's inequality. This trick will be used often in this paper.

Therefore $\mathcal{A}_\zeta^\eta$ can be treated as a Hilbert manifold, actually an affine space, whose tangent space at each $u \in \mathcal{A}_\zeta^\eta$ is $W^{1,2}(\Omega)$, so a critical point of E in $\mathcal{A}_\zeta^\eta$ is a classical solution of (1.1) by the standard elliptic regularity theory. The distance of $u, v \in \mathcal{A}_\zeta^\eta$ is set to be $\|u-v\|_{W^{1,2}(\Omega)}$.

Remark 2.4 *It is easy to prove that for every $u \in \mathcal{A}_\zeta^\eta$, there exists $u' \in \mathcal{A}_\zeta^\eta$ as close to u as I wish so that for some $R>0$, $u'(x)=\zeta$ for $x^1<-R$, and $u'(x)=\eta$ for $x^1>R$.*

Set

$$\mathcal{K} = \{u \in \mathcal{A} : \ E'(u)=0\}. \tag{2.6}$$

$\mathcal{K}$ is the set of all the critical points of E in $\mathcal{A}$. About the functional $E : \mathcal{A}_\zeta^\eta \to R^1$, I have

Lemma 2.5 *For each $\mathcal{A}_\zeta^\eta$, $E : \mathcal{A}_\zeta^\eta \to R^1$ is in $C^2(\mathcal{A}_\zeta^\eta, R^1)$, and*

$$E'(u)\phi = \int_\Omega [\nabla u\cdot\nabla\phi + f(u)\phi]dx, \ \ u\in\mathcal{A}_\zeta^\eta, \phi\in W^{1,2}(\Omega),$$

$$E''(u)(\phi,\psi) = \int_\Omega [\nabla \phi\cdot\nabla\psi + f'(u)\phi\psi]dx, \ \ u\in\mathcal{A}_\zeta^\eta, \phi,\psi\in W^{1,2}(\Omega).$$

Proof. First show the continuity of E. Let $u\in\mathcal{A}_\zeta^\eta$, $\phi\in W^{1,2}(\Omega)$. Then

$$|E(u+\phi)-E(u)| \le \int_\Omega |\nabla u\cdot\nabla\phi + \frac{1}{2}|\nabla\phi|^2|dx + \int_\Omega |W(u+\phi)-W(u)|dx$$

$$\le \|\nabla u\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} + \frac{1}{2}\|\nabla\phi\|^2_{L^2(\Omega)} + \|f(u)\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} + C_1\|\phi\|^2_{L^2(\Omega)} \le$$

$$\|\nabla u\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} + \frac{1}{2}\|\nabla\phi\|^2_{L^2(\Omega)} + C_2[\int_\Omega W(u)dx]^{1/2}\|\phi\|_{L^2(\Omega)} + C_1\|\phi\|^2_{L^2(\Omega)}$$

$$\to 0 \text{ as } \|\phi\|_{W^{1,2}(\Omega)} \to 0.$$

Hence E is continuous.

Follow the argument in [14] to show the differentiability. For every $u \in \mathcal{A}_\zeta^\eta$, $\phi \in W^{1,2}(\Omega)$,

$$\int_\Omega |\nabla u \cdot \nabla \phi + f(u)\phi| dx \le \|\nabla u\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} + C_5 [\int_\Omega W(u) dx]^{1/2} \|\phi\|_{L^2} < 0,$$

so $E'(u)\phi$ is defined. Note

$$|E(u+\phi) - E(u) - E'(u)\phi| \le \frac{1}{2} \int_\Omega |\nabla \phi|^2 dx + \int_\Omega |W(u+\phi) - W(u) - f(u)\phi| dx$$

$$\le \frac{1}{2} \|\nabla \phi\|^2_{L^2(\Omega)} + C_6 \int_\Omega \phi^2 dx \le C_7 \|\phi\|^2_{W^{1,2}(\Omega)} = o(\|\phi\|_{W^{1,2}(\Omega)}).$$

Therefore E is Frechet differentiable. To see that E' is continuous, let $u_n \to u$ in $\mathcal{A}_\zeta^\eta$. Then

$$|E'(u_n)\phi - E'(u)\phi| \le |\int_\Omega \nabla(u_n - u) \cdot \nabla \phi dx| + \|f(u_n) - f(u)\|_{L^2(\Omega)} \|\phi\|_{W^{1,2}(\Omega)}$$

$$\le \|u_n - u\|_{W^{1,2}(\Omega)} \|\phi\|_{W^{1,2}(\Omega)} + C_8 \|u_n - u\|_{W^{1,2}(\Omega)} \|\phi\|_{W^{1,2}(\Omega)},$$

so E' is continuous. The existence and continuity of E'' follow along the same line. □

Lemma 2.6 *There exists $\lambda_0 > 0$ such that for every $u \in \mathcal{A}_\zeta^\zeta \cap (\mathcal{K} \backslash \{\zeta\})$, $\|u - \zeta\|_{W^{1,2}(\Omega)} > \lambda_0$.*

Proof. This is because the symmetric bilinear form $E''(\zeta)$ is positive definite. For every $\phi \in W^{1,2}(\Omega)$,

$$E''(\zeta)(\phi, \phi) = \int_\Omega [|\nabla \phi|^2 + f'(\zeta)\phi^2] dx \ge C \|\phi\|^2_{W^{1,2}(\Omega)}$$

by H-1. ζ is the only critical point in a neighborhood. □

Recall that a sequence $\{u_n\}$ in $\mathcal{A}_\zeta^\eta$ is a Palais-Smale sequence if $\lim_{n\to\infty} E(u_n) = b$ for some $b \ge 0$ and $\lim_{n\to\infty} E'(u_n) = 0$ in $(W^{1,2}(\Omega))^*$ where $(W^{1,2}(\Omega))^*$ is the dual space of $W^{1,2}(\Omega)$. The next lemma says that every Palais-Smale sequence is locally compact.

Lemma 2.7 *For every Palais-Smale sequence $\{u_n\} \in \mathcal{A}_\zeta^\eta$, there exists $u \in \mathcal{K}$ such that along a subsequence of $\{u_n\}$, on every bounded subdomain G of Ω,*

$$u_n \to u \textit{ in } W^{1,2}(G), \textit{ and}$$

$$\nabla u_n \to \nabla u \textit{ weakly in } (L^2(\Omega))^d.$$

(The limit u is not necessarily in $\mathcal{A}^\eta_\zeta$.)
Proof. Let $\{u_n\}$ be a Palais-Smale sequence in $\mathcal{A}^\eta_\zeta$. On an arbitrary bounded subdomain G of Ω, u_n is bounded in $L^2(G)$ since by H-2

$$\int_G u_n^2 \leq \int_G [2 + \frac{1}{\Theta} W(u_n)]dx$$

which is bounded by the energy of u_n. $|\nabla u_n|$ is bounded in $L^2(\Omega)$ by the energy of u_n. Pass to a subsequence of u_n, still denoted by u_n, and find a function $u \in W^{1,2}(G)$ for every bounded subdomain G with $|\nabla u| \in L^2(\Omega)$, such that

$$u_n \to u \text{ weakly in } W^{1,2}(G), \tag{2.7}$$

$$\nabla u_n \to \nabla u \text{ weakly in } (L^2(\Omega))^d, \text{ and} \tag{2.8}$$

$$u_n \to u \text{ in } L^2(G), \tag{2.9}$$

where the last convergence comes from the Sobolev embedding theorem. From the weak lower semicontinuity of the quadratic term in E and Fatou's lemma,

$$E(u) \leq \lim_{n\to\infty} E(u_n),$$

which implies $u \in \mathcal{A}$. Take a smooth cut-off function ξ, $0 \leq \xi \leq 1$ on R^1, such that

$$\xi(x^1) = \begin{cases} 1, & \text{if } x^1 \in [-K, K] \\ 0, & \text{if } x^1 \notin [-K-1, K+1]. \end{cases} \tag{2.10}$$

For $x = (x^1, x') \in \Omega$ set $\xi(x) = \xi(x^1)$. Denoting the norm in $(W^{1,2}(\Omega))^*$ by $\|\cdot\|$, I find

$$2\|E'(u_n)\|^2 + \frac{1}{2}\|(u_n - u)\xi\|^2_{W^{1,2}(\Omega)} \geq E'(u_n)[(u_n - u)\xi]$$

$$= \int_\Omega \nabla u_n \cdot \nabla[(u_n - u)\xi]dx + \int_\Omega (u_n - u)\xi f(u_n)dx$$

$$= \int_\Omega |\nabla(u_n - u)|^2 \xi dx + \int_\Omega [\nabla(u_n - u)\cdot\nabla\xi](u_n - u)dx$$

$$+ \int_\Omega \nabla u \cdot \nabla[(u_n - u)\xi]dx + \int_\Omega (u_n - u)\xi f(u_n)dx.$$

The second, third and forth terms approach 0 because of (2.7), (2.8), (2.9) and the boundedness of $f(u_n)$ in $L^2(\Omega)$ by $\int_\Omega W(u_n)$. I deduce

$$\overline{\lim_{n\to\infty}}[\int_\Omega |\nabla(u_n - u)|^2\xi dx - \frac{1}{2}\int_\Omega |\nabla[(u_n - u)\xi]|^2 dx - \frac{1}{2}\int_\Omega (u_n - u)^2\xi^2 dx] \leq 0.$$

Since the last term on the left side approaches 0 by (2.9),

$$\overline{\lim_{n\to\infty}}[\int_\Omega |\nabla(u_n - u)|^2\xi dx - \frac{1}{2}\int_\Omega |\nabla[(u_n - u)\xi]|^2 dx] \leq 0. \tag{2.11}$$

Note

$$\int_\Omega |\nabla(u_n-u)|^2\xi dx - \frac{1}{2}\int_\Omega |\nabla[(u_n-u)\xi]|^2 dx$$

$$= \int_\Omega |\nabla(u_n-u)|^2\xi dx - \frac{1}{2}\int_\Omega |\nabla(u_n-u)|^2\xi^2 dx$$

$$-\int_\Omega [\nabla(u_n-u)\cdot\nabla\xi](u_n-u)\xi dx - \frac{1}{2}\int_\Omega (u_n-u)^2|\nabla\xi|^2 dx,$$

where the last two terms approach 0 by (2.8) and (2.9). Then (2.11) becomes

$$\overline{\lim_{n\to\infty}}[\int_\Omega |\nabla(u_n-u)|^2\xi dx - \frac{1}{2}\int_\Omega |\nabla(u_n-u)|^2\xi^2 dx] \le 0,$$

which implies

$$\frac{1}{2}\overline{\lim_{n\to\infty}}\int_\Omega |\nabla(u_n-u)|^2\xi^2 dx \le 0, \text{ i.e., } \lim_{n\to\infty}\int_G |\nabla(u_n-u)|^2 dx = 0$$

on each bounded subdomain G.

Finally show $u \in \mathcal{K}$. Let $\phi \in W^{1,2}(\Omega)$ whose support is in a bounded subset of Ω. Such ϕ's are dense in $W^{1,2}(\Omega)$, so I only need to show $E'(u)\phi = 0$. Consider

$$E'(u_n)\phi - E'(u)\phi = \int_\Omega \nabla(u_n-u)\cdot\nabla\phi dx + \int_\Omega [f(u_n)-f(u)]\phi dx, \qquad (2.12)$$

where the first term of the right side approaches 0 by $\nabla u_n \to \nabla u$ weakly in $(L^2(\Omega))^d$, and the second term approaches 0 by $u_n \to u$ strongly in $L^2(G)$ for every bounded G. Therefore, passing to the limit in (2.12), I find

$$E'(u)\phi = \lim_{n\to\infty} E'(u_n)\phi = 0. \square$$

An important feature of Palais-Smale sequences is the following lemma that asserts that uniform energy vanishing on segments of fixed size implies convergence to a trivial solution.

Lemma 2.8 *Let $\{u_n\}$ be a Palais-Smale sequence in $\mathcal{A}^\eta_\zeta$. If there exists $R > 0$ such that*

$$\lim_{n\to\infty}\sup_{x^1\in R^1}\int_{S(x^1,R)}[\frac{1}{2}|\nabla u_n|^2 + W(u_n)]dx = 0,$$

then $\mathcal{A}^\eta_\zeta = \mathcal{A}^\theta_\theta$ for some $\theta \in \{-1,1\}$, and $\lim_{n\to\infty}\|u_n-\theta\|_{W^{1,2}(\Omega)} = 0$.

Proof. Let $\{u_n\}$ be a sequence satisfying the assumption. It follows that for every $R > 0$,

$$\lim_{n\to\infty}\sup_{x^1\in R^1}\int_{S(x^1,R)}[\frac{1}{2}|\nabla u_n|^2 + W(u_n)]dx = 0.$$

Lemma 2.1 implies that for large n, $\widehat{u}_n(x^1)$ lies in a small neighborhood of -1 or 1 for all x^1, uniformly in n. Since $\widehat{u}_n$ is continuous, $\widehat{u}_n$ is in a small neighborhood of $\theta \in \{-1, 1\}$. Therefore $\mathcal{A}_\zeta^\eta = \mathcal{A}_\theta^\theta$. Consider

$$[\min\{\frac{1}{2}, \frac{f'(\theta)}{4}\}]^{-1}\|E'(u_n)\|^2 + \min\{\frac{1}{2}, \frac{f'(\theta)}{4}\}\|u_n - \theta\|^2_{W^{1,2}(\Omega)} \geq \tag{2.13}$$

$$E'(u_n)(u_n-\theta) = \int_\Omega [|\nabla u_n|^2 + \frac{f'(\theta)}{2}(u_n-\theta)^2] + \int_\Omega [f(u_n)(u_n-\theta) - \frac{f'(\theta)}{2}(u_n-\theta)^2].$$

The lemma follows if

$$\overline{\lim_{n\to\infty}} \int_\Omega [f(u_n)(u_n - \theta) - \frac{f'(\theta)}{2}(u_n - \theta)^2]dx \geq 0. \tag{2.14}$$

Claim $\int_\Omega (u_n - \theta)^2 dx$ is bounded uniformly in n, and

$$\lim_{n\to\infty} \sup_{i\in Z} \int_{S(iT^1, T^1/2)} (u_n(x) - \theta)^2 dx = 0,$$

where Z is the set of all the integers.

Take $\epsilon > 0$ so small that $(u-\theta)^2 \leq CW(u)$ for some $C > 0$ and all $u \in [\theta-\epsilon, \theta+\epsilon]$. Set $\Omega_n = \{x \in \Omega : |u_n(x) - \theta| < \epsilon\}$ (the good set). Then on Ω_n

$$\int_{\Omega_n} (u_n - \theta)^2 dx \leq C \int_{\Omega_n} W(u_n)dx, \tag{2.15}$$

$$\int_{S(iT^1,T^1/2)\cap\Omega_n} (u_n - \theta)^2 dx \leq C \int_{S(iT^1,T^1/2)} W(u_n)dx. \tag{2.16}$$

Choose n so large that $|\widehat{u}_n(x^1) - \theta| < \epsilon/2$ for all $x^1 \in R^1$. For $x \in (\Omega\backslash\Omega_n) \cap S(iT^1, T^1/2)$ (the bad set),

$$|u_n(x) - \theta| \leq 2|u_n(x) - \widehat{u}_n(iT^1)|,$$

as in the proof of Lemma 2.3. I deduce

$$\int_{\Omega\backslash\Omega_n} (u_n - \theta)^2 dx \leq 4 \sum_{i=-\infty}^{\infty} \int_{S(iT^1,T^1/2)} (u_n(x) - \widehat{u}_n(iT^1))^2 dx$$

$$\leq C \sum_{i=-\infty}^{\infty} \int_{S(iT^1,T^1/2)} |\nabla u_n|^2 dx = C \int_\Omega |\nabla u_n|^2 dx.$$

Combing this with (2.15), I see that $\int_\Omega (u_n - \theta)^2 dx$ is bounded uniformly in n. The argument also shows, with the help of (2.16), that for every integer i,

$$\int_{S(iT^1,T^1/2)} (u_n - \theta)^2 dx \leq C \int_{S(iT^1,T^1/2)} [\frac{1}{2}|\nabla u_n|^2 + W(u_n)]dx.$$

Since the right side approaches 0 uniformly in i as $n \to \infty$ by the assumption of the lemma, the claim is proved.

I proceed to show (2.14). Find ϵ so small that for all $u \in [\theta - \epsilon, \theta + \epsilon]$, $f(u)(u-\theta) - (f'(\theta)/2)(u-\theta)^2 \geq 0$. Set $\Omega_n = \{x \in \Omega : |u_n(x) - \theta| < \epsilon\}$ (the good set). Then

$$\overline{\lim_{n\to\infty}} \int_{\Omega_n} [f(u_n)(u_n - \theta) - \frac{f'(\theta)}{2}(u_n - \theta)^2] dx \geq 0. \tag{2.17}$$

On $\Omega \backslash \Omega_n$ (the bad set) since $|u_n(x) - \theta| > \epsilon$, by H-1 and H-2

$$|f(u_n)(u_n - \theta)| \leq C(u_n - \theta)^{(2d+4)/d} \text{ and } |\frac{f'(\theta)}{2}(u_n - \theta)^2| \leq C(u_n - \theta)^{(2d+4)/d}$$

where C depends on ϵ. Then with the help of Holder's inequality and the Sobolev embedding theorem I deduce

$$\int_{\Omega\backslash\Omega_n} |f(u_n)(u_n-\theta) - \frac{f'(\theta)}{2}(u_n-\theta)^2| dx \leq C \int_{\Omega\backslash\Omega_n} (u_n-\theta)^{(2d+4)/d} dx$$

$$\leq C \sum_{i=-\infty}^{\infty} \int_{(\Omega\backslash\Omega_n)\cap S(iT^1, T^1/2)} (u_n-\theta)^{(2d+4)/d} dx$$

$$\leq C \sum_{i=-\infty}^{\infty} \int_{S(iT^1, T^1/2)} (u_n-\theta)^{(2d+4)/d} dx$$

$$\leq C \sum_{i=-\infty}^{\infty} \|u_n - \theta\|_{L^2(S(iT^1,T^1/2))}^{((2d+4)/d)(2/(d+2))} \|u_n - \theta\|_{L^{2d/(d-2)}(S(iT^1,T^1/2))}^{((2d+4)/d)(d/(d+2))}$$

$$\leq C \sum_{i=-\infty}^{\infty} \|u_n - \theta\|_{L^2(S(iT^1,T^1/2))}^{4/d} \int_{S(iT^1,T^1/2))} [|\nabla u_n|^2 + (u_n-\theta)^2] dx$$

$$\leq C \sup_{i\in Z} \|u_n - \theta\|_{L^2(S(iT^1,T^1/2))}^{4/d} \int_{\Omega} [|\nabla u_n|^2 + (u_n-\theta)^2] dx.$$

This, the claim and (2.17) imply (2.14) that proves the lemma. □

Corollary 2.9 *There is $\nu_0 > 0$ such that for every $u \in \mathcal{K}\backslash\{-1, 1\}$, $E(u) \geq \nu_0$.*

Proof. Suppose, on the contrary, there is a sequence $\{u_n\}$ in $\mathcal{A}_\zeta^\eta$ of critical points, hence a Palais-Smale sequence, such that $\lim_{n\to\infty} E(u_n) = 0$. Then Lemma 2.8 implies that $\zeta = \eta$ and $\|u_n - \zeta\|_{W^{1,2}(\Omega)} \to 0$, which contradicts Lemma 2.6. □

3 Global Minima

A characterization of Palais-Smale sequences is given in this section. The existence of global minima follows easily from this characterization. First define two operators, the shift operator τ and the pasting operator π. Let k be an integer, and define $\tau_k : \mathcal{A}_\zeta^\eta \to \mathcal{A}_\zeta^\eta$ by

$$\tau_k u(x) = u(x - kT), \ x \in \Omega, \tag{3.1}$$

for $\zeta, \eta \in \{-1, 1\}$. Define $\pi : \mathcal{A}_\zeta^\theta \times \mathcal{A}_\theta^\eta \to \mathcal{A}_\zeta^\eta$ for $\zeta, \theta, \eta \in \{-1, 1\}$ by

$$\pi(u, v) = u + v - \theta. \tag{3.2}$$

Through a recursive use of (3.2), π can be extended to $\pi : \mathcal{A}_\zeta^{\theta_2} \times \mathcal{A}_{\theta_2}^{\theta_3} \times ... \times \mathcal{A}_{\theta_k}^\eta \to \mathcal{A}_\zeta^\eta$, i.e.,

$$\pi(u_1, ..., u_k) = \pi(u_1, \pi(u_2, \pi(..., \pi(u_{k-1}, u_k)))). \tag{3.3}$$

These two operators are often used together. If there is no danger of confusion, I write $\pi_j u$ for $\pi(\tau_{j_1} u_1, ..., \tau_{j_k} u_k)$.

Since Ω is unbounded, a Palais-Smale sequence does not necessarily have a convergent subsequence. The loss of compactness can be caused by the translation invariance of Ω. Take u to be a non-trivial critical point and $\{i(n)\}$ to be a integral sequence with $i(n) \to \infty$ as $n \to \infty$. Then $\{\tau_{i(n)} u\}$ is a Palais-Smale sequence without convergent subsequences.

A more subtle way to lose compactness is to have a sequence in which each function is made of several critical points spreading apart. Let $u_1 \in \mathcal{A}_{-1}^1$, $u_2 \in \mathcal{A}_1^{-1}$ be two critical points, and $\{i_1(n)\}$ and $\{i_2(n)\}$ be two integral sequences with $i_1(n) - i_2(n) \to -\infty$ as $n \to \infty$. Then $\{\pi(\tau_{i_1(n)} u_1, \tau_{i_2(n)} u_2)\}$ is again a Palais-Smale sequence without convergent subsequences.

The next result says that these are basically the only ways to lose compactness. The reader should compare it with Lemma 2.7 that says every Palais-Smale sequence is locally compact.

Proposition 3.1 *Let $\{u_n\}$ be a Palais-Smale sequence in $\mathcal{A}_\zeta^\eta$. If $\lim_{n\to\infty} E(u_n) = 0$, then $\mathcal{A}_\zeta^\eta = A_\theta^\theta$ for some $\theta \in \{-1, 1\}$ and $\lim_{n\to\infty} \|u_n - \theta\|_{W^{1,2}(\Omega)} = 0$.*

If $\lim_{n\to\infty} E(u_n) > 0$, then there exist $w_1, w_2, ..., w_k \in \mathcal{K}\backslash\{-1, 1\}$, $k \geq 1$, $w_i \in \mathcal{A}_{\theta_i}^{\theta_{i+1}}$, $\theta_1 = \zeta$ and $\theta_{k+1} = \eta$, and k integral sequences $I(1, n)$,...,$I(k, n)$ with $\lim_{n\to\infty} (I(i, n) - I(i + 1, n)) = -\infty$ for each i such that along a subsequence

$$\lim_{n\to\infty} \|u_n - \pi_{I(n)} w\|_{W^{1,2}(\Omega)} = 0 \text{ and } \lim_{n\to\infty} E(u_n) = E(w_1) + E(w_2) + ... + E(w_k).$$

Proof. The first part of the proposition follows easily from Lemma 2.8. Now assume $\lim_{n\to\infty} E(u_n) > 0$. I will find the w_i's one at a time. Starting with the left-most one, w_1, I need the following.

Claim 1 There exists an integral sequence $\{i(n)\}$ such that for every $R \geq T^1/2$,

$$\overline{\lim_{n\to\infty}} \int_{S(i(n)T^1,R)} [\frac{1}{2}|\nabla u_n|^2 + W(u_n)]dx > 0,$$

and for every other integral sequence $j(n)$ with $j(n) - i(n) \to -\infty$, every $R > 0$,

$$\overline{\lim_{n\to\infty}} \int_{S(j(n)T^1,R)} [\frac{1}{2}|\nabla u_n|^2 + W(u_n)]dx = 0.$$

Lemma 2.8 implies that there exists an integral sequence $\{i(n)\}$ such that for every $R \geq T^1/2$,

$$\overline{\lim_{n\to\infty}} \int_{S(i(n)T^1,R)} [\frac{1}{2}|\nabla u_n|^2 + W(u_n)]dx > 0,$$

for otherwise $u_n - \theta \to 0$ in $W^{1,2}(\Omega)$ for some $\theta \in \{-1, 1\}$ which implies $E(u_n) \to 0$ contradicting $\lim_{n\to\infty} E(u_n) > 0$.

Suppose there is not a left-most $i(n)$. Then there exist infinitely many sequences,$\{j(1,n)\}$, $\{j(2,n)\}$,..., with $j(m+1,n) - j(m,n) \to -\infty$, such that along a subsequence, for some $R > 0$,

$$\lim_{n\to\infty} \int_{S(j(m,n),R)} [\frac{1}{2}|\nabla u_n|^2 + W(u_n)]dx > 0. \tag{3.4}$$

For each m, $\tau_{-j(m,n)} u_n \to v_m \in \mathcal{K}$ as $n \to \infty$ in the sense of Lemma 2.7. Note $E(v_m) > 0$ because of (3.4) and Lemma 2.7. Now take k many v_m's, and k is so large that $E(v_1) + ... + E(v_k) > 2 \lim_{n\to\infty} E(u_n)$. This can be done by Corollary 2.9. Enlarge R so that for every $m = 1, 2, ..., k$,

$$\int_{S(0,R)} [\frac{1}{2}|\nabla v_m|^2 + W(v_m)] > E(v_m)/2.$$

Then I deduce

$$\lim_{n\to\infty} E(u_n) \geq \sum_{m=1}^{k} \overline{\lim_{n\to\infty}} \int_{S(0,R)} [\frac{1}{2}|\nabla \tau_{-j(m,n)} u_n|^2 + W(\tau_{-j(m,n)} u_n)]dx$$

$$= \sum_{m=1}^{k} \int_{S(0,R)} [\frac{1}{2}|\nabla v_m|^2 + W(v_m)]dx \geq \frac{1}{2} \sum_{m=1}^{k} E(v_m) > \lim_{n\to\infty} E(u_n),$$

which is impossible, and Claim 1 is proved.

Denote the integral sequence $\{i(n)\}$ in Claim 1 by $\{I(1,n)\}$, and let $w_1 \in \mathcal{K}$ be the limit of $\tau_{-I(1,n)} u_n$ given by Lemma 2.7. Note $E(w_1) > 0$ by Claim 1 and Lemma 2.7, so $w_1 \not\equiv -1$ or 1. It follows from Claim 1 and Lemma 2.1 that $\widehat{\tau}_{-I(1,n)} u_n(x^1) \to \zeta$ as $x^1 \to -\infty$, uniformly in n. Then $w_1 \in \mathcal{A}^\theta_\zeta$ for some $\theta \in \{-1, 1\}$. To find the rest of the w_i's set $u'_n = u_n - \tau_{I(1,n)} w_1 + \theta \in \mathcal{A}^\eta_\theta$. I will show that $\{u'_n\}$'s is again a Palais-Smale sequence with energy close to that of $\{u_n\}$ minus $E(w_1)$, so I can repeat the above argument on $\{u'_n\}$.

Claim 2 $\lim_{n\to\infty} E(u_n') = \lim_{n\to\infty} E(u_n) - E(w_1)$.

Write

$$E(u_n) = E(u_n') + E(w_1) + \int_\Omega [W(u_n) - W(u_n') - W(w_1)] + \int_\Omega \nabla u_n' \cdot \nabla \tau_{I(1,n)} w_1.$$

The last term on the right side approaches 0 since $\nabla \tau_{-I(1,n)} u_n \to \nabla w_1$ weakly in $(L^2(\Omega))^d$. I proceed to show

$$\lim_{n\to\infty} \int_\Omega [W(u_n) - W(u_n') - W(w_1)] dx = 0. \tag{3.5}$$

To this end, write $\Omega = \Omega_L \cup \Omega_C \cup \Omega_R$ where

$$\begin{aligned} \Omega_L &= \{x \in \Omega : x^1 < -KT^1\}, \\ \Omega_C &= \{x \in \Omega : -KT^1 \le x^1 \le KT^1\}, \\ \Omega_R &= \{x \in \Omega : KT^1 < x^1\}, \end{aligned} \tag{3.6}$$

and K is a large integer to be determined. Note

$$\int_\Omega [W(u_n) - W(u_n') - W(w_1)] = \int_\Omega [W(\tau_{-I(1,n)} u_n) - W(\tau_{-I(1,n)} u_n') - W(w_1)].$$

On Ω_C

$$\lim_{n\to\infty} \int_{\Omega_C} [W(\tau_{-I(1,n)} u_n) - W(\tau_{-I(1,n)} u_n') - W(w_1)] dx = 0 \tag{3.7}$$

by Lemma 2.7. Let $\epsilon > 0$ be an arbitrary small number. On Ω_R choose K so large that

$$\begin{aligned} &| \int_{\Omega_R} [W(\tau_{-I(1,n)} u_n) - W(\tau_{-I(1,n)} u_n') - W(w_1)] dx| \\ &\le | \int_{\Omega_R} [W(\tau_{-I(1,n)} u_n) - W(\tau_{-I(1,n)} u_n')] dx| + \epsilon \\ &\le \int_{\Omega_R} |f(\tau_{-I(1,n)} u_n')(w_1 - \theta) + \frac{f'(\cdot)}{2} (w_1 - \theta)^2| dx + \epsilon \\ &\le C[\int_{\Omega_R} W(\tau_{-I(1,n)} u_n') dx]^{1/2} \|w_1 - \theta\|_{L^2(\Omega_R)} + \epsilon + \epsilon. \end{aligned}$$

If $\int_{\Omega_R} W(\tau_{-I(1,n)} u_n') dx$ is bounded uniformly in n, then choosing large K makes the first term on the last line smaller than ϵ. To this end, I estimate

$$\int_{\Omega_R} W(\tau_{-I(1,n)} u_n')$$

$$= \int_{\Omega_R} [W(\tau_{-I(1,n)} u_n) + (\theta - w_1) f(\tau_{-I(1,n)} u_n) + \frac{f'(\cdot)}{2} (\theta - w_1)^2]$$

$$\leq \int_\Omega W(u_n) + C[\int_\Omega W(u_n)]^{1/2} \|\theta - w_1\|_{L^2(\Omega_R)} + C\|\theta - w_1\|^2_{L^2(\Omega_R)},$$

which is obviously bounded uniformly in n. Find large K so that for all n,

$$|\int_{\Omega_R} [W(\tau_{-I(1,n)} u_n) - W(\tau_{-I(1,n)} u_n') - W(w_1)] dx| \leq 3\epsilon. \tag{3.8}$$

I now turn to Ω_L. Again by choosing large K I find

$$|\int_{\Omega_L} [W(\tau_{-I(1,n)} u_n) - W(\tau_{-I(1,n)} u_n') - W(w_1)] dx|$$

$$\leq \int_{\Omega_L} W(\tau_{-I(1,n)} u_n) dx + \int_{\Omega_L} W(\tau_{-I(1,n)} u_n') dx + \epsilon.$$

Claim 3 $\|\tau_{-I(1,n)} u_n - \zeta\|_{W^{1,2}(\Omega_L)}$ approaches 0 uniformly in n as $K \to \infty$.

This implies $\|\tau_{-I(1,n)} u_n' - \theta\|_{W^{1,2}(\Omega_L)} \to 0$ uniformly in n as $K \to \infty$. Together with Claim 3, I deduce that for large K and all n

$$\int_{\Omega_L} W(\tau_{-I(1,n)} u_n) dx \leq \epsilon \text{ and } \int_{\Omega_L} W(\tau_{-I(1,n)} u_n') dx \leq \epsilon.$$

Then I can find large K so that for all n

$$|\int_{\Omega_L} [W(\tau_{-I(1,n)} u_n) - W(\tau_{-I(1,n)} u_n') - W(w_1)] dx| \leq 3\epsilon. \tag{3.9}$$

I now proceed to show Claim 3. The proof is similar to that of Lemma 2.8. Take a smooth cut-off function

$$\xi(x^1) = \begin{cases} 1, \text{ if } x^1 < -KT^1 + T^1/2 \\ 0, \text{ if } x^1 > (-K+1)T^1 + T^1/2, \end{cases}$$

and $\xi \in [0, 1]$. For $x = (x^1, x') \in \Omega$ set $\xi(x) = \xi(x^1)$. Consider

$$[\min\{\frac{1}{2}, \frac{f'(\zeta)}{4}\}]^{-1} \|E'(\tau_{-I(1,n)} u_n)\|^2 + \min\{\frac{1}{2}, \frac{f'(\zeta)}{4}\} \|(\tau_{-I(1,n)} u_n - \zeta)\xi\|^2_{W^{1,2}(\Omega)}$$

$$\geq E'(\tau_{-I(1,n)} u_n)[(\tau_{-I(1,n)} u_n - \zeta)\xi]$$

$$= \int_\Omega \nabla(\tau_{-I(1,n)} u_n) \cdot \nabla[(\tau_{-I(1,n)} u_n - \zeta)\xi] + \frac{f'(\zeta)}{2} \int_\Omega (\tau_{-I(1,n)} u_n - \zeta)^2 \xi^2$$

$$+ \int_\Omega [(\tau_{-I(1,n)} u_n - \zeta)\xi f(\tau_{-I(1,n)} u_n) - \frac{f'(\zeta)}{2} (\tau_{-I(1,n)} u_n - \zeta)^2 \xi^2]. \tag{3.10}$$

I first show

$$\overline{\lim_{K\to\infty}} \sup_{n\in Z^+} \int_\Omega [(\tau_{-I(1,n)}u_n - \zeta)\xi f(\tau_{-I(1,n)}u_n) - \frac{f'(\zeta)}{2}(\tau_{-I(1,n)}u_n - \zeta)^2\xi^2] \geq 0 \tag{3.11}$$

where Z^+ is the set of all the positive integers. Note that

$$\int_{\{x^1<(-K+1)T^1+T^1/2\}\cap\Omega} (\tau_{-I(1,n)}u_n - \zeta)^2 \text{ is bounded uniformly in } n, \tag{3.12}$$

and for every $R > 0$ and every integral sequence $i(n) \to -\infty$,

$$\int_{S(i(n)T^1,R)} (\tau_{-I(1,n)}u_n - \zeta)^2 \to 0. \tag{3.13}$$

To see these I argue in a way similar to the claim of Lemma 2.8. Let ϵ_1 be so small that for all $u \in [\zeta - \epsilon_1, \zeta + \epsilon_1]$, $(u-\zeta)^2 \leq CW(u)$ for some $C > 0$. Let $\Omega_n = \{x \in \Omega : x^1 < (-K+1)T^1 + T^1/2, |\tau_{-I(1,n)}u_n(x) - \zeta| < \epsilon_1\}$ (the good set). Then

$$\int_{\Omega_n} (\tau_{-I(1,n)}u_n - \zeta)^2 dx \leq C \int_\Omega W(\tau_{-I(1,n)}u_n)dx,$$

$$\int_{S(iT^1,T^1/2)\cap\Omega_n} (\tau_{-I(1,n)}u_n - \zeta)^2 \leq C \int_{S(iT^1,T^1/2)\cap\Omega_n} W(\tau_{-I(1,n)}u_n)dx \tag{3.14}$$

for all $i \leq -K+1$. Divide $\{x \in \Omega : x^1 < (-K+1)T^1 + T^1/2\}\backslash\Omega_n$ (the bad set) into $\cup_{i=-\infty}^{-K+1}(S(iT^1,T^1/2)\backslash\Omega_n)$. Claim 1 implies

$$\|\widehat{\tau}_{-I(1,n)}u_n - \zeta\|_{L^\infty(-\infty,(-K+1)T^1+T^1/2)} \to 0, \text{ as } K \to \infty$$

uniformly in n. Choose K so large that

$$\|\widehat{\tau}_{-I(1,n)}u_n - \zeta\|_{L^\infty(-\infty,(-K+1)T^1+T^1/2)} < \frac{\epsilon}{2}$$

for all n. Then for $x \in S(iT^1,T^1/2)\backslash\Omega_n$, $i \leq -K+1$,

$$|\tau_{-I(1,n)}u_n(x) - \zeta| \leq 2|\tau_{-I(1,n)}u_n(x) - \widehat{\tau}_{-I(1,n)}u_n(iT^1)|$$

as in the proof of Lemma 2.3. Then

$$\begin{aligned}
&\int_{S(iT^1,T^1/2)\backslash\Omega_n} |\tau_{-I(1,n)}u_n(x) - \zeta|^2 \\
&\leq 4\int_{S(iT^1,T^1/2)} |\tau_{-I(1,n)}u_n(x) - \widehat{\tau}_{-I(1,n)}u_n(iT^1)|^2 \\
&\leq C\int_{S(iT^1,T^1/2)} |\nabla(\tau_{-I(1,n)}u_n)|^2 dx.
\end{aligned}$$

This, (3.14) and Claim 1 imply (3.13).

To see (3.11) let $\epsilon_1 > 0$ be such that for all $|u-\zeta| < \epsilon_1$, $f(u)(u-\zeta) \geq (f'(\zeta)/2)(u-\zeta)^2$. Set $\Omega_n = \{x \in \Omega : |u_n(x) - \zeta| < \epsilon_1\}$ (the good set). I deduce

$$\int_{\Omega_n} [(\tau_{-I(1,n)}u_n - \zeta)\xi f(\tau_{-I(1,n)}u_n) - \frac{f'(\zeta)}{2}(\tau_{-I(1,n)}u_n - \zeta)^2\xi^2]$$

$$= \int_{\Omega_n \cap \{x^1 < -KT^1 + T^1/2\}} (\tau_{-I(1,n)}u_n - \zeta)[f(\tau_{-I(1,n)}u_n) - \frac{f'(\zeta)}{2}(\tau_{-I(1,n)}u_n - \zeta)]$$

$$+ \int_{\Omega_n \cap \{-KT^1 + T^1/2 < x^1 < (-K+1)T^1 + T^1/2\}} (\tau_{-I(1,n)}u_n - \zeta)\xi$$

$$[f(\tau_{-I(1,n)}u_n) - \frac{f'(\zeta)}{2}(\tau_{-I(1,n)}u_n - \zeta)\xi].$$

The first term of the right side is nonnegative and the second is bounded by

$$C\int_{\{-KT^1 + T^1/2 < x^1 < (-K+1)T^1 + T^1/2\}} (\tau_{-I(1,n)}u_n - \zeta)^2 \tag{3.15}$$

that vanishes uniformly in n if K is large by (3.13). Therefore

$$\overline{\lim_{K\to\infty}} \sup_{n \in Z^+} \int_{\Omega_n} [(\tau_{-I(1,n)}u_n - \zeta)\xi f(\tau_{-I(1,n)}u_n) - \frac{f'(\zeta)}{2}(\tau_{-I(1,n)}u_n - \zeta)^2\xi^2] \geq 0. \tag{3.16}$$

On $\Omega \backslash \Omega_n$ (the bad set) note

$$\int_{\Omega\backslash\Omega_n} [(\tau_{-I(1,n)}u_n - \zeta)\xi f(\tau_{-I(1,n)}u_n) - \frac{f'(\zeta)}{2}(\tau_{-I(1,n)}u_n - \zeta)^2\xi^2]$$

$$= \int_{(\Omega\backslash\Omega_n)\cap\{x^1 < -KT^1 + T^1/2\}} [...] + \int_{(\Omega\backslash\Omega_n)\cap\{-KT^1 + T^1/2 < x^1 < (-K+1)T^1 + T^1/2\}} [...].$$

The second term approaches 0 uniformly in n as $K \to \infty$ as in (3.15). The first term is estimated as follows.

$$\int_{(\Omega\backslash\Omega_n)\cap\{x^1 < -KT^1 + T^1/2\}} [...]dx$$

$$\leq C\int_{(\Omega\backslash\Omega_n)\cap\{x^1 < -KT^1 + T^1/2\}} (\tau_{-I(1,n)}u_n - \zeta)^{(2d+4)/d}dx$$

$$\leq \sum_{m=-\infty}^{-K} \int_{S(mT^1, T^1/2)} (\tau_{-I(1,n)}u_n - \zeta)^{(2d+4)/d}dx$$

$$\leq C \sup_{m \leq -K} \|\tau_{-I(1,n)}u_n - \zeta\|^{4/d}_{L^2(S(mT^1, T^1/2))}$$

$$\cdot \int_{\{x^1 < (-K+1)T^1 + T^1/2\}} [|\nabla(\tau_{-I(1,n)}u_n)|^2 + (\tau_{-I(1,n)}u_n - \zeta)^2]dx$$

(as in the claim of Lemma 2.8), which approaches 0 uniformly in n as $K \to \infty$ by (3.12) and (3.13). This and (3.16) imply (3.11).

Back to (3.10), I find

$$\int_\Omega \nabla(\tau_{-I(1,n)}u_n)\cdot\nabla[(\tau_{-I(1,n)}u_n-\zeta)\xi]-\int_\Omega|\nabla[(\tau_{-I(1,n)}u_n-\zeta)\xi]|^2$$

$$\geq \int_{S((-K+1)T^1,T^1/2)}\{[\nabla(\tau_{-I(1,n)}u_n)\cdot\nabla\xi](\tau_{-I(1,n)}u_n-\zeta)-$$

$$2[\nabla(\tau_{-I(1,n)}u_n)\cdot\nabla\xi](\tau_{-I(1,n)}u_n-\zeta)\xi-|\nabla\xi|^2(\tau_{-I(1,n)}u_n-\zeta)^2\}dx.$$

By (3.13)

$$\overline{\lim_{K\to\infty}}\sup_{n\in Z^+}\int_\Omega \nabla(\tau_{-I(1,n)}u_n)\cdot\nabla[(\tau_{-I(1,n)}u_n-\zeta)\xi]-\int_\Omega|\nabla[(\tau_{-I(1,n)}u_n-\zeta)\xi]|^2\geq 0.$$

Together with (3.11) and (3.10) I find

$$\|(\tau_{-I(1,n)}u_n-\zeta)\xi\|_{W^{1,2}(\Omega)}\to 0 \text{ as } K\to\infty,$$

uniformly in n. This proves Claim 3.

From (3.7), (3.8) and (3.9) I deduce (3.5). Hence the proof of Claim 2 is complete.

Claim 4 $\{u_n'\}$ is a Palais-Smale sequence in $\mathcal{A}_\theta^\eta$.

Let $\phi \in W^{1,2}(\Omega)$. Consider

$$E'(\tau_{-I(1,n)}u_n')\phi=\int_\Omega\nabla(\tau_{-I(1,n)}u_n')\cdot\nabla\phi dx+\int_\Omega f(\tau_{-I(1,n)}u_n')\phi dx$$

$$=\int_\Omega\nabla(\tau_{-I(1,n)}u_n)\cdot\nabla\phi-\int_\Omega\nabla w_1\cdot\nabla\phi+\int_\Omega f(\tau_{-I(1,n)}u_n')\phi$$

$$=E'(\tau_{-I(1,n)}u_n)\phi+\int_\Omega[f(w_1)+f(\tau_{-I(1,n)}u_n')\phi-f(\tau_{-I(1,n)}u_n)\phi]dx.$$

Claim 4 follows if

$$\lim_{n\to\infty}\sup_{\|\phi\|_{W^{1,2}(\Omega)}\leq 1}|\int_\Omega[f(w_1)\phi+f(\tau_{-I(1,n)}u_n')\phi-f(\tau_{-I(1,n)}u_n)\phi]dx|=0. \quad (3.17)$$

I again divide Ω into Ω_L, Ω_C and Ω_R by (3.6). On Ω_C, I have

$$\lim_{n\to\infty}\sup_{\|\phi\|_{W^{1,2}(\Omega)}\leq 1}\int_{\Omega_C}[f(w_1)\phi+f(\tau_{-I(1,n)}u_n')\phi-f(\tau_{-I(1,n)}u_n)\phi]dx=0$$

since $\tau_{-I(1,n)}u_n'\to\theta$ and $\tau_{-I(1,n)}u_n\to w_1$ in $L^2(\Omega_C)$ by Lemma 2.7. On Ω_R, I have

$$|\int_{\Omega_R}f(w_1)\phi|\leq C[\int_{\Omega_R}W(w_1)]^{1/2}\|\phi\|_{L^2(\Omega_R)}$$

that is a small quantity times the $W^{1,2}(\Omega)$ norm of ϕ if K is large, and

$$|\int_{\Omega_R}[f(\tau_{-I(1,n)}u_n') - f(\tau_{-I(1,n)}u_n)]\phi dx| \leq C\|w_1 - \theta\|_{L^2(\Omega_R)}\|\phi\|_{L^2(\Omega)},$$

that again is a small quantity times the $W^{1,2}(\Omega)$ norm of ϕ if K is large. On Ω_L, I have

$$|\int_{\Omega_L} f(w_1)\phi| \leq C[\int_{\Omega_L} W(w_1)]^{1/2}\|\phi\|_{L^2(\Omega_L)},$$

$$|\int_{\Omega_L} f(\tau_{-I(1,n)}u_n)\phi| \leq C\|\tau_{-I(1,n)}u_n - \zeta\|_{L^2(\Omega_L)}\|\phi\|_{L^2(\Omega)}, \text{ and}$$

$$|\int_{\Omega_L} f(\tau_{-I(1,n)}u_n')\phi| \leq C\|\tau_{-I(1,n)}u_n' - \theta\|_{L^2(\Omega_L)}\|\phi\|_{L^2(\Omega)}.$$

Claim 3 implies that the right sides of the above three inequalities are small (uniformly in n) quantities times the $W^{1,2}(\Omega)$ norm of ϕ. This gives (3.17), and Claim 4 is proved.

To prove the proposition, repeat the argument on u_n' and find w_2, w_3,.... Lemma 2.7 and Claim 3 imply that for every $K > 0$, $\tau_{-I(1,n)}u_n \to w_1$ in $W^{1,2}(\Omega \cap \{x^1 < KT^1\})$ as $n \to \infty$. This forces $I(1,n) - I(2,n) \to -\infty$. Corollary 2.9 and Claim 2 ensure that there are only finitely many w_m's. I therefore have w_1, w_2,...,w_k, and $\{I(1,n)\}$, $\{I(2,n)\}$,...,$\{I(k,n)\}$ such that extracting the w_m's along the $\{I(m,n)\}$'s implies

$$\lim_{n\to\infty} E(u_n - \tau_{I(1,n)}w_1 + \theta_2 - \tau_{I(2,n)}w_2 + \theta_3 ... - \tau_{I(k,n)}w_k + \theta_{k+1}) = 0,$$

where $\theta_{m+1} = \lim_{n\to\infty} \widehat{w}_m(x^1)$. Then the first part of the proposition implies

$$\lim_{n\to\infty} \|u_n - \pi(\tau_{I(1,n)}w_1, \tau_{I(2,n)}w_2, ..., \tau_{I(k,n)}w_k)\|_{W^{1,2}(\Omega)} = 0.$$

The proof of Proposition 3.1 is complete. □

Theorem 3.2 *In each $\mathcal{A}_\zeta^\eta$, there is a global minimum of E, i.e., there exists $u \in \mathcal{A}_\zeta^\eta$ such that*

$$E(u) = \inf_{v\in\mathcal{A}_\zeta^\eta} E(v).$$

Proof. Clearly $u \equiv -1$ is the global minimum in $\mathcal{A}_{-1}^{-1}$ and $u \equiv 1$ is the global minimum in $\mathcal{A}_1^1$. I need only to consider $\mathcal{A}_{-1}^1$. The case of $\mathcal{A}_1^{-1}$ is analogous. Let $u_n \subset \mathcal{A}_{-1}^1$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} E(u_n) = \inf_{u\in\mathcal{A}_{-1}^1} E(u).$$

According to Corollary 4.1 [12], there exists another minimizing sequence $\{v_n\} \subset \mathcal{A}_{-1}^1$ such that

$$E(v_n) \leq E(u_n)$$

$$\|v_n - u_n\|_{W^{1,2}(\Omega)} \to 0 \text{ as } n \to \infty$$

$$\|E'(v_n)\| \to 0 \text{ as } n \to \infty.$$

The first part of Proposition 3.1 implies that $\lim_{n\to\infty} E(v_n) > 0$ since $\{v_n\} \subset \mathcal{A}^1_{-1}$, and the second part then implies that there exist $w_1, w_2, ..., w_k \in \mathcal{K}$, $k \geq 1$, $w_i \in \mathcal{A}^{\theta_{i+1}}_{\theta_i}$, $\theta_1 = -1$ and $\theta_{k+1} = 1$ such that

$$\lim_{n\to\infty} E(v_n) = E(w_1) + ... + E(w_k).$$

Obviously at least one of the w_i's, say w_j, is in $\mathcal{A}^1_{-1}$, and $E(w_j) \geq \lim_{n\to\infty} E(v_n)$. Therefore $k = 1$ and along a subsequence of $\{v_n\}$

$$\|v_n - \tau_{I(j,n)} w_j\|_{W^{1,2}(\Omega)} \to 0 \text{ as } n \to \infty$$

for an appropriate integral sequence $I(j,n)$. Hence w_j is a global minimum. □

Remark 3.3 *Corollary 4.1 [12] is stated for a Banach space. However the same proof works for $\mathcal{A}^1_{-1}$, a complete affine space.*

4 Multiple Layer Solutions

Take $U_1 \in \mathcal{A}^1_{-1}$, $U_2 \in \mathcal{A}^{-1}_1$,...,$U_M \in \mathcal{A}^{(-1)^{M+1}}_{(-1)^M}$ to be M global minima in their own subclasses. The existence of these minima is guaranteed by Theorem 3.2. In each $\mathcal{A}^\eta_\zeta$ set

$$\mathcal{B}(u, R) = \{v \in \mathcal{A}^\eta_\zeta : \ \|v - u\|_{W^{1,2}(\Omega)} < R\}. \tag{4.1}$$

A key assumption in this section is that the U_i's are isolated, i.e., there exists $\mu_0 > 0$ such that $\mathcal{B}(U_i, \mu_0) \cap \mathcal{K} = \{U_i\}$ for all $i = 1, 2, ..., M$.

Note that given integers $j_1, ..., j_M$ I write $\pi_j U$ for $\pi(\tau_{j_1} U_1, ..., \tau_{j_M} U_M)$. The reader may think it as an approximate solution near which a real solution is to be found if $j_1, ..., j_M$ are chosen properly.

Proposition 4.1 *Let $0 < r < \lambda_0$, where λ_0 is defined in Lemma 2.6, be fixed. For integers $j_1 < j_2 < ... < j_M$, there exists $\delta = \delta(j, U, r)$ depending on $j_1, ..., j_M$, $U_1, ..., U_M$, and r such that either $\|E'(u)\| > \delta$ for all $u \in \mathcal{B}(\pi_j U, r) \subset \mathcal{A}^{(-1)^{M+1}}_{-1}$, or $E'(u) = 0$ for some $u \in \overline{\mathcal{B}}(\pi_j U, r)$, the closure of $\mathcal{B}(\pi_j U, r)$.*

If the U_i's are isolated and μ_0 is the corresponding small number, then for every $0 < r < \min\{\lambda_0, \mu_0\}$ and every positive integer L there exists $\delta = \delta(U, r, L)$ depending on $U_1, ..., U_M$, r and L such that for all integers $j_1 < j_2 < ... < j_M$ with $l < j_{i+1} - j_i < Ll$, $i = 1, 2, ..., M-1$, for some positive integer l, either $\|E'(u)\| > \delta$ for all $u \in \mathcal{B}(\pi_j U, r) \backslash \mathcal{B}(\pi_j U, r/2)$, or $E'(u) = 0$ for some $u \in \overline{\mathcal{B}}(\pi_j U, r)$.

The second part of the proposition is the key step of this paper. It basically implies that if there were no critical points in $\overline{\mathcal{B}}(\pi_j U, r)$, then E' would significantly pushes the energy down under the deformation (4.10) in $\mathcal{B}(\pi_j U, r)\backslash\mathcal{B}(\pi_j U, r/2)$. I will show in the proof of Theorem 4.4 that this could not happen since the energy of a function $u \in \mathcal{B}(\pi_j U, r)$ is not much lower than $E(\pi_j U)$.

Proof. Suppose there exists a sequence $\{u_n\}$ in $\mathcal{B}(\pi_j U, r)$ with $\lim_{n\to\infty} \|E'(u_n)\| = 0$. By Proposition 3.1, there exist $w_1, ..., w_k$ in $\mathcal{K}$ and sequences $I(1, n), ..., I(k, n)$ with $I(i, n) - I(i+1, n) \to -\infty$ such that along a subsequence of $\{u_n\}$

$$\lim_{n\to\infty} \|u_n - \pi(\tau_{I(1,n)} w_1, ..., \tau_{I(k,n)} w_k)\|_{W^{1,2}(\Omega)} = 0. \tag{4.2}$$

Therefore

$$\overline{\lim_{n\to\infty}} \|\pi(\tau_{j_1} U_1, ..., \tau_{j_M} U_M) - \pi(\tau_{I(1,n)} w_1, ..., \tau_{I(k,n)} w_k)\|_{W^{1,2}(\Omega)} \leq r. \tag{4.3}$$

If one of the $I(i, n)$'s, say $I(j, n)$, is unbounded in n, then for given $R > 0$ observe that for some $\theta \in \{-1, 1\}$

$$r^2 \geq \overline{\lim_{n\to\infty}} \|\pi_{I(n)} w - \pi_{J(n)} U\|^2_{W^{1,2}(\Omega)}$$

$$\geq \overline{\lim_{n\to\infty}} \int_{S(I(j,n)T^1, R)} [|\nabla(\pi_{I(n)} w) - \nabla(\pi_{J(n)} U)|^2 + |\pi_{I(n)} w - \pi_{J(n)} U|^2]$$

$$= \int_{S(0,R)} [|\nabla w_j|^2 + |w_j - \theta|^2].$$

Sending $R \to \infty$, I deduce

$$r \geq \overline{\lim_{n\to\infty}} \|\pi_{I(n)} w - \pi_{J(n)} U\|_{W^{1,2}(\Omega)} \geq \|w_j - \theta\|_{W^{1,2}(\Omega)} \geq \lambda_0$$

by Lemma 2.6, but $r \geq \lambda_0$ is inconsistent with the assumption. So $k = 1$, $\pi_{I(n)} w = \tau_{I(1,n)} w_1$ and $I(1, n)$ is bounded in n Therefore $\tau_{I(1,n)} w_1 \in \mathcal{K}$ is in $\overline{\mathcal{B}}(\pi_j U, r)$ for large n. This proves the first part.

Assume that the U_i's are isolated. Suppose there exist integral sequences $J(1, n), ..., J(M, n)$ with $l(n) < J(i+1, n) - J(i, n) < Ll(n)$, and $\{u_n\} \in \mathcal{B}(\pi_{J(n)} U, r)\backslash\mathcal{B}(\pi_{J(n)} U, r/2)$ with $\|E'(u_n)\| \to 0$. If $\{l(n)\}$ is bounded in n, the first part of the proposition implies the conclusion of the second part.

Suppose $l(n) \to \infty$. To use Proposition 3.1, I have to show that $E(u_n)$ is bounded in n. First note

$$\lim_{n\to\infty} E(\pi_{J(n)} U) = E(U_1) + ... + E(U_M). \tag{4.4}$$

Then consider

$$|E(u_n) - E(\pi_{J(n)} U)|$$

$$= |\int_\Omega \frac{1}{2} |\nabla(u_n - \pi_{J(n)} U) + \nabla \pi_{J(n)} U|^2 dx - \int_\Omega \frac{1}{2} |\nabla \pi_{J(n)} U|^2 dx$$

$$+\int_\Omega [W(u_n) - W(\pi_{J(n)}U)]dx|$$

$$= |\int_\Omega \frac{1}{2}|\nabla(u_n - \pi_{J(n)}U)|^2 dx + \int_\Omega \nabla(u_n - \pi_{J(n)}U)\cdot\nabla\pi_{J(n)}U dx$$

$$+\int_\Omega [W(u_n) - W(\pi_{J(n)}U)]dx|$$

$$\le \frac{1}{2}\|\nabla(u_n - \pi_{J(n)}U)\|^2_{L^2(\Omega)} + \|\nabla(u_n - \pi_{J(n)}U)\|_{L^2(\Omega)}\|\nabla\pi_{J(n)}U\|_{L^2(\Omega)}$$

$$+\|f(\pi_{J(n)}U)\|_{L^2(\Omega)}\|u_n - \pi_{J(n)}U\|_{L^2(\Omega)} + C\|u_n - \pi_{J(n)}U\|^2_{L^2(\Omega)}$$

$$\le C_1(\|\nabla\pi_{J(n)}U\|_{L^2(\Omega)} + \|f(\pi_{J(n)}U)\|_{L^2(\Omega)})\|u_n - \pi_{J(n)}U\|_{W^{1,2}(\Omega)}$$

$$+C_2\|u_n - \pi_{J(n)}U\|^2_{W^{1,2}(\Omega)}$$

$$\le C_1\sqrt{E(\pi_{J(n)}U)}\|u_n - \pi_{J(n)}U\|_{W^{1,2}(\Omega)} + C_2\|u_n - \pi_{J(n)}U\|^2_{W^{1,2}(\Omega)}.$$

The last inequality follows from the fact $f^2(u) \le CW(u)$. And the last line is bounded uniformly in n by (4.4) and $\|u_n - \pi_{J(n)}U\|^2_{W^{1,2}(\Omega)} < r$. Hence $E(u_n)$ is bounded, and along a subsequence of $\{u_n\}$ $E(u_n)$ converges. Applying Proposition 3.1, I find $w_1, ..., w_k$, $I(1,n), ..., I(k,n)$ with

$$\lim_{n\to\infty}\|u_n - \pi_{I(n)}w\|_{W^{1,2}(\Omega)} = 0, \text{ and} \tag{4.5}$$

$$\overline{\lim_{n\to\infty}}\|\pi_{J(n)}U - \pi_{I(n)}w\|_{W^{1,2}(\Omega)} \le r. \tag{4.6}$$

Claim 1 $I(1,n) - J(1,n)$ is bounded in n.

Otherwise $I(1,n) - J(1,n) \to -\infty$ or ∞ along a subsequence. Assume $I(1,n) - J(1,n) \to -\infty$. The other case is similar. Fix $R > 0$ and consider

$$\lambda_0^2 > r^2 \ge \overline{\lim_{n\to\infty}}\int_{S(I(1,n)T^1,R)} [|\nabla(\pi_{I(n)}w) - \nabla(\pi_{J(n)}U)|^2$$

$$+|\pi_{I(n)}w - \pi_{J(n)}U|^2] = \int_{S(0,R)} [|\nabla w_1|^2 + |w_1 + 1|^2].$$

Sending $R \to \infty$, I find $\|w_1 + 1\|_{W^{1,2}(\Omega)} < \lambda_0$ that violates Lemma 2.6.

Once I know $I(1,n) - J(1,n)$ is bounded, I can further shift w_1 and assume $I(1,n) = J(1,n)$ for all n.

Claim 2 $w_1 = U_1$.

Fix $R > 0$ and consider

$$\mu_0^2 > r^2 \ge \overline{\lim_{n\to\infty}}\int_{S(I(1,n)T^1,R)} [|\nabla(\pi_{I(1,n)}w) - \nabla(\pi_{J(1,n)}U)|^2$$

$$+|\pi_{I(1,n)}w - \pi_{J(1,n)}U|^2] = \int_{S(0,R)} [|\nabla w_1 - \nabla U_1|^2 + |w_1 - U_1|^2].$$

Sending $R \to \infty$, I find $\|w_1 - U_1\|_{W^{1,2}(\Omega)} < \mu_0$. Since U_1 is isolated, $w_1 = U_1$.

Now take $\pi(\tau_{J(2,n)}U_2, ..., \tau_{J(M,n)}U_M)$ and $\pi(\tau_{I(2,n)}w_2, ..., \tau_{I(k,n)}w_k)$ to be the new $\pi_{J(n)}U$ and $\pi_{I(n)}w$. Clearly

$$\overline{\lim_{n\to\infty}} \|\pi_{I(n)}w - \pi_{J(n)}U\|_{W^{1,2}(\Omega)} \leq r.$$

Repeat the same argument to deduce $w_2 = U_2$. This process ends up with one of the following three cases.

Case 1 $k > M$.

After M steps of extraction, $\pi_{J(n)}U$ becomes $(-1)^{M+1}$ and $\pi_{I(n)}w$ becomes $\pi(\tau_{I(M+1,n)}w_{M+1}, ..., \tau_{I(k,n)}w_k)$. Fix $R > 0$ and observe

$$\lambda_0^2 > r^2 \geq \overline{\lim_{n\to\infty}} \int_{S(I(M+1,n)T^1,R)} [|\nabla(\pi_{I(n)}w)|^2 + |\pi_{I(n)}w - (-1)^{M+1}|^2]$$

$$= \int_{S(0,R)} [|\nabla w_{M+1}|^2 + |w_{M+1} - (-1)^{M+1}|^2].$$

Sending $R \to \infty$, I find $\|w_{M+1} - (-1)^{M+1}\|_{W^{1,2}(\Omega)} < \lambda_0$ that is inconsistent with Lemma 2.6.

Case 2 $k < M$.

This case can be ruled out in the same way.

Case 3 $k = M$.

For large n, I find $\pi_{I(n)}w = \pi_{J(n)}U$. Therefore for large n, $u_n \in \mathcal{B}(\pi_{J(n)}U, r/2)$ that violates the assumption about u_n. Proposition 4.1 is proved. $\square$

Remark 4.2 *The role of Ll may be replaced by $g(l)$ where g is a positive increasing function of l. Then δ in the second part depends on U, r, g.*

Set $\mathcal{U}$ to be the function class in $\mathcal{A}_{-1}^{(-1)^{M+1}}$ such that u is in $\mathcal{U}$ if and only if there exist intervals $-\infty = s_1 < t_1 < s_2 < t_2 < ... < ...s_M < t_M = \infty$ such that

$$u(x^1, x') = (-1)^{i+1} \text{ if } t_i < x^1 < s_{i+1},\ i = 1, 2, ..., M-1. \tag{4.7}$$

Lemma 4.3 $\inf_{u\in\mathcal{U}} E(u) = E(U_1) + E(U_2) + ... + E(U_M)$.

Proof. Clearly for every $u \in \mathcal{U}$, $i = 1, 2, ..., M$,

$$\int_{\Omega\cap\{x^1\in(s_i,t_i)\}} [\frac{1}{2}|\nabla u|^2 + W(u)]dx \geq \inf_{v\in\mathcal{A}_{(-1)^i}^{(-1)^{i+1}}} E(v) = E(U_i).$$

Then $E(u) \geq E(U_1) + ... + E(U_M)$. With the help of Remark 2.4, for every $\epsilon > 0$, take a function U_i' near U_i and choose a large integer K independent of i so that $U_i'(x) = (-1)^i$ for $x^1 < -KT^1$, $U_i'(x) = (-1)^{i+1}$ for $x^1 > KT^1$, and $|E(U_i) - E(U_i')| \leq \epsilon/M$. Then if $U^* = \pi(\tau_{2KT^1}U_1', \tau_{4KT^1}U_2', ..., \tau_{2MKT^1}U_M')$,

$$E(U^*) = E(U_1') + E(U_2') + ... + E(U_M') \leq E(U_1) + ... + E(U_M) + \epsilon,$$

which proves the lemma. □

Theorem 4.4 *Assume the U_i's are isolated. Then for every r, L in the second part of Proposition 4.1, there exists a positive integer l_0 such that for all integers $j_1 < j_2 < ... < j_M$ with $l_0 \leq l < j_{i+1} - j_i < Ll$, $i = 1, 2, ..., M-1$, for some positive integer l, $\overline{\mathcal{B}}(\pi_j U, r) \cap \mathcal{K} \neq \emptyset$.*

Proof. Let r and L be the same as in Proposition 4.1, and $\delta = \delta(U, r, L)$ be the number in the second part of Proposition 4.1.

Note that as $l_0 \to \infty$, $E(\pi_j U) \to E(U_1) + ... + E(U_M) = \inf_{u \in \mathcal{U}} E(u)$. Find l_0 so large that

$$E(\pi_j U) < \inf_{u \in \mathcal{U}} E(u) + \frac{r\delta}{8}. \tag{4.8}$$

Take a vector field $F(u)$ on $\mathcal{A}_{-1}^{(-1)^{M+1}} \backslash \mathcal{K}$ to be

$$F(u) = -\frac{r\delta}{4} \frac{\widetilde{E}'(u)}{\|E'(u)\|^2} \tag{4.9}$$

where $\widetilde{E}'(u)$ is the Riesz representation of $E'(u)$ in $W^{1,2}(\Omega)$. Assume $\pi_j U \notin \mathcal{K}$, otherwise the theorem is proved. Consider in $\mathcal{A}_{-1}^{(-1)^{M+1}}$ the ordinary differential equation

$$z'(t) = F(z),\ z(0) = \pi_j U. \tag{4.10}$$

It serves as a deformation along which the energy decreases. Note that Lemma 2.5 implies that (4.10) gives a unique local solution. Let σ be the maximum time of the existence of (4.10). Consider two cases:

Case 1 $\sigma \leq 1$.

If $z(t) \in \partial\mathcal{B}(\pi_j U, r)$ for some $t < \sigma$, then there exist $0 < s_1 < s_2 < \sigma$ such that $z(s_1) \in \partial\mathcal{B}(\pi_j U, r/2)$, $z(s_2) \in \partial\mathcal{B}(\pi_j U, r)$, and

$$r/2 < \|z(t) - \pi_j U\|_{W^{1,2}(\Omega)} < r$$

for $s_1 < t < s_2$. Then

$$r/2 \leq \|z(s_1) - z(s_2)\|_{W^{1,2}(\Omega)} = \|\int_{s_1}^{s_2} z'(t)dt\|_{W^{1,2}(\Omega)} = \|\int_{s_1}^{s_2} F(z)dt\|_{W^{1,2}(\Omega)}$$

$$\leq \int_{s_1}^{s_2} \|F(z)\|_{W^{1,2}(\Omega)}dt \leq \int_{s_1}^{s_2} \frac{r\delta}{4\delta}dt \leq r/4, \tag{4.11}$$

a contradiction. Here I have assumed $\mathcal{B}(\pi_j U, r) \cap \mathcal{K}$ is empty, otherwise the theorem is proved.

Therefore $z(t) \in \mathcal{B}(\pi_j U, r)$ for all $t < \sigma$. Then along a sequence $t_n \to \sigma$, $z(t_n)$ satisfies

$$\|E'(z(t_n))\| \to 0 \text{ as } n \to \infty$$

which yields the theorem by the first part of Proposition 4.1.

Case 2 $\sigma > 1$.

If $z(t) \in \partial\mathcal{B}(\pi_j U, r)$ for some $t \le 1$, the argument in (4.11) gives a contradiction. Therefore $z(t) \in \mathcal{B}(\pi_j U, r)$ for all $t \le 1$ (in particular $z(1) \in \mathcal{B}(\pi_j U, r)$), and

$$E(z(1)) - E(\pi_j U) = \int_0^1 E'(z)z'(t)dt = \int_0^1 E'(z)F(z)dt = \int_0^1 -\frac{r\delta}{4}dt = -\frac{r\delta}{4},$$

which implies, with the help of (4.8),

$$E(z(1)) < \inf_{u \in \mathcal{U}} E(u) - \frac{r\delta}{8}. \tag{4.12}$$

What has been proved so far is that if $\mathcal{B}(\pi_j U, r) \cap \mathcal{K} = \emptyset$, there is $z(1) \in \mathcal{B}(\pi_j U, r)$ whose energy is less than $\inf\limits_{u \in \mathcal{U}} E(u) - (r\delta)/8$. If $z(1) \in \mathcal{U}$, I would have a contradiction. In general $z(1) \notin \mathcal{U}$, so I have to show that $z(1)$ is almost in $\mathcal{U}$.

Fix $R > 0$ and large l_0, and decompose Ω, from left to right, into segments Ω_1, $S(j_1 T^1, R)$, Ω_2, $S(j_2 T^1, R)$,..., $S(j_M T^1, R)$, Ω_{M+1} so that

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{S}(j_1 T^1, R) \cup \overline{\Omega}_2 \cup \overline{S}(j_2 T^1, R) \cup ... \cup \overline{S}(j_M T^1, R) \cup \overline{\Omega}_{M+1}$$

where the Ω_i's and S's are mutually disjoint. This decomposition depends on l_0. When l_0 gets larger, the Ω_i's get longer.

Claim For each $i = 1, 2..., M+1$, $\int_{\Omega_i} [|\nabla(\pi_j U)|^2 + |\pi_j U - (-1)^i|^2] < \overline{C}$ where $\overline{C}$ is independent of l_0.

To see this, send $l_0 \to \infty$. Then

$$\int_{\Omega_i} [|\nabla(\pi_j U)|^2 + |\pi_j U - (-1)^i|^2] \to$$

$$\int_{\{x^1 > R\}} [|\nabla U_{i-1}|^2 + |U_{i-1} - (-1)^i|^2]dx + \int_{\{x^1 < -R\}} [|\nabla U_i|^2 + |U_i - (-1)^i|^2]dx.$$

This proves the claim.

Note for each i, $\|z(1) - (-1)^i\|_{W^{1,2}(\Omega_i)} \le \overline{C} + r$. For large l_0, I can find $M+1$ integers $m_1, m_2, ..., m_{M+1}$ such that $S(m_i T^1, T^1/2)$ is contained in Ω_i and

$$\|z(1) - (-1)^i\|_{W^{1,2}(S(m_i T^1, T^1/2))} \to 0 \text{ as } l_0 \to \infty. \tag{4.13}$$

Such integers m_i exist since the Ω_i's get longer as l_0 gets larger. Define a smooth function $\xi \in [0,1]$ satisfying

$$\xi(x^1) = \begin{cases} 0, & \text{if } |x^1| < T^1/4 \\ 1, & \text{if } |x^1| > T^1/2. \end{cases}$$

For $x = (x^1, x') \in \Omega$ set $\xi(x) = \xi(x^1)$. Define

$$\overline{z}(x) = \begin{cases} z(1)(x), & \text{if } x \notin S(m_i T^1, T^1/2) \\ (z(1)(x) - (-1)^i)\tau_{m_i}\xi + (-1)^i, & \text{if } x \in S(m_i T^1, T^1/2). \end{cases}$$

Clearly $\overline{z} \in \mathcal{U}$. Also

$$\|\overline{z} - z(1)\|^2_{W^{1,2}(\Omega)} = \sum_{i=1}^{M+1} \int_{S(m_i T^1, T^1/2)} [|\nabla(z(1) - (z(1) - (-1)^i)\tau_{m_i}\xi)|^2$$

$$+|(z(1) - (-1)^i)(1 - \tau_{m_i}\xi)|^2] \to 0 \text{ as } l_0 \to 0$$

by (4.13). Then as in the argument following (4.4)

$$|E(\overline{z}) - E(z(1))| \le \|\nabla z(1)\|_{L^2(\Omega)}\|\overline{z} - z(1)\|_{W^{1,2}(\Omega)} + \frac{1}{2}\|\overline{z} - z(1)\|^2_{L^2(\Omega)}$$

$$+\|f(z(1))\|_{L^2(\Omega)}\|\overline{z} - z(1)\|_{W^{1,2}(\Omega)} + C\|\overline{z} - z(1)\|^2_{L^{1,2}(\Omega)}$$

$$\le C_1\sqrt{E(z(1))}\|\overline{z} - z(1)\|_{W^{1,2}(\Omega)} + C_2\|\overline{z} - z(1)\|^2_{W^{1,2}(\Omega)}$$

that approaches 0 as $l_0 \to \infty$ if $E(z(1))$ is bounded uniformly in l_0. To see that $E(z(1))$ is bounded uniformly in l_0, observe that $\|z(1) - \pi_j U\|_{W^{1,2}(\Omega)} < r$ and $E(\pi_j U) \to E(U_1) + ... + E(U_M)$ as $l_0 \to \infty$. The same argument as before shows that $|E(z(1)) - E(\pi_j U)|$ and $E(z(1))$ are bounded uniformly in l_0. Therefore $E(\overline{z}) - E(z(1)) \to 0$ as $l_0 \to \infty$. Combining this with (4.12), I find that for l_0 large enough

$$E(\overline{z}) < \inf_{u \in \mathcal{U}} E(u) - \frac{r\delta}{16}.$$

This is impossible since $\overline{z} \in \mathcal{U}$, and the proof of Theorem 4.4 is complete. □

Corollary 4.5 *Let V be a solution constructed in Theorem 4.4, and r, l be the corresponding numbers. As $r \to 0$ and $l_0 \le l \to \infty$, $E(V) \to E(U_1) + ... + E(U_M)$.*

Remark 4.6 *The solution constructed in Theorem 4.4 is close to $\pi_j U$ and each $\tau_{j_i} U_i$, $i = 1, ..., M$, corresponds to a transition layer. l_0 measures the distance of the layers. The more isolated the U_i's are, the smaller l_0 can be.*

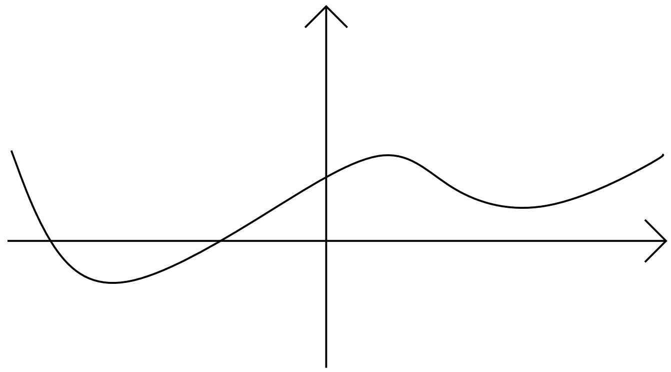

Figure 4: An example of unbalanced W.

5 Further Remarks

Remark 5.1 *In H-1, the function class C^2 may be weakened to C^1.*

I have used E' to construct the differentiable vector field F in the proof of Theorem 4.4 to perform the deformation argument. Such a vector field can be replaced by a pseudo-gradient field if E' is only continuous, and the argument also goes through (see [14]).

Remark 5.2 *The gradient term in (1.2) can be generalized.*

Proposed by Bates, Fife, Ren and Wang in [2], phase transitions can be modeled by the Helmholtz free energy functional

$$F(u) = \int_\Omega \int_\Omega J(x-y)(u(x)-u(y))^2 dxdy + \int_\Omega W(u)dx \tag{5.1}$$

where J is nonnegative, even, and scaled so that $\int_{R^d} J(x)dx = 1$. As calculated in Fife [9], the gradient flow in $L^2(\Omega)$ associated to (5.1) is

$$u_t = \int_\Omega J(x-y)u(y)dy - K(x)u - f(u) \tag{5.2}$$

where $K(x) = \int_\Omega J(x-y)dy$. If $\Omega = R^1$ and W is not necessarily balanced (see Figure 4), traveling waves of (5.2) are found in [2]. When W is balanced and $\Omega \subset R^d$ is periodic in x^1-direction, in [15] Ren and Winter find global minima of (5.1) in the class of Young measure valued functions.

If change variables in the first integral of (5.1) using $\eta = (x-y)/2$, $\xi = (x+y)/2$, then expand $u(x) = u(\xi+\eta)$ and $u(y) = u(\xi-\eta)$ about ξ to get

$$2^{d+2} \int_\Pi J(2\eta)(\sum_{k=1}^{\infty} \sum_{|\alpha|=2k-1} \frac{1}{\alpha!} D^\alpha u(\xi)\eta^\alpha)^2 d\xi d\eta,$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$, $D^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}...\partial_{x_d}^{\alpha_d}$, $\alpha! = \alpha_1!\alpha_2!...\alpha_d!$, and $\eta^\alpha = \eta_1^{\alpha_1}\eta_2^{\alpha_2}...\eta_d^{\alpha_d}$. Π is the corresponding region for (ξ, η) in $R^d \times R^d$.

Truncate the infinite sum at $k = N$ and integrate with respect to η. This "truncated " free energy is given by

$$E(u) = \int_\Omega [\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=2i-1\\|\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u + W(u)]dx \tag{5.3}$$

where

$$A_{\alpha\beta} = A_{\alpha\beta}(x) = 2^{d+2} \int_{\{\eta:(x,\eta)\in\Pi\}} \frac{J(2\eta)\eta^{\alpha+\beta}}{\alpha!\beta!} d\eta$$

is convex in all its indices. Note that $A_{\alpha\beta}$ is periodic in x^1-direction if Ω is so.

As in (1.3), define $\mathcal{A} = \{u \in L^1_{loc}(\Omega) : \ E(u) < \infty\}$. $\mathcal{A}^\eta_\zeta$ is again defined by (2.5). When $N > (d+2)/4$, it is not necessary to introduce (2.2). One can just use the continuity of u to define $\mathcal{A}^\eta_\zeta$. See [4] for this simpler case.

The Euler-Lagrange equation of (5.3) is

$$\begin{cases} \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=2i-1\\|\beta|=2j-1}} D^\alpha[(A_{\alpha\beta} + A_{\beta\alpha})D^\beta u] - W'(u) = 0 \text{ in } \Omega, \\ B_j(u) = 0 \text{ on } \partial\Omega, \ j = 1, 2, ..., 2N-1; \end{cases}$$

where the B_j's are the natural boundary conditions. All the results in this paper are valid for this higher order problem.

Remark 5.3 *One may study unbounded Ω that is only periodic near infinity.*

It has been shown in [4] that for $N > (d+2)/4$, when Ω is not periodic in one direction, the existence of global minima depends on the shape of Ω. Let me reproduce a sufficient condition for all $N \geq 1$. Assume that Ω is a smooth unbounded domain. Suppose it can be divided into three disjoint open domains, Ω_1, Ω_2 and Ω_0, such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_0$. Further assume that Ω_0 is bounded and Ω_1 and Ω_2 are "half periodic". That Ω_m, $m = 1, 2$, is half periodic means there exists $T_m \in R^d$, $m = 1, 2$, such that for every $x_m \in \Omega_m$, $x_m + T_m \in \Omega_m$, and for every $x_m \in \Omega_m$ there exist $y_m \in \Omega_0$ and a non-negative integer n_m satisfying $x_m = y_m + n_m T_m$. Ω_1 and Ω_2 are indeed the two half periodic tails of Ω. I associate the end e_1 to Ω_1 and the other end e_2 to Ω_2 (see Figure 5).

Treat Ω as a metric space equipped with a distance function $\rho : \Omega \times \Omega \to R$ defined by

$$\rho(x, y) = \inf_{\gamma\in\Gamma} \int_0^1 |\gamma'(t)| dt$$

where Γ is the class of all smooth paths $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. Define

$$B_\Omega(x, r) = \{y \in \Omega : \rho(y, x) < r\}$$

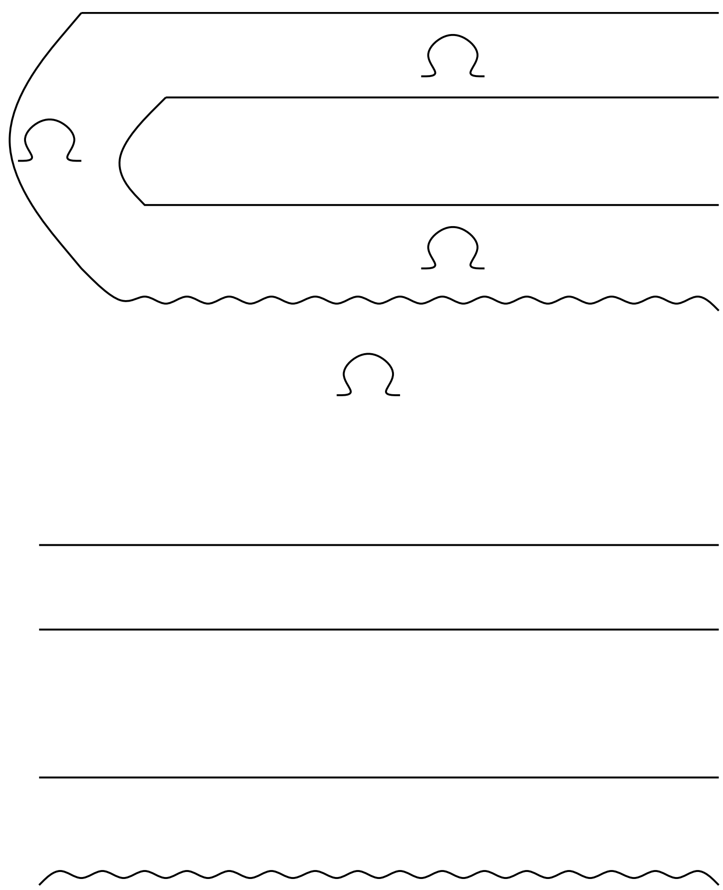

Figure 5: An example of Ω, G_1 and G_2.

to be the ball centered at x with radius r in the metric space (Ω, ρ).

Define $\mathcal{A}$ and $\mathcal{A}_\zeta^\eta$ by some formulae similar to (1.3) and (2.5) where $\zeta, \eta \in \{-1, 1\}$. Loosely speaking $\mathcal{A}_\zeta^\eta = \{u \in \mathcal{A} : \lim_{x \to e_1} u(x) = \zeta,\ \lim_{x \to e_2} = \eta\}$. To state an existence theorem in $\mathcal{A}_{-1}^1$, define G_m, $m = 1, 2$, to be the periodic extension of Ω_m, i.e.,

$$G_m = \{x \in R^d : x + n_m T_m \in \Omega_m \text{ for some integer } n_m\}.$$

I require that G_m has bounded cross sections perpendicular to T_m. Denote the other end of G_m by e'_m (see Figure 5). Define two auxiliary functionals:

$$E_m(u) = \int_{G_m} [\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u + W(u)] dx, \tag{5.4}$$

where $u \in \mathcal{A}_{-1}^1(m)$, $m = 1, 2$. Here $\mathcal{A}_{-1}^1(m)$ is defined to be

$$\{u \in L^1_{loc}(G_m) : E_m(u) < \infty, \lim_{x \to e_m} u(x) = (-1)^m, \lim_{x \to e'_m} u(x) = (-1)^{m+1}\}$$

where the limits are interpreted in a way similar to (2.5). The following theorem gives a sufficient condition for the existence of a global minimum in $\mathcal{A}_{-1}^1$.

Theorem 5.4 *If Ω is a tube half-periodic near the two ends, then*

$$\inf_{u \in \mathcal{A}_{-1}^1} E(u) \le \min\{\inf_{u \in \mathcal{A}_{-1}^1(1)} E_1(u), \inf_{u \in \mathcal{A}_{-1}^1(2)} E_2(u)\}.$$

If the strict inequality

$$\inf_{u \in \mathcal{A}_{-1}^1} E(u) < \min\{\inf_{u \in \mathcal{A}_{-1}^1(1)} E_1(u), \inf_{u \in \mathcal{A}_{-1}^1(2)} E_2(u)\}$$

holds, then (5.3) has a global minimum in $\mathcal{A}_{-1}^1$.

A similar result holds for $\mathcal{A}_1^{-1}$. I give the outline of the proof. The reader may consult [4] for details and applications.

Sketch of the proof. The first part follows from a comparison argument.

The second part is proved with the help of the concentration-compactness principle (see [11]). Take an energy minimizing sequence $\{u_n\}$ from $\mathcal{A}_{-1}^1$. Define the concentration function of u_n to be

$$Q_n(t) = \sup_{y \in \Omega} \int_{B_\Omega(y,t)} [\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n + W(u_n)] dx$$

for $t \ge 0$. $\{Q_n\}$ is a sequence of nonnegative, nondecreasing, uniformly bounded functions on $[0, \infty)$ and $\lim_{n \to \infty} \lim_{t \to \infty} Q_n(t) = \Lambda$ where $\Lambda = \inf_{u \in \mathcal{A}_{-1}^1} E(u)$.

By a classical result about monotone functions, there exist a subsequence of Q_n, again denoted by Q_n, and a nonnegative, nondecreasing function $Q(t)$ such that $\lim_{n\to\infty} Q_n(t) = Q(t)$ for all $t \geq 0$. Let $\lambda = \lim_{t\to\infty} Q(t)$. Then $\lambda \in [0, \Lambda]$. Consider three cases.

Case 1 $\lambda = 0$ (Vanishing).

This implies for every $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\Omega} \int_{B_\Omega(y,R)} [\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n + W(u)] dx = 0.$$

It does not occur by a result similar to Lemma 2.1 and the fact $u_n \in \mathcal{A}^1_{-1}$.

Case 2 $\lambda \in (0, \Lambda)$ (Dichotomy).

It does not occur since $\{u_n\}$ is a minimizing sequence. The proof involves a careful estimate on the energies of the sequence. Roughly speaking, for a sequence with $\lambda \in (0, \Lambda)$ one can always construct another sequence with lower energies. Therefore an energy minimizing sequence can not be in this case.

Case 3 $\lambda = \Lambda$ (Compact).

In this case for every $\epsilon > 0$, find $R > 0$ and a sequence $\{y_n\} \subset \Omega$ such that

$$\overline{\lim_{n\to\infty}} \int_{B_\Omega(y_n,R)} [\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n + W(u_n)] dx > \Lambda - \epsilon.$$

The sequence $\{u_n\}$ converges to a limit in $\mathcal{A}^1_{-1}$ if y_n is bounded in Ω. It turns out by a comparison argument that under the strict inequality in Theorem 5.4 y_n is bounded. This gives the convergence and a global minimum. $\square$

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