Homoclinic Solutions of an Integral Equation: Existence and Stability

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January 13, 2000

Abstract

We study existence and stability of homoclinic type solutions of a bistable integral equation. These are stationary solutions of an integrodifferential equation, which is a gradient flow for a free energy functional with general nonlocal integrals penalizing spatial nonuniformity.

1 Introduction

We study the integral equation

$$(J * u)(x) - u(x) - f(u(x)) = 0, \ x \in R^1 \tag{1.1}$$

with the decay condition $u(-\infty) = u(\infty) = 0$, where $J * u$ is the convolution of J and u. We assume $J > 0$ in R^1, $\int_{R^1} J(z)dz = 1$ and f is bistable, e.g., $f(u) = u(u-1)(u-a)$. Solutions to (1.1) are stationary solutions of the evolution equation

$$u_t = J * u - u - f(u). \tag{1.2}$$

Equation (1.2), recently proposed in [2], can model a variety of physical and biological phenomena, e.g., a material whose state is described by an order

*Part of the work was done between September, 95 and August, 96, when the authors were postdoctoral members of the IMA, University of Minnesota. The second named author was supported in part by NSF grant DMS-9703727 and Utah State University, New Faculty Research Grant, SC45978.

parameter. Note that (1.2) is the L^2-gradient flow of the free energy functional

$$E(u) = \frac{1}{4}\int_{R^1}\int_{R^1} J(x-y)(u(x)-u(y))^2 dxdy + \int_{R^1} W(u(x))dx, \tag{1.3}$$

which it is sometimes convenient to write as

$$E(u) = -\frac{1}{2}\int_{R^1}\int_{R^1} J(x-y)u(x)u(y)dxdy + \int_{R^1}\Big(\frac{1}{2}u^2(x) + W(u(x))\Big)dx.$$

Here, W is a double-well function with two (not necessarily equal) local minima (say, at 0 and 1), and $J(r)$ is a measure of the energy resulting from $u(x)$ being different from $u(x+r)$. The first term in (1.3) penalizes spatially inhomogeneous materials, and the second term (bulk term) penalizes states which take values other than the two minima of W. Note that $W' = f$.

Functional (1.3) is a natural generalization of the well-known and studied functional

$$E^l(u) = \frac{1}{2}\int_{R^1} |\nabla u(x)|^2 dx + \int_{R^1} W(u(x))dx. \tag{1.4}$$

Namely, if we change variables in the first integral using $\eta = \frac{x-y}{2}$, $\xi = \frac{x+y}{2}$ and then expand $u(x) = u(\xi+\eta)$ and $u(y) = u(\xi-\eta)$ about ξ, we formally write

$$E(u) = 2\int_{R^1}\int_{R^1} J(2\eta)\Big(\sum_{j=0}^{\infty}\frac{D^{2j+1}u(\xi)\eta^{2j+1}}{(2j+1)!}\Big)^2 d\xi d\eta + \int_{R^1} W(u(x))dx. \tag{1.5}$$

Note that if we truncate the summation in (1.5) and replace it by the first term, we get

$$E^l(u) = \int_{R^1} \left[c\,|\nabla u|^2 + W(u)\right]\,dx,$$

where $c = \int_{R^1} J(2\eta)\eta^2\,d\eta$. Thus the right side is the same as $E^l(u)$ up to the constant c which can be stretched out by a further change of variable, and $E^l(u)$ can be treated as a first order approximation of $E(u)$. Consequently,

$$u_t = \Delta u - f(u). \tag{1.6}$$

can be regarded as the first order approximation of (1.2).

In [2], the authors constructed traveling wave solutions $u(x-ct)$ of (1.2) and studied their stability with respect to (1.2). More recently, Chen [8] extended the stability results of [2], and also gave some examples of non-monotone stationary waves.

In our paper, we build on the work of [2]. We construct homoclinic solutions of (1.1), i.e., even solutions with $u(\pm\infty) = 0$. Physically such solutions represent a threshold between the domains of attraction of the two local minima of W.

Let us recall that for the local model (1.6), the solutions of

$$u'' - f(u) = 0 \tag{1.7}$$

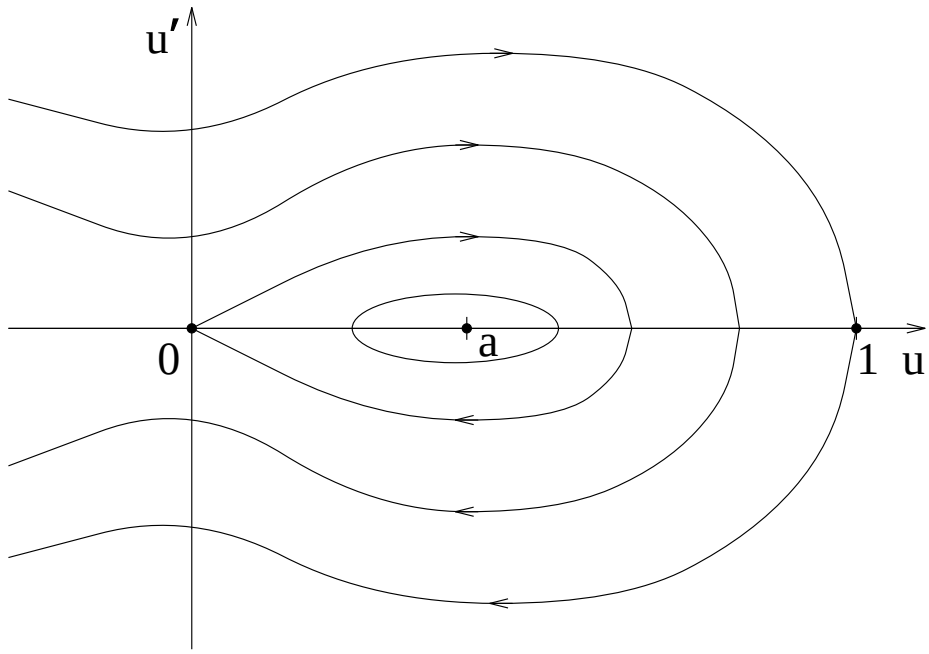

Figure 1: A phase plane.

are easily obtained from a phase plane analysis. In particular, if W has two wells at 0 and 1 with $W(1) < W(0) = 0$, there exists a (unique) homoclinic solution of (1.7), such that $u(\pm\infty) = 0$ (see Figure 1).

For the equation (1.2) the nonlocal term $J * u$ causes considerable difficulty in constructing solutions. In [2], the authors overcame this by a clever homotopy argument. However, both in [2] and Chen's work [8], the underlying stability of traveling waves was very strongly put to use. In our problem, we expect homoclinic solutions to be unstable, as is the case for (1.6). Thus the methods used in [2] and [8] will not work here, and we are forced to solve the problem in a more complicated way. We obtain our solutions by an intuitively clear minimax argument. However, lack of compactness causes considerable technical difficulties in our construction. Some of them are overcome by a rearrangement argument.

One of the most striking differences between (1.6) and (1.2) is the existence of discontinuous stationary waves ([2],[8]) for a class of nonlinearities f. We observe a similar phenomena in our work, namely, if $u + f(u)$ is not monotone, (1.1) can admit discontinuous homoclinic solutions.

In a companion paper [9], we will generalize our work to higher space di-

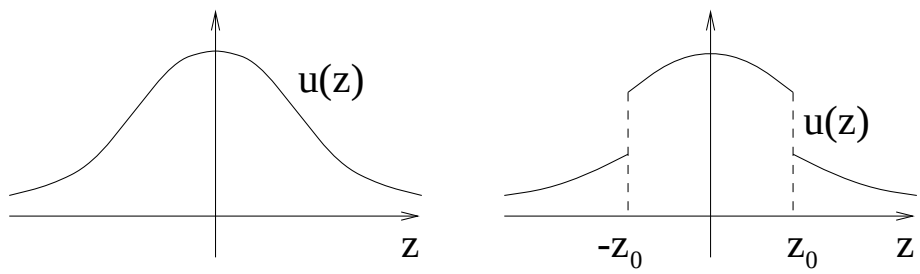

Figure 2: Solutions of Theorem 2.1.

mensions. Bates, Chmaj and Presutti study in a forthcoming paper [3] general solutions of the higher dimensional version of (1.2) for a class of f's. Other papers that address similar nonlocal problems include [4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22].

Our paper is organized as follows. We prove the existence of a homoclinic solution in Section 2, and discuss its smoothness in Section 3. We consider a special case of J in Section 4, where the reader can find more differences between the homoclinic solution of (1.1) and that of (1.7). Finally in Section 5 we study the stability of the solution, where we prove a stability result and an instability result.

2 The existence theorem

In this paper we assume that $f \in C^r \cap C^{0,1}, r \geq 0$, and has three zeros 0, a and 1. We also assume $\int_0^1 f(z)dz < 0$, $f'(0) > 0$, $f'(a) < 0$ and $f'(1) > 0$. For technical reasons we assume $f(u)$ is linear in u for $u < -1$ and $u > 2$. For a f not satisfying the last condition one can always modify f to satisfy the condition since the solutions considered here are bounded between 0 and 1. Also, for simplicity we assume that $u + f(u)$ has at most three intervals of monotonicity.

About J we assume that $J > 0$, $\int_{R^1} J(z) = 1$, $J \in W^{k,1}(R^1)$, $k \geq 1$, and J is even, J is strictly decreasing in $(0, \infty)$.

Let $g(u) \equiv u + f(u)$, $m \equiv \min\{r, k\}$.

Theorem 2.1 *There exists an even, positive solution u of (1.1), nonincreasing on $(0, \infty)$, with $u(-\infty) = u(\infty) = 0$ and $u \in L^2(R^1)$ (see Figure 2).*

We begin the proof of the theorem with some properties of the functional E defined in (1.3).

Lemma 2.2 *The functional $E : L^2(R^1) \to R^1$ defined in (1.3) is in the class $C^{1,1}$.*

Proof. Straightforward calculation. □

We next discuss the stability of the trivial solution $u \equiv 0$.

Lemma 2.3 *$u \equiv 0$ is a strict local minimum of E if $u^2/2 + W(u) > 0$ for $u \neq 0$. $u \equiv 0$ is not a local minimum of E if $u_0^2/2 + W(u_0) < 0$ for some $u_0 > 0$. There always exists $e \in L^2(R^1)$ with large $\|e\|$ such that $E(e) \leq 0$.*

Proof. First suppose $u^2/2 + W(u) > 0$ for $u \neq 0$. Let u_1 and u_2 be the two zeros of W other than 0. Take $A < u_1$ and $B > u_2$ to be two positive numbers. Observe that only the second term in (1.3) can be negative. For each $u \in L^2(R^1)$ set

$$u_g(x) = \begin{cases} u(x) & if\ u(x) \notin [A, B] \\ 0 & if\ u(x) \in [A, B] \end{cases},$$

and $u_b(x) = u(x) - u_g(x)$. Then

$$\begin{aligned} E(u) &= E(u_g + u_b) \\ &= \frac{1}{2}\int_{R^1} [u_g^2(x) - J * u_g(x)u_g(x) + u_b^2(x) - J * u_b(x)u_b(x) \\ &\quad -2\ J * u_g(x)u_b(x)] + \int_{R^1} W(u_g) + \int_{R^1} W(u_b) \\ &\geq \int_{R^1} \Big(\frac{1}{2}u_b^2 + W(u_b)\Big) - \frac{1}{2}\int_{R^1} J * (u(x) + u_g(x))u_b(x) + \int_{R^1} W(u_g). \end{aligned}$$

Since we assume that $W''(0) > 0$ and W is quadratic in large u, there exists $\delta > 0$ such that

$$E(u) \geq \delta\|u\|^2 - \frac{1}{2}\int_{R^1} J * (u(x) + u_g(x))u_b(x).$$

Let $\Omega \equiv \text{supp } u_b$. Then $\int_{R^1} u_b^2 \geq A^2|\Omega|$, so $|\Omega| \leq (\int_{R^1} u_b^2)/A^2$. Also

$$\begin{aligned} |\int_{R^1} J * (u(x) + u_g(x))u_b(x)| &\leq |J * (u + u_g)|_{L^\infty(R^1)} \int_\Omega |u_b| \\ &\leq \|J\|_{L^2(R^1)}\|u + u_g\|B|\Omega| \leq 2\|J\|_{L^2(R^1)}\|u\|^3 \frac{B}{A^2}. \end{aligned}$$

Thus we have

$$E(u) \geq \delta\|u\|^2 - C\|u\|^3 \tag{2.1}$$

for some $C > 0$, which proves that 0 is a strict local minimum of (1.3).

Now let us suppose that there exists $u_0 > 0$ such that $u_0^2/2 + W(u_0) < 0$. Define

$$u_L(x) = \begin{cases} u_0, & if\ x \in (-L, L) \\ 0, & if\ x \notin (-L, L) \end{cases}$$

We calculate $\lim_{L\to 0} E(u_L)/L$. To calculate the double integral part of $E(u_L)$, we separate R^2 into:

$$\begin{cases} \Omega_1 \equiv \{(x,y) : x \in (-L,L), y \in (-L,L)\} \\ \Omega_2 \equiv \{(x,y) : x \notin (-L,L), y \notin (-L,L)\} \\ \Omega_3 \equiv \{(x,y) : x \in (-L,L), y \notin (-L,L)\} \\ \Omega_4 \equiv \{(x,y) : x \notin (-L,L), y \in (-L,L)\} \end{cases}.$$

We find

$$E(u_L) = \frac{u_0^2}{2} \int_{-L}^{L} dx \int_{y \notin (-L,L)} J(x-y)dy + 2LW(u_0).$$

Setting $K(x) = \int_{-\infty}^{x} J(z)dz$, we deduce

$$E(u_L) = \frac{u_0^2}{2} \int_{-L}^{L} [K(x-L) + 1 - K(x+L)]dx + 2LW(u_0)$$

which implies

$$\begin{aligned} \lim_{L\to 0} \frac{E(u_L)}{2L} &= \frac{u_0^2}{2} + W(u_0) + \lim_{L\to 0} \frac{u_0^2}{2} \frac{\int_{-2L}^{0} K(z)dz - \int_0^{2L} K(z)dz}{2L} \\ &= \frac{u_0^2}{2} + W(u_0) + \frac{u_0^2}{2} \lim_{L\to 0} (K(-2L) - K(2L)) = \frac{u_0^2}{2} + W(u_0) < 0 \end{aligned}$$

which together with the fact $\|u_L\| \to 0$ as $L \to 0$ proves the second part.

We set

$$e_L(x) = \begin{cases} 1, & if\ x \in (-L, L) \\ 0, & if\ x \notin (-L, L) \end{cases}.$$

We show that $E(e_L) < 0$ if L is large enough. By dividing R^2 into four parts, as in the calculation of $E(u_L)$, we find

$$E(e_L) = \frac{1}{2}(2L + \int_{-2L}^{0} K(z)dz - \int_0^{2L} K(z)dz) + 2LW(1).$$

Integration by parts yields

$$2L + \int_{-2L}^{0} K(z)dz - \int_0^{2L} K(z)dz$$

$$= 2L(1 - K(2L)) + \int_0^{2L} zJ(z)dz + 2LK(-2L) - \int_{-2L}^{0} zJ(z)dz$$

L'Hospital's Rule and the assumption $\int_{R^1} J(z)\, dz = 1$ implies that

$$\lim_{L\to\infty} \frac{E(e_L)}{2L} = W(1) < 0,$$

thus $E(e_L) < 0$ for L large enough. $\square$

Remark 2.4 *We think Lemma 2.3 is surprising, by comparing it to the local case, where $u \equiv 0$ is always a local minimum of (1.4). This easily follows from the fact that (1.4) is defined on $H^1(R)$ and Sobolev's Imbedding Theorem.*

At this moment we assume that $u^2/2 + W(u) > 0$ if $u \neq 0$. Indeed we will strengthen this condition to that $u + f(u)$ is nondecreasing in u. The case that $u + f(u)$ is not nondecreasing will be reduced to the first case later in this section.

We proceed as follows. We 'anticipate' the solution we are seeking to be a 'saddle' point of (1.3). However, we are unable to use the classical Mountain-Pass Theorem [21] directly, since the Palais-Smale condition is not satisfied. We are thus forced to solve the problem in an indirect way. We first construct a Palais-Smale sequence, then show by a rearrangement argument that it can be taken to consist of uniformly bounded, even functions nonincreasing in $(0, \infty)$. At this stage, we are able to overcome the lack of Palais-Smale condition by passing to the pointwise limit (we can apply Helly's Theorem to our sequence). After ruling out some undesired cases, we show that the limit is a non-trivial solution of (1.1).

We set

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t)) \tag{2.2}$$

where $\Gamma = \{\gamma \in C([0,1], L^2(R^1)) : \gamma(0) = 0, \gamma(1) = e\}$. Note that (2.1) in the proof of Lemma 2.3 implies that $c > 0$. We are to construct a sequence u_n of even functions nonincreasing in $(0, \infty)$ with $u_n(x) \in [0, k]$ for all $x \in R^1$, where k is the greatest zero of W. u_n is to satisfy

$$\lim_{n\to\infty} E(u_n) = c, \ \lim_{n\to\infty} E'(u_n) = 0. \tag{2.3}$$

To this end we show that if γ_n is a sequence of paths in Γ satisfying

$$\lim_{n\to\infty} \max_{t\in[0,1]} E(\gamma_n(t)) = c,$$

then γ_n^{T*}, the $[0, k]$ truncated and spherically rearranged γ_n, has the same properties. First define γ_n^T by

$$\gamma_n^T(t)(x) = \begin{cases} \gamma_n(t)(x) & if\ \gamma_n(t)(x) \in [0,k] \\ k & if\ \gamma_n(t)(x) > k \\ 0 & if\ \gamma_n(t)(x) < 0 \end{cases}.$$

It is easily verified that

$$||\gamma_n^T(t) - \gamma_n^T(s)|| \leq ||\gamma_n(t) - \gamma_n(s)||$$

thus $\gamma_n^T \in \Gamma$. Also, obviously $E(\gamma_n^T(t)) \leq E(\gamma_n(t))$. Next we recall the notion of spherical rearrangement. Suppose $u : R^1 \to [0, \infty]$. Set

$$\chi_a^u(x) = \begin{cases} 1 & if\ u(x) \geq a \\ 0 & otherwise \end{cases}.$$

Define

$$(\chi_a^u)^*(x) = \begin{cases} 1 & if\ |x| \leq \frac{1}{2}\int_{R^1} \chi_a^u(s)ds \\ 0 & otherwise \end{cases}.$$

The spherical rearrangement of u is

$$u^*(x) = \int_0^\infty (\chi_a^u)^*(x)da.$$

It is shown in [23] that

$$\int_{R^1} u(x)v(x)dx \leq \int_{R^1} u^*(x)v^*(x)dx.$$

It is also easily seen that for $u \geq 0$

$$\int_{R^1} F(u^*(x))dx = \int_{R^1} F(u(x))dx$$

when one of the integrals makes sense. Thus, if we define $\gamma_n^{T*}(t) = (\gamma_n^T(t))^*$, then

$$||\gamma_n^{T*}(t) - \gamma_n^{T*}(s)|| \leq ||\gamma_n^T(t) - \gamma_n^T(s)||$$

so that $\gamma_n^{T*}(t) \in \Gamma$. Also, it is shown in [23] that for $u, v, w \geq 0$

$$\int_{R^1}\int_{R^1} u(x)v(x-y)w(y)dxdy \leq \int_{R^1}\int_{R^1} u^*(x)v^*(x-y)w^*(y)dxdy.$$

which implies that

$$E(\gamma_n^{T*}(t)) \leq E(\gamma_n^T(t)) \leq E(\gamma_n(t)).$$

We deduce that

$$\lim_{n\to\infty} \max_{t\in[0,1]} E(\gamma_n^{T*}(t)) = c.$$

We now quote a theorem from [21] (A corollary of Ekeland's Variational Principle).

Theorem 2.5 *[21] Let K be a compact metric space, $K_0 \in K$ a closed set, X a Banach space, $\chi \in C(K_0, X)$ and let us define the complete metric space M by*

$$M = \{g \in C(K, X) : g(s) = \chi(s) \ if \ s \in K_0\}$$

with the usual distance d. Let $\phi \in C^1(X, R^1)$ and let us define

$$c = \inf_{g \in M} \max_{s \in K} \phi(g(s)), \ c_1 = \max_{\chi(K_0)} \phi.$$

If $c > c_1$, then for each $\epsilon > 0$ and each $f \in M$ such that

$$\max_{s \in K} \phi(f(s)) \leq c + \epsilon,$$

there exists $v \in X$ such that

$$c - \epsilon \leq \phi(v) \leq \max_{s \in K} \phi(f(s)), \ dist(v, f(K)) \leq \epsilon^{1/2}, \ |\phi'(v)| \leq \epsilon^{1/2}.$$

In our setting $M = \Gamma$, $K_0 = \{0, 1\}$, $\chi(0) = 0$, and $\chi(1) = e$. Applying this theorem to $f = \gamma_n^{T*}$ we can find a sequence v_n in $L^2(R^1)$ with

$$\lim_{n \to \infty} E(v_n) = c, \ \lim_{n \to \infty} E'(v_n) = 0.$$

At this moment v_n is not necessarily even, nonincreasing in $(0, \infty)$ and bounded between 0 and k. The following argument shows that v_n can be replaced by u_n which is even, nonincreasing in $(0, \infty)$ and bounded between 0 and k.

Given positive $\epsilon_n \to 0$, let v_n be the function given by the theorem. According to the same theorem there exists $u_n \in \{\gamma_n^{T*}(t) : t \in [0, 1]\}$ such that $\|v_n - u_n\| \leq \epsilon_n^{1/2}$. Clearly $E(u_n) \leq \max_{t \in [0,1]} E(\gamma_n^{T*}(t))$, and by Lemma 2.2

$$\|E'(u_n)\| \leq \|E'(v_n)\| + C\|u_n - v_n\| \leq \epsilon_n^{1/2} + C\epsilon_n^{1/2},$$

and for some $\theta \in (0, 1)$

$$\begin{aligned} E(u_n) &\geq E(v_n) - \|u_n - v_n\| \cdot \|E'(v_n + \theta(u_n - v_n))\| \\ &\geq E(v_n) - \|u_n - v_n\|(\|E'(v_n)\| + C\theta\|u_n - v_n\|) \\ &\geq c - \epsilon_n - \epsilon_n^{1/2}(\epsilon_n^{1/2} + C\epsilon_n^{1/2}). \end{aligned}$$

Therefore we have proved

Lemma 2.6 *If $u^2/2 + W(u) > 0$ for $u \neq 0$, there exists a sequence u_n of even, nonincreasing in $(0, \infty)$ functions with $0 \leq u_n \leq k$ where k is the greatest zero of W such that*

$$\lim_{n \to \infty} E(u_n) = c > 0, \ \lim_{n \to \infty} E'(u_n) = 0.$$

Let us assume $u^2/2 + W(u) > 0$ for $u \neq 0$, and u_n be the sequence given in Lemma 2.6. We can now apply Helly's theorem to u_n. Recall that Helly's Theorem states that a uniformly bounded sequence of monotone functions has a pointwise convergent subsequence. Thus we find u such that

$$\lim_{n\to\infty} u_n(x) = u(x),\ 0 \leq u(x) \leq k,$$

for every $x \in R^1$. Now, $E'(u_n) \to 0$ is equivalent to

$$\int_{R^1} |J * u_n - u_n - f(u_n)|^2 \to 0,$$

so it easily follows that

$$J * u - u - f(u) = 0 \tag{2.4}$$

We need to show that $u \not\equiv 0$ and $u(-\infty) = u(\infty) = 0$. Since the convergence in Helly's Theorem is only pointwise, it may well happen that $u \equiv 0, a$ or 1. To show u is not constant, we argue in the following way.

Let μ be a positive number less than a. Let $x_n > 0$ be such that $u_n(x_n) = \mu$. Of course u_n may be discontinuous and miss μ there. In this case just redefine u_n at the discontinuous point so $u_n = \mu$ there. Then one of the following three cases occurs.

A. there exist b_1, $b_2 > 0$ with $b_1 \leq x_n \leq b_2$;

B. $x_n \to 0$;

C. $x_n \to \infty$.

Case A implies $u(\infty) \leq \mu < a$. But $f(u(\infty)) = 0$ since u solves (1.1). Therefore, $u(\infty) = 0$. Also $u \not\equiv 0$ since for $x \in [-b_1, b_1]$, $u(x) \geq \mu$. Therefore u is a desired solution of (1.1).

Cases B and C are ruled out by the following lemmas. Note that the fact $u + f(u)$ is nondecreasing in u implies that $u^2/2 + W(u) > 0$ if $u \neq 0$.

Lemma 2.7 *Under the condition of Lemma 2.6, Case B cannot occur.*

Proof. We show that $E'(u_n) \to 0$ and $x_n \to 0$ together imply $\|u_n\| \to 0$ which in turn implies $E(u_n) \to 0$ contradicting $E(u_n) \to c > 0$. Note

$$\int_{R^1} u_n(-J * u_n + u_n + f(u_n)) \leq \|u_n\| \cdot \|E'(u_n)\| \leq \epsilon\|u_n\|^2 + \frac{1}{\epsilon}\|E'(u_n)\|^2. \tag{2.5}$$

On the other hand for some $\delta > 0$

$$\int_{R^1} u_n(-J * u_n + u_n + f(u_n)) \quad \geq \quad \int_{R^1} u_n f(u_n)$$

$$
\begin{aligned}
&= \int_{R^1\setminus(-x_n,x_n)} u_n f(u_n) + \int_{(-x_n,x_n)} u_n f(u_n) \\
&\geq \int_{(-x_n,x_n)} \delta u_n^2 + \int_{(-x_n,x_n)} u_n f(u_n) \\
&\geq \delta \int_{R^1} u_n^2 + \int_{(-x_n,x_n)} (u_n f(u_n) - \delta u_n^2)
\end{aligned}
$$

The last term of the last line can be bounded by $x_n M$ for some M independent of n. Therefore,

$$\int_{R^1} u_n(-J * u_n + u_n + f(u_n)) \geq \delta \|u_n\|^2 - x_n M.$$

Combining this with (2.5) we find

$$(\delta - \epsilon)\|u_n\|^2 \leq \frac{1}{\epsilon}\|E'(u_n)\|^2 + x_n M. \tag{2.6}$$

Taking $\epsilon < \delta$, we deduce $\|u_n\| \to 0$. □

Lemma 2.8 *Case C can not occur if $u + f(u)$ is nondecreasing.*

Proof. Set $w_n(x) = u_n(x - x_n)$. Then w_n is nondecreasing in $(-\infty, x_n)$. By using Helly's theorem and a diagonal argument we find that along a subsequence $w_n(x) \to w(x)$ for each $x \in R^1$ and w is a nondecreasing solution of

$$J * w - w - f(w) = 0 \tag{2.7}$$

with $w(-\infty) = 0$ and $w(\infty) = a$ or 1 where a is the second zero of f.

Now we show that such a solution of (2.7) does not exist. To this end, set $g(w) = w + f(w)$ and write (2.7) as

$$J * w = g(w). \tag{2.8}$$

We integrate both sides against measure dw in R^1, i.e.,

$$\int_{R^1} J * w dw = \int_{R^1} g(w) dw.$$

We first show that

$$\int_{R^1} g(w) dw = \int_0^{w(\infty)} g(z) dz, \tag{2.9}$$

where the integral on the right side of (2.9) is just the usual Lebesgue integral of g against the Lebesgue measure dz. (2.9) can be written as

$$\int_{R^1} g(w) dw = \frac{w^2(\infty)}{2} + \int_0^{w(\infty)} f(z) dz. \tag{2.10}$$

To show (2.9) we recall the notion of distribution measures. Consider $g(w)$ as a function from R^1 equipped with measure dw to R^1. The distribution measure $\mu_{g(w)}$ of $g(w)$ is a measure defined on the target space of $g(w)$ such that

$$\mu_{g(w)}(A) = dw((g(w))^{-1}(A))$$

for each Borel measurable $A \in R^1$. One of the properties of $\mu_{g(w)}$ is

$$\int_{-\infty}^{\infty} g(w)dw = \int_{-\infty}^{\infty} \alpha d\mu_{g(w)}(\alpha).$$

The same argument can be applied to g, a function from R^1, equipped with the usual Lebesgue measure dz, to R^1. Then

$$\int_0^{w(\infty)} g(z)dz = \int_{-\infty}^{\infty} \alpha d\mu_g(\alpha)$$

where μ_g is the induced distribution measure of g. Therefore to prove (2.9) we need only to show that $\mu_{g(w)} = \mu_g$. It suffices to show that for each $b \in [0, w(\infty)]$

$$\mu_{g(w)}((-\infty, b]) = \mu_g((-\infty, b]). \tag{2.11}$$

By the continuity and monotonicity of g we have

$$\mu_g((-\infty, b]) = dz(g^{-1}((-\infty, b])) = \max\{y : g(y) = b\}. \tag{2.12}$$

By the continuity, which comes from (2.8) and the fact that $J * w$ is continuous, and monotonicity of $g(w)$, we find, setting $X = \max\{x : g(w(x)) = b\}$ and $Y = \max\{y : g(y) = b\}$,

$$\mu_{g(w)}((-\infty, b]) = dw((-\infty, X]) = \lim_{x \to X+} w(x).$$

We show that the last limit is Y. Note that $\lim_{x \to X+} w(x) \geq Y$ by the monotonicity of g. Assume

$$\lim_{x \to X+} w(x) > Y.$$

Then the continuity and monotonicity of g implies

$$\lim_{x \to X+} g(w(x)) > b,$$

contradicting the continuity of $g(w)$. This proves $\mu_{g(w)}((-\infty, b]) = Y$. Together with (2.12) we find (2.11) which implies (2.9).

We next show

$$\int_{R^1} J * w dw = \frac{w^2(\infty)}{2}. \tag{2.13}$$

The proof of (2.13) concludes the proof of the lemma since (2.13) and (2.10) force

$$\int_0^{w(\infty)} f(z)dz = 0.$$

But this is not true in either $w(\infty) = a$ case or $w(\infty) = 1$ case.

To see (2.13) we integrate by parts to obtain

$$\int_{R^1} J * w\, dw = J * w \cdot w|_{-\infty}^{\infty} - \int_{R^1} w\, dJ * w. \tag{2.14}$$

We next show

$$\int_{R^1} w\, dJ * w = \int_{R^1} J * w\, dw. \tag{2.15}$$

Clearly (2.14) and (2.15) imply (2.13).

To prove (2.15) we take a smooth function ϕ with compact support in R^1 and observe

$$\begin{aligned} \int_{R^1} J * \phi dw &= -\int_{R^1} [J * \phi]' w dx = -\int_{R^1} J * \phi' w\, dx \\ &= -\int_{R^1} J * w\phi' dx = \int_{R^1} \phi dJ * w. \end{aligned}$$

Here we have used the fact that J is even. So we deduce

$$\int_{R^1} J * \phi dw = \int_{R^1} \phi dJ * w.$$

If we approximate w by ϕ, we find (2.15). □

Remark 2.9 *The solution u is in $L^2(R^1)$.*

This is a byproduct of Lemmas 2.7 and 2.8. Because of (2.6) and the fact established after the proof of Lemma 2.8 that x_n is bounded, the Palais-Smale sequence u_n is bounded in $L^2(R^1)$. This L^2 bound ensures that the solution u, the pointwise limit of u_n, is also a weak L^2 limit of u_n.

We have obtained the existence of a homoclinic solution if $g(u) = u + f(u)$ is non-decreasing in u. The following argument completes the proof of Theorem 2.1.

Let us assume that g is not nondecreasing. We 'truncate' $g(u)$ into a modified $g^t(u)$, and construct a homoclinic solution of

$$J * u = g^t(u). \tag{2.16}$$

Then we show that the solution of (2.16) also satisfies

$$J * u = g(u). \tag{2.17}$$

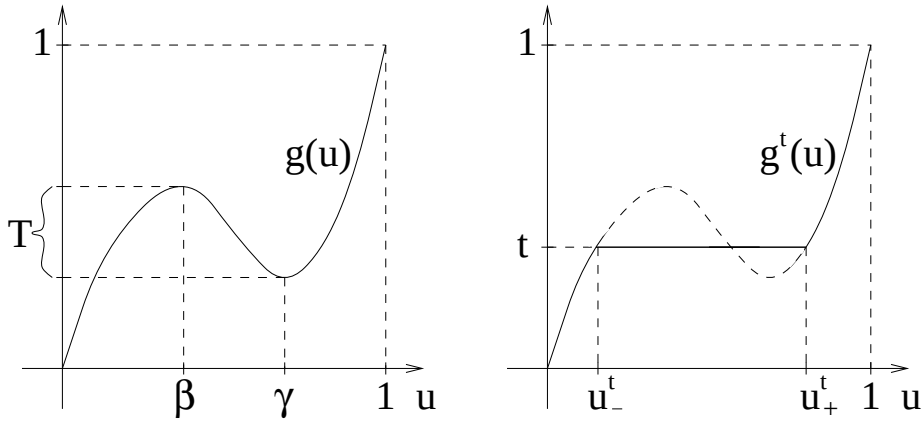

Figure 3: g and g^t.

Let $T \equiv \{\{g(u) : u \in [0, \beta]\} \cap \{g(u) : u \in [\gamma, 1]\}\} \cap \{(0, 1)\}$. For any $t \in T$, we define $g^t(u)$ to be the continuous nondecreasing function obtained by modifying g to be the constant t between the ascending branches of g (see Figure 3).

Let u^t_- and u^t_+ be such that

$$g^t(u) = \begin{cases} g(u), & u \in [0, u^t_-] \cup [u^t_+, 1] \\ t, & u \in [u^t_-, u^t_+] \end{cases}$$

Note that to each g^t there corresponds a modification of f defined by

$$f^t(u) = \begin{cases} f(u), & u \notin [u^t_-, u^t_+] \\ -u + t, & u \in [u^t_-, u^t_+] \end{cases}$$

(see Figure 4).

We restrict the t's to those for which

$$\int_0^1 f^t(u)du < 0. \tag{2.18}$$

Let $I \equiv T \cap \{t : \int_0^1 f^t(u)du < 0\}$. For any $t \in I$, consider equation (2.16). Note that g^t is Lipshitz continuous, and nondecreasing, so according to the earlier results there is a homoclinic solution u^t of (2.16). We need only to show that this u^t also solves (2.17).

We prove this by showing that there is no interval on which u^t takes values in (u^t_-, u^t_+). Otherwise we assume that for $x \in (x_1, x_2)$, $0 < x_1 < x_2$, $u^t(x) \in (u^t_-, u^t_+)$. This implies that for $x \in (x_1, x_2)$, $J * u^t(x)$ is constant. Note that,

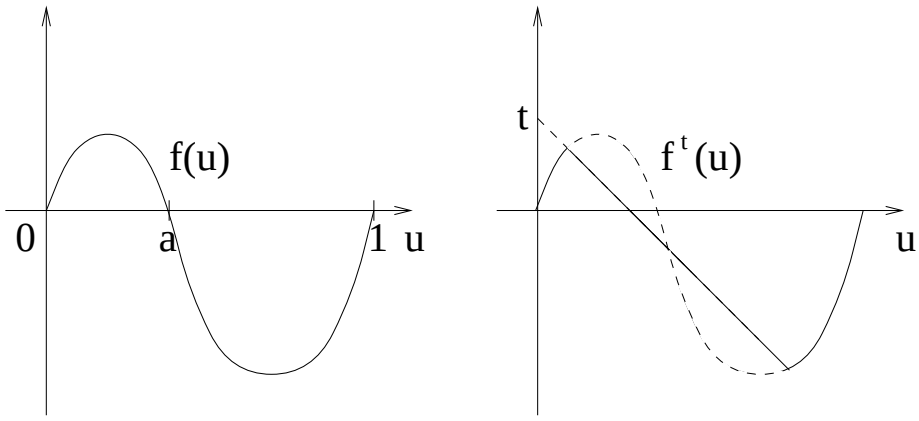

Figure 4: f and f^t.

setting $x_0 = (x_1 + x_2)/2$,

$$\begin{aligned} 0 &= \int_{R^1} J(y)[u^t(x_2 - y) - u^t(x_1 - y)]dy \\ &= \int_{R^1} J(y + x_0)[u^t(\frac{x_2 - x_1}{2} - y) - u^t(\frac{x_1 - x_2}{2} - y)]dy. \end{aligned}$$

Set $U(y) = u^t(\frac{x_2-x_1}{2} - y) - u^t(\frac{x_1-x_2}{2} - y)$. Then $U(-y) = U(y)$, and

$$\int_0^\infty [J(y + x_0) - J(-y + x_0)]U(y)dy = 0.$$

Since J is strictly decreasing and u^t is nonincreasing in $(0, \infty)$, we find that for $y > 0$, $U(y) \geq 0$ and $J(y + x_0) - J(-y + x_0) < 0$. We deduce $U(y) = 0$ for $y > 0$. This implies that u^t is constant, and contradicts the fact that u^t is a nontrivial solution of (2.16).

3 More properties of the solution

The smoothness of the solution u constructed in Section 2 depends on g. Let us discuss a few cases.

Proposition 3.1 *If g is strictly increasing, u is continuous. If $g' > 0$, u is in $C^m(R^1)$ where m is defined at the beginning of Section 2.*

This is because $u(x) = g^{-1}(J * u(x))$.

Proposition 3.2 *If $g' > 0$ on $[0,\beta) \cup (\gamma, 1]$, constant on $[\beta, \gamma]$, one of the following three cases occurs:*

1. *u is in the class $C^m(R^1)$.*
2. *u is in the class C^m on $(-\infty, 0) \cup (0, +\infty)$.*
3. *u is discontinuous at $-z_0$ and z_0, for some $z_0 > 0$, and in the class C^m on $(-\infty, -z_0) \cup (-z_0, z_0) \cup (z_0, +\infty)$. Moreover, $u(-z_0-) = \beta$, $u(-z_0+) = \gamma$.*

From the arguments at the end of last section, $g(u(x))$ cannot be constant on any interval $[x_1, x_2]$, thus there is a possibility of our solution having a jump discontinuity at $-z_0$ and z_0, for some $z_0 > 0$. However, this jump will take place only if $u(0) > \gamma$. More precisely: case 1 occurs if $u(0) < \beta$, case 2 occurs if $u(0) = \beta$, case 3 occurs if $u(0) > \gamma$.

Proposition 3.3 *If $g' > 0$ on $[0,\beta) \cup (\gamma, 1]$, and $g' < 0$ on (β, γ), one of the following three cases occurs:*

1. *u is in the class $C^m(R^1)$.*
2. *there is a solution which is in the class $C^m(R^1)$ or C^m on $(-\infty, 0) \cup (0, +\infty)$, and a family of discontinuous solutions.*
3. *there is a family of discontinuous solutions.*

Again, from the arguments at the end of last section and the previous proposition, we have three possibilities: case 1 occurs if $u^t(0) < u^t_-$ for all $t \in I$, case 2 occurs if $u^t(0) < u^t_-$ for $t \in I_0$, where I_0 is a subset of I, and $u^t(0) > u^t_+$ for $t \in I \backslash I_0$, case 3 occurs if $u^t(0) > u^t_+$ for all $t \in I$.

4 Case $J(z) = \frac{1}{2}e^{-|z|}$

In the special case $J(z) = \frac{1}{2}e^{-|z|}$, integration by parts yields

$$(J * u)'' = J * u - u. \tag{4.1}$$

Thus a homoclinic solution of (1.1) is also a solution of the local equation

$$(g(u))'' = f(u), \tag{4.2}$$

where $g(u) = u + f(u)$. If we let $w = g(u)$ then (4.2) becomes

$$w'' = f(g^{-1}(w)). \tag{4.3}$$

Assume that w is a homoclinic solution of (4.3). By subtracting w and applying $J*$ to each side of (4.3), we see that $u \equiv g^{-1}(w)$ is then a homoclinic

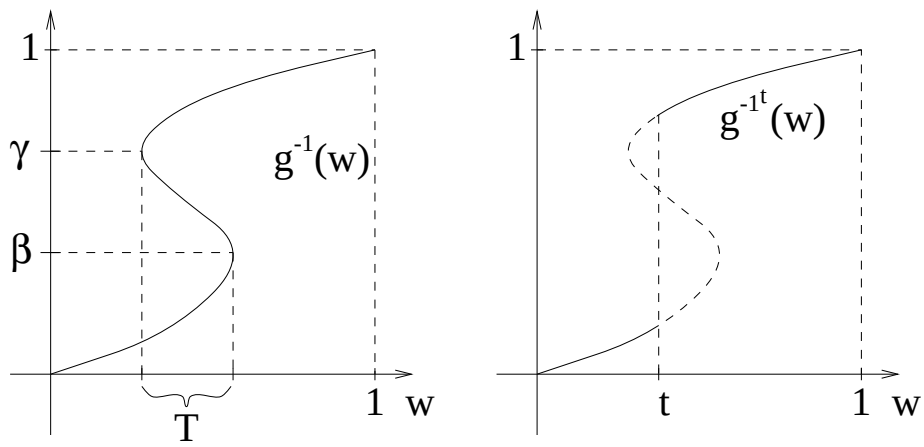

Figure 5: g^{-1} and g^{-1^t}.

solution of (1.1). Thus, in this special case, it suffices to consider solutions of (4.3).

First, let g be invertible. Then, homoclinic solutions of (4.3) are easily constructed from a phase-plane analysis, similar to Figure 1.

If g is not invertible, let $g' > 0$ on $[0, \beta) \cup (\gamma, 1]$, $g' < 0$ on (β, γ), as in Section 2. Then, g^{-1} is not well-defined, namely, in the interval $[\max\{0, g(\gamma)\}, \min\{1, g(\beta)\}]$ it is triple-valued (see Figure 5). Let T be as in Section 2. For any $t \in T$, let g^{-1^t} be the single-valued function defined by

$$g^{-1^t}(w) = \begin{cases} \textit{lowest branch of } g^{-1}, & 0 \le w \le t \\ \textit{highest branch of } g^{-1}, & t < w \le 1 \end{cases}$$

(see Figure 5). Then (4.3) is piecewise well-defined, and a homoclinic solution is obtained by 'gluing' $W^u(0,0) \cap \{(w, w') : 0 \le w \le t\}$ and $W^s(0,0) \cap \{(w, w') : 0 \le w \le t\}$ with a connecting orbit (see Figure 6). Here $W^u(0,0)$ ($W^s(0,0)$, respectively) is the unstable (stable, respectively) manifold of $(0,0)$.

We conclude this section with the following proposition.

Proposition 4.1 *Let u be an even homoclinic solution of (1.1). Then*

1. $a < u(0) < 1$.
2. *If u is continuous and b is the second zero of W, then $u(0) \ge b$.*
3. *If u is continuous and $J(z) = \frac{1}{2}e^{-|z|}$, $u(0)$ satisfies*

$$W(u(0)) + \frac{1}{2} f(u(0))^2 = 0.$$

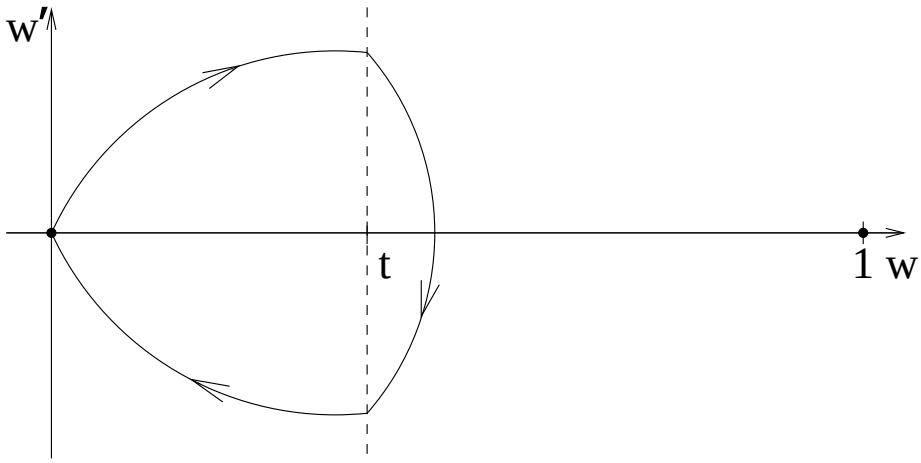

Figure 6: The 'gluing' method.

4. If u is discontinuous at $-z_0$ and z_0 for some $z_0 > 0$ and $J(z) = \frac{1}{2}e^{-|z|}$, $u(0)$ satisfies

$$\begin{aligned} W(u(0)) - W(u(-z_0+)) + W(u(-z_0-)) + \\ \frac{1}{2}f(u(0))^2 - \frac{1}{2}f(u(-z_0+))^2 + \frac{1}{2}f(u(-z_0-))^2 \quad = \quad 0. \end{aligned}$$

Proof. It is easily seen that $J * u(0) - u(0) < 0$, thus $f(u(0)) < 0$ which implies $a < u(0) < 1$.

Next, assume that u is continuous and $u(0) < b$. We can redefine $W(u)$ in such a way that a new $W_r(v) = W(v)$ for $v \leq u(0)$, and W_r is a double well function with the second well (say, at u_r) > 0 (see Figure 7).

Let $f_r = W_r'$. Then u is a homoclinic solution of $J * u - u - f_r(u) = 0$. From [2] we know that

$$u_t = J * u - u - f_r(u) \tag{4.4}$$

has a traveling wave solution $U(x - ct)$ with $U' > 0$, $c > 0$, $U(-\infty) = 0$ and $U(\infty) = u_r$. A stability result in [2] implies that there exists constants $\xi, K, \nu > 0$ such that

$$u(x) < U(x - ct + \xi) + Ke^{-\nu t}.$$

But this inequality can not hold for t large enough, thus we reach a contradiction.

If we now assume that u is continuous and $J(z) = \frac{1}{2}e^{-|z|}$, then if we multiply

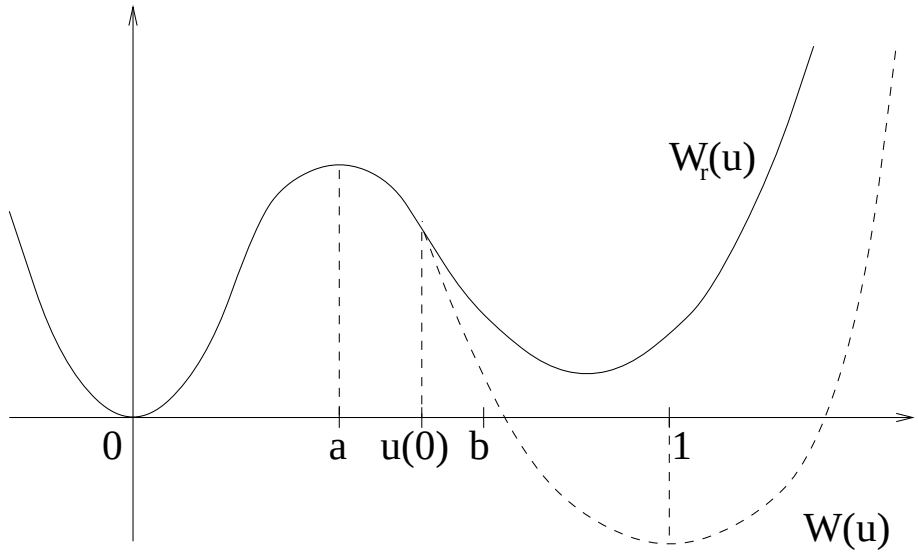

Figure 7: W and W_r.

(4.1) by $(J * u)'$ and integrate over $(-\infty, 0)$, we get

$$\int_{-\infty}^{0} (J * u)'(J * u)'' = \int_{-\infty}^{0} f(u)(u + f(u))'.$$

Since $J * u'(0) = 0$, we have $0 = \int_0^{u(0)} f(u)(1 + f'(u))du = W(u(0)) + \frac{1}{2} f(u(0))^2$. If u is discontinuous at $-z_0$ and z_0, then a similar calculation easily yields

$$W(u(0)) - W(u(-z_0+)) + W(u(-z_0-)) + \frac{1}{2} f(u(0))^2 - \frac{1}{2} f(u(-z_0+))^2$$

$$+ \frac{1}{2} f(u(-z_0-))^2 = 0.$$

□

Remark 4.2 *It is worth recalling here that if u is an even homoclinic solution of the local equation (1.7), then $u(0) = b$, where b is the second zero of W.*

One simply multiplies (1.7) by u' and integrates over $(-\infty, 0)$:

$$0 = \int_{-\infty}^{0} u'' u' = \int_{-\infty}^{0} f(u) u' = W(u(0)).$$

5 Stability of the solution

We discuss the stability of the solution u constructed in Section 2. As opposed to the case of the stationary homoclinic solution of (1.6) where the homoclinic solution is unstable, the stationary homoclinic solution of (1.2), in the discontinuous case, can be stable in $L^\infty(R^1)$. In this section we assume $f \in C^r, r \geq 2$. We start with an instability result.

Theorem 5.1 *Let u be a homoclinic solution constructed in Theorem 2.1. Then if u is continuous, or u is discontinuous at $-z_0$ and z_0 for some $z_0 > 0$ and*

$$\int_{R^1_-} J * (|u'| - u')u' > \tag{5.1}$$

$$\int_{R^1_-} (u(-z_0+) - u(-z_0-))(J(x+z_0) - J(x-z_0))u'(x)$$

then u is unstable in $L^\infty(R^1)$ norm.

Proof. We study the spectrum of the linear operator

$$Lv = J * v - v - f'(u)v.$$

We first consider L as a bounded self-adjoint operator from $L^2(R^1)$ to itself, and show that L has a positive eigenvalue. To this end we consider three cases.

Case 1. $u \in C^m(R^1)$, $m \geq 1$.

Then $Lu' = 0$. It is also true that $u' \in L^2(R^1)$. This is because that

$$\int_{R^1} |u'|^2 dx = 2\int_0^\infty u' du \leq 2 \max_{x \in R^1_+} |u'(x)| \cdot [u(0) - u(\infty)].$$

The second term of the last is clearly finite. To show that $\max_{x \in R^1_+} |u'(x)|$ is finite, we note from $Lu' = 0$ that

$$u'(x) = \frac{J' * u(x)}{1 + f'(u(x))} \to \frac{0}{1 + f'(0)} = 0$$

as $x \to \infty$. This implies that $\max_{x \in R^1_+} |u'(x)|$ is finite. Therefore 0 is an eigenvalue of L in $L^2(R^1)$. An easy calculation gives

$$\langle L|u'|, |u'|\rangle_{L^2} = 2\int_{R^1_-} J * (|u'| - u)u' > 0.$$

Since the largest spectral point λ_1 is characterized by

$$\sup_{\|v\|=1} \langle Lv, v\rangle_{L^2},$$

we deduce $\lambda_1 > 0$.

We now proceed to show that λ_1 is an eigenvalue by proving that the essential spectrum of L is contained in $(-\infty, 0]$. Let $\lambda > 0$. Note that for $v \in L^2(R^1)$,

$$(L-\lambda)v = J * v - (1 + f'(u) + \lambda)v$$

$$= (1+f'(u)+\lambda)\Big[\Big(\frac{1}{1+f'(u)+\lambda} - \frac{1}{1+f'(0)+\lambda}\Big)J*v + \Big(\frac{1}{1+f'(0)+\lambda}J*v - v\Big)\Big]$$

$$\equiv (1 + f'(u) + \lambda)(L_1 v + L_2 v).$$

According to the construction of u, $1 + f'(u(x)) \geq 0$ for all $x \in R^1$. From the fact that

$$\frac{1}{1+f'(u)+\lambda} - \frac{1}{1+f'(0)+\lambda} \to 0 \quad as\ |x| \to \infty,$$

one can prove that for every bounded set $B \subset L^2(R)$ and any $\epsilon > 0$ there is $M_\epsilon > 0$ such that

$$\int_{|x|>M_\epsilon} |L_1\phi|^2 dx < \epsilon \quad for\ all\ \phi \in B$$

and for all $\epsilon > 0$ there is $h_\epsilon > 0$ such that

$$\int_{R^1} |L_1\phi(x+h) - L_1\phi(x)|^2 dx < \epsilon \quad for\ all\ h < h_\epsilon,\ \phi \in B.$$

The compactness criterion of L^p spaces (see for instance [1] Theorem 2.21, p.31) implies that $L_1(B)$ is precompact in $L^2(R)$, so L_1 is a compact operator. Since $1 + f'(0) + \lambda > 1$, L_2 is an invertible operator. Thus $L_1 + L_2$ is a Fredholm operator of index 0. Then $L - \lambda$, as the product of the invertible operator $1 + f'(u) + \lambda$ and the Fredholm operator $L_1 + L_2$, is also Fredholm of index 0. Therefore λ is not in the essential spectrum of L, and the essential spectrum is contained in $(-\infty, 0]$.

Case 2. u is in $C^0(R^1)$ but not in $C^m(R^1), m \geq 1$.

According to Section 3, $u \in C^m((-\infty, -z_0) \cup (-z_0, z_0) \cup (z_0, \infty)), m \geq 1$ for some $z_0 \geq 0$. Let u' denote the derivative of u in $(-\infty, -z_0) \cup (-z_0, z_0) \cup (z_0, \infty)$ and arbitrarily defined at $-z_0$ and z_0. Define, for $\epsilon > 0$,

$$w_\epsilon(x) = \begin{cases} |u'(x)|,\ x \in R^1 \backslash \{(-z_0-\epsilon, -z_0+\epsilon) \cup (z_0-\epsilon, z_0+\epsilon)\} \\ 0,\ x \in (-z_0-\epsilon, -z_0+\epsilon) \cup (z_0-\epsilon, z_0+\epsilon) \end{cases}.$$

It is easy to see that

$$\langle Lw_\epsilon, w_\epsilon\rangle \to 2\int_{R^1_-} J*(|u'|-u')u' > 0$$

as $\epsilon \to 0$. Then λ_1, the largest spectral point, is positive. Following the argument in case 1, one can show that λ_1 is an eigenvalue.

Case 3. u is discontinuous at $-z_0$ and z_0.

Again let u' be the derivative of u away from $-z_0$ and z_0, and be arbitrarily defined at $-z_0$ and z_0. Then a straightforward computation yields

$$\langle L|u'|, |u'|\rangle = 2\int_{R^1_-} (J*(|u'|-u')u'$$

$$-2\int_{R^1_-} (u(-z_0+)-u(-z_0-))(J(x+z_0)-J(x-z_0))u'(x)dx. \tag{5.2}$$

Under the condition (5.1) of this theorem, the right side of (5.2) is positive, so the largest spectral point is positive. Again the same argument as in Case 1. shows that this spectral point is an eigenvalue.

Remark 5.2 *The condition (5.1) holds if either z_0 or $u(-z_0+)-u(-z_0-)$ is sufficiently small.*

Let ϕ_1 be an eigenfunction, as guaranteed in Case 1-3, corresponding to λ_1. Note that ϕ_1 is of one sign. This is because that if ϕ_1 changes sign, then $\langle L|\phi_1|, |\phi_1|\rangle > \langle L\phi_1, \phi_1\rangle$ which violates the characterization of λ_1. Indeed $|\phi_1| > 0$ almost everywhere. At a point where $|\phi_1(x)| = 0$ we have $(J*|\phi_1|-|\phi_1|)(x) > 0$ but $(1+f'(u(x))+\lambda_1)|\phi_1(x)| = 0$. The equation $(L-\lambda_1)|\phi_1| = 0$ is violated there. Therefore the set of points where $\phi_1 = 0$ must have measure 0.

It is also clear, by writing

$$\phi_1 = \frac{J*\phi_1}{1+f'(u)+\lambda_1},$$

that ϕ_1 is an eigenfunction of L as an operator from $L^\infty(R^1)$ to itself.

We are now ready to apply the spectral information so far obtained to the stability question. Let $\lambda_1 > 0$, and $\phi_1 \in L^\infty(R^1)$ be such that $L\phi_1 = \lambda_1\phi_1$ and assume $\phi_1 > 0$.

For some positive number β to be chosen later, let

$$\underline{v}(x,t) = u(x) + \epsilon\phi_1(x)e^{\beta t}.$$

Define $Nv = v_t - (J*v - v - f(v))$. Then it is easily seen that

$$N\underline{v}(x,t) = \epsilon\phi_1(x)e^{\beta t}(\beta-\lambda_1)$$

$$+[f(u(x)+\epsilon\phi_1(x)e^{\beta t})-f(u(x)-\epsilon\phi_1(x)e^{\beta t}f'(u(x))].$$

Let us call the quantity in the brackets R. Then by Taylor expansion, we have

$$|R| \le C\phi_1(x)(\epsilon e^{\beta t})^2$$

for some $C>0$. Choose $\beta=\frac{1}{2}\lambda_1$ (for instance), and so get

$$N\underline{v}(x,t) \le \epsilon\phi_1(x)e^{\beta t}\big(-\frac{1}{2}\lambda_1+C\epsilon e^{\beta t}\big).$$

Thus $N\underline{v}\le 0$ provided that

$$2C\epsilon e^{\beta t} \le \lambda_1 \tag{5.3}$$

and $\underline{v}$ is a subsolution of the initial value problem (1.2) up till time t_0, at which (5.3) holds with the equal sign. It is easily seen that the t-independent function $\underline{v}(x,t_0)$ satisfies:

$$N\underline{v}(x,t_0) \le \epsilon\phi_1(x)e^{\beta t_0}(-\lambda_1+C\epsilon e^{\beta t_0})<0$$

so $\underline{v}(x,t_0)$ is also a subsolution. By the comparison principle for (1.2), which is similar to the usual comparison principle for (1.6), this means that for all small $\epsilon>0$, the solution $u(x,t)$ of the initial value problem (1.2) with $u_0(x)\ge u(x)+\epsilon\phi_1(x),\ x\in R^1$, can never get closer to $u(x)$ than $\epsilon\phi_1(x)e^{\beta t_0}$, which by (5.3) with equal sign can be written as $\frac{\lambda_1}{2C}\phi_1(x)$. Since the last expression is independent of ϵ, u is unstable in $L^\infty(R^1)$ norm. □

Remark 5.3 *If λ_1 is an eigenvalue, $\lambda_1>0$ and $\phi_1>0$, an easy computation shows that*

$$E(u+\epsilon\phi_1)=E(u)-\frac{1}{2}\epsilon^2\lambda_1\int_{R^1}{\phi_1}^2+O(\epsilon^3),$$

thus

$$E(u+\epsilon\phi_1)<E(u)$$

for small $\epsilon\ne 0$ and u is also unstable in a variational sense.

We now state a stability result.

Theorem 5.4 *Let u be a homoclinic solution of (1.1). Then if u is discontinuous at $-z_0$ and z_0 for some $z_0>0$ and $f'(u(x))>0$ for all $x\in R^1$ then u is locally, asymptotically exponentially stable in $L^\infty(R^1)$ norm.*

Proof. The proof is similar in spirit to the proof of Theorem 5.1. Namely, for some positive δ and α to be chosen later, define

$$\overline{u}(x,t)=u(x)+\delta e^{-\alpha t}.$$

A similar calculation as above shows that for some constant $C > 0$

$$\begin{aligned} N\overline{u}(x,t) &\geq -\delta\alpha e^{-\alpha t} + f(u(x) + \delta e^{-\alpha t}) - f(u(x)) \\ &\geq \delta e^{-\alpha t}(-\alpha + f'(u(x)) - C\delta e^{-\alpha t}). \end{aligned}$$

By our assumption $f'(u(x)) > 0$ for all $x \in R^1$, we can now choose small positive δ and α such that

$$N\overline{u}(x,t) \geq 0,$$

thus $\overline{u}$ is a supersolution for all $t > 0$.

Similarly,

$$\underline{u}(x,t) = u(x) - \delta e^{-\alpha t}$$

is a subsolution for positive δ and α chosen as before.

Now, consider the initial value problem (1.2) with initial data u_0 such that

$$u(x) - \delta \leq u_0(x) \leq u(x) + \delta.$$

The comparison principle for (1.2) easily implies that the solution $u(x,t)$ of (1.2) is sandwiched between the subsolution $\underline{u}(x,t)$ and supersolution $\overline{u}(x,t)$:

$$u(x) \leq \liminf_{t\to\infty} u(x,t) \leq \limsup_{t\to\infty} u(x,t) \leq u(x).$$

Thus u is locally, asymptotically, exponentially stable in $L^\infty(R^1)$ norm. □

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