Asymptotic Behavior of Energy Solutions to a Two Dimensional Semilinear Problem with Mixed Boundary Condition

Xiaofeng Ren
School of Mathematics, University of Minnesota
Minneapolis, MN 55455

Juncheng Wei
School of Mathematics, University of Minnesota
Minneapolis, MN 55455
Key words and phrases: Mixed boundary value problem, energy solution, asymptotic behavior.

1 Introduction

This work is concerned with the asymptotic behavior of the energy solutions of the mixed boundary value problem

$$\begin{cases} \Delta u + u^p = 0 \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1 \end{cases} \tag{1.1}$$

where

- Ω is a $C^{0,1}$ and bounded domain in R^2,
- $\partial\Omega$ consists of two pieces Γ_0 and Γ_1, where the 1-dimensional Hausdorff measure of Γ_0 is greater than 0,
- Γ_0 is smooth and Γ_1 is piecewise smooth,
- Γ_0 and Γ_1 are relatively closed in $\partial\Omega$,
- ν is the unit outer normal of Ω,
- p is a large parameter.

In this work, we shall only consider the least energy solutions, although the method can be used to study other solutions with the same decay rate of energies. Let

$$\mathcal{A}_p = \{v \in W^{1,2}(\Omega) : \ v = 0 \text{ on } \Gamma_0, \ \|v\|_L^{p+1}(\Omega) = 1\}$$

be the admissible set. Define the energy

$$J_p(v) := \int_\Omega |\nabla v|^2 dx$$

on the admissible set $\mathcal{A}_p$. Standard argument shows that for any $p > 1$ J_p is bounded from below and the infimum is obtained by a function u'_p in $\mathcal{A}_p$. By the inhomogeneity of (1.1) we know that a positive multiple of u'_p solves (1.1). Throughout the rest of this paper we denote such least energy solutions by u_p.

Our goal here is to understand the asymptotic behavior of u_p as p, serving as a parameter, approaches ∞. It is known in [10] that for the pure Dirichlet problem, i.e. $\Gamma_1 = \emptyset$, the solutions u_p develop single or double bounded peaks in the interior of Ω as $p \to \infty$. In the current mixed problem, we shall see peaks on the Neumann boundary Γ_1 and show that u_p can develop no more than either one interior peak or two boundary peaks on Γ_1. We start to investigate c_p where

$$c_p := \inf\{[\int_\Omega |\nabla u|^2 dx]^{1/2} : \ u \in \mathcal{A}_p\}. \tag{1.2}$$

According to the construction of least energy solution u_p,

$$c_p^2 = \frac{\int_\Omega |\nabla u_p|^2 dx}{[\int_\Omega u_p^{p+1} dx]^{2/(p+1)}}, \tag{1.3}$$

and c_p^{-1} is the optimal constant of the Sobolev embedding

$$V(\Gamma_1, \Omega) \hookrightarrow L^{p+1}(\Omega)$$

where $V(\Gamma_1, \Omega) = \{v \in W^{1,2}(\Omega) : \ v = 0 \text{ on } \Gamma_0\}$ is a Hilbert space equipped with the inner product

$$< u, v >= \int_\Omega < \nabla u, \nabla v > dx.$$

We shall see that c_p possesses nice decay property as $p \to \infty$. Next we extend some L^1 estimates of H. Brezis and F. Merle for Δ with Dirichlet boundary condition in R^2 to mixed boundary condition. After these preparations we shall prove

Theorem 1.1 *There exist C_1, C_2, independent of p, such that*

$$0 < C_1 < \|u_p\|_{L^\infty} < C_2 < \infty$$

for large p. Indeed

$$1 \le \liminf_{p\to\infty} \|u_p\|_{L^\infty(\Omega)} \le \limsup_{p\to\infty} \|u_p\|_{L^\infty(\Omega)} \le \exp \frac{1+\alpha_0}{2}$$

where α_0, defined later in (4.4) section 4, is a constant depending on the pair (Γ_1, Ω) only.

To state our second result, we need a few definitions. Let

$$v_p = \frac{u_p}{\int_\Omega u_p^p}. \tag{1.4}$$

For a sequence $\{u_{p_n}\}$ of $\{u_p\}$ with $p_n \to \infty$ as $n \to \infty$, we define the blow-up set S to be the subset of $\overline{\Omega}$ such that $x \in S$ if there exist a subsequence, still denoted by $\{p_n\}$, and a sequence x_n in Ω with

$$v_{p_n}(x_n) \to \infty \text{ and } x_n \to x. \tag{1.5}$$

Define

$$\begin{aligned} &S_I = S \cap \Omega, \\ &S_C = S \cap (\Gamma_0 \cap \Gamma_1), \\ &S_D = S \cap (\Gamma_0 \backslash (\Gamma_0 \cap \Gamma_1)), \\ &S_N = S \cap (\Gamma_1 \backslash (\Gamma_0 \cap \Gamma_1)). \end{aligned} \tag{1.6}$$

So every blow-up point must fall in one and only one of the above 4 classes. We shall see later that S contains the set of peaks of sequence $\{u_{p_n}\}$. By a peak $P \in \overline{\Omega}$ we mean that $\{u_{p_n}\}$ doesn't vanish in L^∞ norm in any small neighborhood of P. Theorem 1.1 in particular implies that the set of peaks of $\{u_p\}$ is not empty. In this paper we are mainly concerned with S_I and S_N. We will use $\#S_I$ ($\#S_N$) to denote the cardinality of S_I (S_I respectively). Our second result says

Theorem 1.2 *For a domain Ω with the properties stated in the beginning of this article, we have*

1. $$S_D = \emptyset, \ \#(S_I \cup S_C \cup S_N) \ge 1;$$

2. $$\#S_I + \frac{1}{2}\#S_N \le 1$$

   if Γ_1 is smooth;

3. $$S_I = \emptyset, \text{ and } \#S_N = 1$$

   if Γ_1 has convex corners; furthermore in this case if x_0 is the point in S_N, x_0 must be a corner point with the least angle among all the corners on Γ_1.

Here by a convex corner, we mean a corner having angle less than π.

We shall also see that under extra condition of Ω, Γ_0 and Γ_1, u_p can develop only one peak on the Neumann boundary Γ_1. We would like to point out that like what we did in [11], most of our results can be extended to higher dimensions with Δ replaced by Δ_N, the N-Laplacian operator ($\Delta_N u = \mathrm{div}(|\nabla u|^{N-2}\nabla u)$), in (1.1) if Ω is a domain in R^N. However, we don't know any thing about S_C if $\Gamma_0 \cap \Gamma_1$ is non-empty.

Our paper is organized as follows. In section 2, we give some background materials for the mixed boundary value problem. Then in section 3, we prove the decay rate of c_p. We prove theorem 1.1 in section 4. In section 5, we present some L^1 estimates. Section 6 is devoted to the proof of theorem 1.2. Finally we consider some special domains and some examples in section 7.

2 Preliminaries

Let Ω be a domain in R^2 with conditions stated in the beginning of this article. Let Γ_0 and Γ_1 be two parts of the boundary of Ω with Γ_0 having positive one dimensional Hausdorff measure. We recall that *the isoperimetric constant of* Ω *relative to* Γ_1, $Q(\Gamma_1, \Omega)$, is defined to be

$$Q(\Gamma_1, \Omega) = \sup \frac{|E|^{1/2}}{P_\Omega(E)} \tag{2.1}$$

where the supremum is taken over all measurable sets of Ω such that $\partial E \cap \Gamma_0$ has 1-dimensional Hausdorff measure 0, and $P_\Omega(E)$ denotes the De Giorgi perimeter of E relative to Ω, i.e.

$$P_\Omega(E) = \sup\{|\int_E \mathrm{div}\psi dx| : \ \psi \in [C_0^\infty(\Omega)]^2, \ |\psi| \le 1\}. \tag{2.2}$$

Some properties of $P_\Omega(E)$ are stated in [8] and [6]. We also refer to [14] and [3] for more information about De Giorgi perimeter and isoperimetric inequalities. In particular we notice that

$$Q(\Gamma_1, \Omega) \ge (2\pi^{1/2})^{-1}$$

where the second is the *absolute* isoperimetric constant; and if $H^1(\Gamma_1) > 0$,

$$Q(\Gamma_1, \Omega) \ge (2\pi/2)^{-1/2}.$$

From here we deduce that if $H^1(\Gamma_1) > 0$ and $Q(\Gamma_1, \Omega) < \infty$, there exists $\alpha \in [0, \pi]$ such that $Q(\Gamma_1, \Omega) = (\sqrt{2\alpha})^{-1}$ where α is the angle of the unitary sector $\Sigma(\alpha, 1) = \{x = (r, \theta) \in R^2 : \ 0 \le r \le 1, \ \theta \in [0, \alpha]\}$. We denote by $\mathcal{E}_\alpha$ the class of all pairs (Γ_1, Ω) of the type considered above such that

$$Q(\Gamma_1, \Omega) = (\sqrt{2\alpha})^{-1}. \tag{2.3}$$

By virtue of an isoperimetric inequality described in [6], any pair of a convex sector and its non-circular boundary $(\Gamma_1, \Sigma(\alpha, 1))$ belongs to $\mathcal{E}_\alpha$ once we denote by Γ_0 the circular part of $\Sigma(\alpha, 1)$. Therefore

$$Q(\Gamma_1, \Sigma(\alpha, 1)) = (\sqrt{2\alpha})^{-1}$$

if $\Sigma(\alpha, 1)$ is a convex sector. By the way, if $(\Gamma_1, \Omega) \in \mathcal{E}_\alpha$ and β is the smallest angle among all convex corners on Γ_1,

$$\beta \geq \alpha. \tag{2.4}$$

Recall $V(\Gamma_1, \Omega)$ the Hilbert space defined in section 1. Assuming $(\Gamma_1, \Omega) \in \mathcal{E}_\alpha$ for some $\alpha \in [0, \pi]$, we have the following two dimensional Moser type embedding while the proof of this result in any dimension can be found in [6]. Also see [7].

Proposition 2.1 *There exists a universal constant C such that*

$$\int_\Omega \exp[\frac{(2\alpha)|u|^2}{\|\nabla u\|^2_{L^2(\Omega)}}] \leq C|\Omega|$$

for any $u \in V(\Gamma_1, \Omega)$ with $(\Gamma_1, \Omega) \in \mathcal{E}_\alpha$.

We also need some results concerning the relative isoperimetric constants near the boundary Γ_1. Let us fix our notation first. For each smooth point $x \in \Gamma_1$, we can associate a smooth flattening map Φ_x in a neighborhood of x that maps the neighborhood of x to a neighborhood of $(0, 0)$ in

$$\{y \in R^2 : \; y = (y_1, y_2), \; y_2 > 0\}$$

and maps Γ near x to

$$\{y \in R^2 : \; y = (y_1, y_2), \; y_2 = 0\}$$

near $(0, 0)$. For a corner point x on Γ_1 we associate a similar map Φ_x in a neighborhood of x that maps the neighborhood of x to a neighborhood of $(0, 0)$ in

$$\{y \in R^2 : \; y = (\rho\cos\theta, \rho\sin\theta), \; 0 \leq \theta \leq \beta\}$$

where β is the angle of corner at x and that maps the boundary near x to the boundary near $(0, 0)$. We further require that $D\Phi_x = I$ at x, and Φ_x varies smoothly with respect to x. From now on throughout the rest of this paper, for any x on Γ_1, by a ball $B_r(x_0)$, we mean $\Phi_x^{-1}(B_r(0, 0))$. Clearly it is well-defined if r is small. We can now state the following result concerning the asymptotic behavior of the relative isoperimetric constants and the quantities α defined in (2.3) of $(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0))$.

Proposition 2.2 *1. Let $x_0 \in \Gamma_2$ such that Γ_2 is smooth near x_0. Then as $r \to 0$,*

$$Q(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \frac{1}{\sqrt{2\pi}},$$

i.e.

$$\alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \pi$$

where $\alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0))$ is the angle of the unit sector whose relative isoperimetric constant is the same as the one of $(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0))$.

2. Let $x_0 \in \Gamma_2$ such that x_0 is the vertex of a convex corner with angle β_0 in Γ_2. Then as $r \to 0$,

$$Q(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \frac{1}{\sqrt{2\beta_0}}$$

i.e.

$$\alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \beta_0.$$

To prove, one just invokes the variable change formula in standard integration theory to compare the relative isoperimetric constants above with the relative isoperimetric constants of sectors computed in [6]. We leave the details of this argument to reader.

3 Some Estimates for c_p

Recall c_p defined in (1.2). We have the following refined Sobolev embedding.

Lemma 3.1 *For every $t \geq 2$ there is D_t such that*

$$\|u\|_{L^t} \leq D_t t^{1/2} \|\nabla u\|_{L^2}$$

for all $u \in V(\Gamma_1, \Omega)$ with $(\Gamma_1, \Omega) \in \mathcal{E}_\alpha$; furthermore

$$\lim_{t \to \infty} D_t = (4\alpha e)^{-1/2}.$$

Proof. Let $u \in V(\Gamma_1, \Omega)$. We know

$$\frac{1}{\Gamma(s+1)} x^s \leq e^x$$

for all $x \geq 0$, $s \geq 0$ where Γ is the Γ function. Using proposition 2.1, we have

$$\int_\Omega \exp[2\alpha(\frac{u}{\|\nabla u\|_{L^2}})^2] dx \leq C|\Omega|$$

where C doesn't depend on any thing and $|\Omega|$ is the Lebesgue measure of Ω. Therefore

$$\frac{1}{\Gamma(\frac{t}{2}+1)}\int_\Omega u^t dx$$

$$= \frac{1}{\Gamma(\frac{t}{2}+1)}\int_\Omega [2\alpha(\frac{u}{\|\nabla u\|_{L^2}})^2]^{t/2} dx (2\alpha)^{-t/2}\|\nabla u\|_{L^2}^t$$

$$\leq \int_\Omega \exp[2\alpha(\frac{u}{\|\nabla u\|_{L^2}})^2] dx (2\alpha)^{-t/2}\|\nabla u\|_{L^2}^t$$

$$\leq C|\Omega|(2\alpha)^{-t/2}\|\nabla u\|_{L^2}^t$$

Hence

$$(\int_\Omega u^t dx)^{1/t} \leq (\Gamma(\frac{t}{2}+1))^{1/t} C^{1/t}(2\alpha)^{-1/2}|\Omega|^{1/t}\|\nabla u\|_{L^2(\Omega)}$$

Notice that, according to Stirling's formula,

$$(\Gamma(\frac{t}{2}+1))^{1/t} \sim ((\frac{\frac{t}{2}}{e})^{t/2}\sqrt{te}e^{\theta_t})^{1/t} \sim (\frac{1}{2e})^{1/2}t^{1/2}$$

where $0 < \theta_t < \frac{1}{12}$. Choosing D_t to be

$$(\Gamma(\frac{t}{2}+1))^{1/t} C^{1/t}(2\alpha)^{-1/2}|\Omega|^{1/t} t^{-1/2}$$

we get the desired result. □

An immediate consequence is

Corollary 3.2

$$\liminf_{p\to\infty} p^{1/2}c_p \geq (4\alpha e)^{1/2}.$$

Next we prove an upper bound for $p^{1/2}c_p$.

Lemma 3.3 *For domains Ω with smooth Γ_1*

$$\limsup_{p\to\infty} p^{1/2}c_p \leq (4\pi e)^{1/2};$$

if the domain Ω has convex corners on Γ_1,

$$\limsup_{p\to\infty} p^{1/2}c_p \leq (4\beta e)^{1/2}$$

where β is the smallest angle among all convex corners on Γ_1.

Proof. Let us first assume that Ω contains $\{(x_1, x_2) : \; x_2 > 0, \; x_1^2 + x_2^2 \leq L\}$ with $\{(x_1, x_2) : \; x_2 = 0, \; x_1^2 + x_2^2 \leq L\}$ being part of the Neumann boundary. We construct a Moser type test function near $(0, 0)$. Letting

$$m_l(x) = \frac{1}{\sqrt{\pi}} \begin{cases} (\log L - \log l)^{1/2}, \; 0 \leq |x| \leq l \\ \dfrac{\log l - \log |x|}{[\log L - \log l]^{1/2}}, \; l \leq |x| \leq L \\ 0, \; |x| \geq L, \end{cases} \tag{3.1}$$

we have $m_l \in V(\Gamma_1, \Omega)$, $\|\nabla m_l\|_{L^2(\Omega)} = 1$ and

$$\int_\Omega m_l^{p+1}(x) dx$$

$$= [\frac{1}{\sqrt{\pi}} (\log \frac{L}{l})^{1/2}]^{p+1} |B_l|$$

$$+ [\frac{1}{\sqrt{2\pi}} (\log \frac{L}{l})^{-1/2}]^{p+1} \int_{l<|x|<L} (\log \frac{L}{|x|})^{p+1} dx$$

$$:= I_1 + I_2$$

where

$$I_1 = [\frac{1}{\sqrt{\pi}} (\log \frac{L}{l})^{1/2}]^{p+1} \pi l^2$$

$$I_2 = [\frac{1}{\sqrt{\pi}} (\log \frac{L}{l})^{-1/2}]^{p+1} \int_{l<|x|<L} (\log \frac{L}{|x|})^{p+1} dx.$$

Choosing $l = Le^{-(p+1)/4}$, we have

$$\|m_l\|_{L^{p+1}} \geq I_1^{1/(p+1)}$$

$$\geq [\frac{1}{4\pi e}]^{1/2} (p+1)^{1/2} (\pi L^2)^{1/(p+1)}.$$

Hence

$$c_p \leq [4\pi e]^{1/2} (p+1)^{-1/2} (\pi L^2)^{-1/(p+1)},$$

i.e.

$$\limsup_{p \to \infty} p^{1/2} c_p \leq (4\pi e)^{1/2}.$$

For a domain Ω with smooth Γ_1, we can first flatten the boundary and construct the same test function with small L. Sending L to 0, we still get the desired result.

If the domain Ω has a corner on Γ_1, we can first transform it to a sector by a smooth map. Then we construct a similar test function on that sector. Finally we let L tend to 0. □

Corollary 3.4 *1. For domains Ω with smooth Γ_1,*

$$\limsup_{p\to\infty} p\int_\Omega u_p^{p+1} \le (4\pi e) \text{ and } \limsup_{p\to\infty} p\int_\Omega |\nabla u_p|^2 \le 4\pi e.$$

2. For domains Ω having convex corners on Γ_1,

$$\limsup_{p\to\infty} p\int_\Omega u_p^{p+1} \le 4\beta e \text{ and } \limsup_{p\to\infty} p\int_\Omega |\nabla u_p|^2 \le 4\beta e$$

where β is the smallest angle among all convex corners on Γ_1.

Proof. From (1.3), we know

$$c_p = \frac{\|\nabla u_p\|_{L^2(\Omega)}}{\|u_p\|_{L^{p+1}(\Omega)}}.$$

If we multiply (1.1) by u_p and integrate by parts, we have

$$\int_\Omega |\nabla u_p|^2 = \int_\Omega u_p^{p+1}.$$

Therefore

$$\int_\Omega u_p^{p+1} = c_p^{\frac{2(p+1)}{p-1}} \text{ and } \int_\Omega |\nabla u_p|^2 = c_p^{\frac{2(p+1)}{p-1}}.$$

The results follow immediately from lemma 3.3. □

As another consequence of lemma 3.3, we prove a crucial estimate for quantity

$$L_0 = \limsup_{p\to\infty} \frac{p\int_\Omega u_p^p}{e}. \tag{3.2}$$

The proof follows easily from lemma 3.3 and the Holder's inequality.

Corollary 3.5 *1. For domains Ω with smooth Γ_1*

$$L_0 \le 4\pi;$$

2. For domains Ω having convex corners on Γ_1,

$$L_0 \le 4\beta$$

where β is the smallest angle among all convex corners on Γ_1.

4 Proof of Theorem 1.1

A uniform lower bound indeed exists for *any* positive solutions to (1.1). Let λ_1 be the first eigenvalue of $-\Delta$ with the same boundary condition as the one in (1.1) and φ be a corresponding positive eigenfunction. Then for any solution u

$$\int_\Omega [u\Delta\varphi - \varphi\Delta u] = \int_{\partial\Omega} [u\frac{\partial\varphi}{\partial\nu} - \varphi\frac{\partial u}{\partial\nu}] = 0. \tag{4.1}$$

Therefore

$$\int_\Omega (u^p - \lambda_1 u)\varphi = 0.$$

Hence

$$\|u\|_{L^\infty(\Omega)} \geq \lambda_1^{1/(p-1)} \to 1 \tag{4.2}$$

as $p \to \infty$ which yields a uniform lower bound in p for $\|u\|_{L^\infty(\Omega)}$ when $p > 1+\epsilon$, $\epsilon > 0$.

To get an upper bound for $\{u_p\}$, we use an iteration argument. Define

$$\gamma_0 = \beta/\alpha \tag{4.3}$$

where β is the smallest angle among all convex corners on Γ_1 and (Γ_1, Ω) is in class $\mathcal{E}_\alpha$. Then $\gamma_0 \geq 1$ by (2.4). Let α_0 be such that

$$\exp\alpha_0 = \gamma_0(1+\alpha_0). \tag{4.4}$$

Fix t and ϵ that will be chosen later. Letting $\nu = (1+t)(p+1)$, from lemma 3.1, we have

$$[\int_\Omega u_p^\nu]^{1/\nu} \leq (4\alpha e)^{-1/2} E_{(1+t)(p+1)} \nu^{1/2} \|\nabla u_p\|_{L^2(\Omega)}$$

where

$$\lim_{p\to\infty} E_{(1+t)(p+1)} = 1.$$

But from corollary 3.4 we know

$$\limsup_{p\to\infty} p \int_\Omega |\nabla u_p|^2 \leq 4\beta e.$$

Hence there exists P_0 such that for all $p > P_0$,

$$\int_\Omega u_p^\nu \leq [\gamma_0(1+t+\epsilon)]^{\nu/2}. \tag{4.5}$$

Multiplying (1.1) both side by u_p^{2s-1}, we get, after integrating by parts,

$$\frac{2s-1}{s^2}\int_\Omega |\nabla u_p^s|^2 = \int_\Omega u_p^{p-1+2s}. \tag{4.6}$$

Using lemma 3.1 again, we have

$$[\int_\Omega u_p^{\nu s}]^{1/\nu} \le D_{\nu s}\nu^{1/2}\|\nabla u_p^s\|_{L^2(\Omega)};$$

$$\begin{aligned}[\int_\Omega u_p^{\nu s}]^{2/\nu} &\le C_0\nu\frac{s^2}{2s-1}\int_\Omega u_p^{p-1+2s}\\ &\le C_1\nu s\int_\Omega u_p^{p-1+2s}\end{aligned}$$

where $D_{\nu s}$ is defined in lemma 3.1 and C_0 and C_1 are constant independent of $p > P_0$. Hence we have

$$[\int_\Omega u_p^{\nu s}]^{2/\nu} \le C_1\nu s\int_\Omega u_p^{p-1+2s}. \tag{4.7}$$

We now define two sequences $\{s_j\}$ and $\{M_j\}$ by

$$\begin{cases} p-1+2s_0=\nu \\ p-1+2s_{j+1}=\nu s_j \\ M_0=[\gamma_0(1+t+\epsilon)]^{\nu/2} \\ M_{j+1}=[C_1\nu s_j M_j]^{\nu/2} \end{cases} \tag{4.8}$$

where C_1 is the constant in (4.7). From (4.5) and (4.7), we have, by induction,

$$\int_\Omega u_p^{\nu s_{j-1}} \le M_j. \tag{4.9}$$

Next we claim

$$M_j \le \exp[m(\gamma_0,t,p,\epsilon)\nu s_{j-1}] \tag{4.10}$$

where $m(\gamma_0,t,p,\epsilon)$ is a constant depending on γ_0, t, p, ϵ and

$$\lim_{p\to\infty} m(\gamma_0,t,p,\epsilon) = \frac{1+t}{2t}\log[\gamma_0(1+t+\epsilon)].$$

In fact, we can write down $\{s_j\}$ explicitly.

$$s_j = \frac{1}{\nu-2}\{(\frac{\nu}{2})^{j+1}(\nu-1-p-1)+p-1\}. \tag{4.11}$$

Put

$$\sigma_j = \frac{\nu}{2}\log(C_1\nu s_j),\ \ \mu_j=\log M_j.$$

Hence

$$\mu_{j+1} = \frac{\nu\mu_j}{2}+\sigma_j.$$

Therefore it is easy to see

$$\sigma_j = \frac{\nu}{2}(\log[\frac{C_1\nu}{\nu-2}] + \log[(\frac{\nu}{2})^{j+1}(\nu-p-1)+p-1])$$
$$\leq [\nu\log\sqrt{2C_1}\nu](j+1).$$

Now we define $\{\tau_j\}$ by

$$\begin{aligned} &\tau_0 = \mu_0 \\ &\tau_{j+1} = \tfrac{1}{2}\nu\tau_j + (\nu\log\sqrt{2C_1}\nu)(j+1). \end{aligned} \tag{4.12}$$

Clearly $\mu_j \leq \tau_j$. Moreover we have

$$\tau_j = (\frac{\nu}{2})^j[\mu_0 + 2\nu\log(\sqrt{2C_1}\nu)\frac{\nu}{(\nu-2)^2}] - \frac{2}{\nu-2}(\nu\log(\sqrt{2C_1}\nu)(j+\frac{\nu}{\nu-2})$$
$$\leq \frac{\mu_0 + 2\nu\log(\sqrt{2C_1}\nu)\frac{\nu}{(\nu-2)^2}}{(\nu-2)^{-1}(\nu-p-1)}s_{j-1}$$
$$\leq \frac{\mu_0 + 2\nu\log(\sqrt{2C_1}\nu)\frac{\nu}{(\nu-1)^2}}{\nu-p-1}\frac{\nu-2}{\nu}\nu s_{j-1}$$
$$:= m(\gamma_0, t, p, \epsilon)\nu s_{j-1}$$

where

$$\lim_{p\to\infty} m(\gamma_0, t, p, \epsilon) = \frac{1+t}{2t}\log[\gamma_0(1+t+\epsilon)].$$

Therefore we get

$$\|u_p\|_{L^{\nu s_{j-1}}(\Omega)} \leq \exp[m(\gamma_0, t, p, \epsilon)].$$

Sending $j \to \infty$, we see

$$\|u_p\|_{L^\infty(\Omega)} \leq \exp[m(\gamma_0, t, p, \epsilon)].$$

Sending $p \to \infty$, we have

$$\limsup_{p\to\infty} \|u_p\|_{L^\infty} \leq [\gamma_0(1+t+\epsilon)]^{\frac{1+t}{2t}}.$$

Sending $\epsilon \to 0$, we deduce

$$\limsup_{p\to\infty} \|u_p\|_{L^\infty} \leq [\gamma_0(1+t)]^{\frac{1+t}{2t}}.$$

If we let $f(t) = [\gamma_0(1+t)]^{\frac{1+t}{2t}}$, standard calculus argument shows that $\log f(t)$ achieves its minimum at α_0 where

$$\alpha_0 = \log[\gamma_0(1+\alpha_0)].$$

defined in (4.4). So we obtain

$$\limsup_{p\to\infty} \|u_p\|_{L^\infty} \leq \exp \frac{1+\alpha_0}{2}.$$

□

We include a consequence of theorem 1.1 here which will be used later.

Corollary 4.1 *There exist C_1 and C_2 such that*

$$\frac{C_1}{p} \leq \int_\Omega u_p^p \leq \frac{C_2}{p}$$

Proof. The first inequality follows from theorem 1.1 and the first limit of corollary 2.3; the second inequality follows from the first limit of corollary 2.3 through an interpolation argument. □

5 Some Apriori Estimates

In this section we collect some less well-known estimates for Δ on two dimensional domains.

We first state a boundary estimate lemma. The proof of the lemma is standard. One combines the moving plane method in [4] with a Kelvin transform. We refer to [2] and [4] for details. This lemma actually excludes the possibility that u_p develop a peak on Γ_0. See Remark 6.5.

Lemma 5.1 *Let u be a positive solution of*

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega \subset R^2 \\ u|_{\Gamma_0} = 0 \end{cases}$$

where Γ_0 is a smooth piece of $\partial\Omega$ and f is a smooth function. Then for every $\Gamma \subset\subset$ int (Γ_0) with respect to the relative topology of $\partial\Omega$ there exist a neighborhood ω of Γ and a constant C both depending on the geometry of Ω and Γ only such that

$$\|u\|_{L^\infty(\omega)} \leq C\|u\|_{L^1(\Omega)}.$$

Next we state an L^1 estimate of H. Brezis and F. Merle, theorem 1 [1].

Lemma 5.2 *Let u be a solution of*

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where Ω is a smooth bounded domain in R^2. We have for $0 < \epsilon < 4\pi$

$$\int_\Omega \exp\left[\frac{(4\pi - \epsilon)|u(x)|}{\|f\|_{L^1}}\right]dx \leq \frac{4\pi \text{Area}(\Omega)}{\epsilon}.$$

Remark 5.3 *In their paper, Brezis and Merle used* $(\text{Diameter}(\Omega))^2$ *instead of* $\text{Area}(\Omega)$ *in lemma 5.2. It turns out from the following symmetrization approach that* $\text{Area}(\Omega)$ *is more appropriate.*

We need a similar L^1 estimate as above to take care of the mix boundary condition.

Lemma 5.4 *Let* u *be a solution of*

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u|_{\Gamma_0} = 0 \\ \dfrac{\partial u}{\partial \nu}|_{\Gamma_1} = 0 \end{cases}$$

where the boundary condition is the same as the one in (1.1) and $(\Gamma_1, \Omega) \in \mathcal{E}_\alpha$. *Then for every* $0 < \epsilon < 2\alpha$,

$$\int_\Omega \exp[\frac{(2\alpha - \epsilon)|u(x)|}{\|f\|_{L^1}}]dx \le \frac{2\alpha \text{Area}(\Omega)}{\epsilon}.$$

Proof. Because of the maximum principle, we may assume $f \ge 0$. Otherwise we just replace f by $|f|$. We use the symmetrization approach here. Let $\Sigma(\alpha, R)$ be the sector having the same area as Ω and the same relative isoperimetric constant as Ω. Define as in [8] the α-symmetrization to be the transformation that associates $u(x)$ with

$$u_\alpha := u_*(\frac{\alpha}{2}|x|^2)$$

for $x \in \Sigma(\alpha, R)$ where u_* is the standard decreasing rearrangement. Namely

$$u_* := \inf\{t \ge 0 : \ \mu(s) < t\}$$

and

$$\mu(t) = \text{meas}\{x \in \Omega : \ |u(x)| > t\}.$$

u_α has the similar properties to those of standard Schwartz symmetrization. In particular

$$\int_\Omega F(u(x))dx = \int_{\Sigma(\alpha,R)} F(u_\alpha(x))dx. \tag{5.1}$$

for real Borel function F. Moreover, let u be a solution to the equation in lemma 5.4, and v be the solution of

$$\begin{cases} -\Delta v = f_\alpha \text{ in } \Sigma(\alpha, R) \\ v|_{\widetilde{\Gamma}_0} = 0 \\ \dfrac{\partial v}{\partial \nu}|_{\widetilde{\Gamma}_1} = 0 \end{cases}$$

where

$$\widetilde{\Gamma}_0 = \{x \in \partial\Sigma(\alpha, R) : \ |x| = R\},$$

$$\widetilde{\Gamma}_1 = \{x \in \partial\Sigma(\alpha, R) : |x| \leq R\}$$

and f_α is the α-symmetrization of f. Standard argument shows that v is radially symmetric. From [8], we assert

$$u_\alpha(x) \leq v(x) \tag{5.2}$$

where u_α is the α-symmetrization of the solution u in lemma 5.4. But since it is radially symmetric, v satisfies

$$\begin{cases} v''(t) + \dfrac{1}{t}v'(t) + f_\alpha(t) = 0 \\ v'(0) = 0 \\ v(R) = 0. \end{cases}$$

Therefore solving the O.D.E., we have

$$v(r) \leq \log(\frac{R}{r}) \int_0^R s f_\alpha(s) ds;$$

$$\int_{\Sigma(\alpha,R)} \exp[\frac{(2\alpha - \epsilon)v}{\|f_\alpha\|_{L^1(\Omega)}}] \leq \frac{2\alpha \text{Area}(\Sigma(\alpha, R))}{\epsilon} = \frac{2\alpha \text{Area}(\Omega)}{\epsilon}.$$

Combining this with (5.1) and (5.2), we have the desired result. □

6 Proof of Theorem 1.2

lemma 5.4 implies that $\{v_p\}$ is uniformly bounded in $L^1(\Omega)$. Therefore lemma 5.1 implies that $\{v_p\}$ is uniformly bounded in $L^\infty(\omega)$ where ω is a neighborhood of any compact subset of $int(\Gamma_0)$. Since

$$\max_{x \in \overline{\Omega}} v_n(x) \geq \frac{C}{\nu_{p_n}} \to \infty,$$

from theorem 1.1 and corollary 4.1, we deduce $S \neq \emptyset$. However, since $S_D = \emptyset$, we conclude that $\#(S_I \cup S_C \cup S_N) \geq 1$. This proves part 1. To prove the rest of the theorem, define

$$L_0 = \overline{\lim_{p \to \infty}} \frac{p\nu_p}{e} \tag{6.1}$$

where

$$\nu_p = \int_\Omega u_p^p. \tag{6.2}$$

We denote any sequence u_{p_n} of u_p with $p_n \to \infty$ by u_n. Let

$$v_n := v_{p_n} := \frac{u_n}{\nu_{p_n}}; \tag{6.3}$$

$$f_n := f_{p_n} := \frac{u_n^{p_n}}{\int_\Omega u_n^{p_n}} = \nu_{p_n}^{p_n-1} v_n. \tag{6.4}$$

Because

$$\int_{\Omega\cup\Gamma_1} f_n = 1,$$

we can subtract a subsequence of f_n, still denoted by f_n, so that there is a positive bounded measure μ in $M(\Omega \cup \Gamma_1)$, the set of all real bounded Borel measures on $\Omega \cup \Gamma_1$, such that

$$\int_{\Omega\cup\Gamma_1} f_n\varphi \to \int_{\Omega\cup\Gamma_1} \varphi d\mu \tag{6.5}$$

for all

$$\varphi \in C_0^\infty(\Omega \cup \Gamma_1).$$

Recall S_I and S_N defined in (1.6). For any $\delta > 0$ we call $x_0 \in \Omega \cup (\Gamma_1 \backslash (\Gamma_1 \cap \Gamma_0))$ a δ-regular point if

- $x_0 \in \Omega$ and there is $\varphi \in C_0(\Omega)$, $0 \le \varphi \le 1$, $\varphi = 1$ in a neighborhood of x_0, such that
$$\int_{\Omega\cup\Gamma_1} \varphi d\mu \le \frac{4\pi}{L_0 + 2\delta} \tag{6.6}$$
where L_0 is the quantity defined in (3.2), or
- $x_0 \in \Gamma_1 \backslash (\Gamma_1 \cap \Gamma_0)$ and there is $\varphi \in C_0(\Omega \cup \Gamma_1)$, $0 \le \varphi \le 1$, $\varphi = 1$ in a neighborhood of x_0, such that
$$\int_{\Omega\cup\Gamma_1} \varphi d\mu \le \frac{2\alpha(x_0)}{L_0 + 2\delta} \tag{6.7}$$
where $\alpha(x_0) := \lim_{r\to 0} \alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)$ considered in proposition 2.2.

We let $\alpha(x_0) = 2\pi$ if $x_0 \in \Omega$. We say $x_0 \in \Omega \cup \Gamma_1 \backslash (\Gamma_0 \cap \Gamma_1)$ is δ-irregular if x_0 is not δ-regular. Clearly

$$\mu(x_0) \ge \frac{2\alpha(x_0)}{L_0 + 2\delta}$$

for all δ-irregular point x_0.

Lemma 6.1 *If x_0 is a δ-regular point for $\delta > 0$, then $\{v_n\}$ is uniformly bounded in $L^\infty(B_{R_0}(x_0) \cup \overline{\Omega})$ for some $R_0 > 0$.*

Proof. We first consider the case where $x_0 \in \Gamma_1 \backslash (\Gamma_0 \cap \Gamma_1)$. Let x_0 be a δ-regular point on $\Gamma_1 \backslash (\Gamma_0 \cap \Gamma_1)$. Then there exists R_0 such that

$$\int_{B_{R_0}(x_0)\cup\overline{\Omega}} f_n \le \frac{2\alpha(x_0)}{L_0 + \delta}$$

for n large enough.

Split v_n into two parts, $v_n = v_{1n} + v_{2n}$ where v_{1n} solves

$$\begin{cases} \Delta v_{1n} + f_n = 0 \text{ in } B_{R_0}(x_0) \cap \Omega \\ v_{1n} = 0 \text{ on } \partial B_{R_0}(x_0) \cap \Omega \\ \dfrac{\partial v_{1n}}{\partial \nu} = 0 \text{ on } B_{R_0}(x_0) \cap \Gamma_1 \end{cases} \tag{6.8}$$

and v_{2n} solves

$$\begin{cases} \Delta v_{2n} = 0 \text{ in } B_{R_0}(x_0) \cap \Omega \\ v_{2n} = v_n \text{ on } \partial B_{R_0}(x_0) \cap \Omega \\ \dfrac{\partial v_{2n}}{\partial \nu} = 0 \text{ on } B_{R_0}(x_0) \cap \Gamma_1. \end{cases} \tag{6.9}$$

Then $v_{1n} \leq v_n$ and $v_{2n} \leq v_n$ by the maximum principle. Now from the standard elliptic boundary estimate for harmonic functions with Neumann data, we have

$$\|v_{2n}\|_{L^\infty(B_{\frac{R_0}{2}}(x_0) \cap \overline{\Omega})} \leq C \|v_{2n}\|_{L^1(B_{R_0} \cap \overline{\Omega})} \leq C'$$

where C' is a constant independent of n and the last inequality follows from lemma 5.4. So we need only to estimate v_{1n}.

We first claim that when n is large enough

$$f_n(x) \leq \exp(L_0 + \delta/2) v_n(x) \tag{6.10}$$

for all $x \in \Omega$.

Now observe

$$\log x \leq \frac{x}{e} \tag{6.11}$$

for $x > 0$. We have

$$p_n \log \frac{u_n}{\nu_n^{1/p_n}} \leq \frac{p_n}{e} \frac{u_n}{\nu_n^{1/p_n}}$$

$$\leq \frac{L_0 + \delta/3}{\nu_n} \frac{u_n}{\nu_n^{1/p_n}} \leq \frac{t' - \delta/6}{\nu_n^{1/p_n}} \frac{u_n}{\nu_n} \leq t' \frac{u_n}{\nu_n}$$

for n large enough because

$$\lim_{n \to \infty} \nu_n^{1/p_n} = 1$$

which follows from corollary 4.1. Hence

$$f_n \leq \exp[(L_0 + \delta/2) v_n]$$

Next we claim that $\{f_n\}$ is uniformly bounded in $L^{1+\delta_0}(B_{R_1/2})$ for δ_0 sufficiently small. Because $\{v_{2n}\}$ is uniformly bounded in $B_{R_1/2}(x_0)$, we see from the previous claim

$$\int_{B_{R_1/2}} f_n^{1+\delta_0} \leq \int_{B_{R_1/2}} \exp[(1+\delta_0)(L_0 + 0.5\delta) v_n]$$

$$\leq C\int_{B_{R_1/2}} \exp[(1+\delta_0)(L_0+0.5\delta)v_{1n}]$$

$$\leq C\int_{B_{R_1/2}} \exp\frac{4\pi(1+\delta_0)\frac{L_0+0.5\delta}{L_0+\delta}v_{1n}}{\int_{B_{R_1/2}(x_0)} f_n} \leq C'$$

with the aid of lemma 5.4 if we choose δ_0 sufficiently small. So we have proved the claim 6.10.

Now take $B_{R_1/4}(x_0)$. We conclude from the weak Hanack inequality (Theorem 8.17, [5]),

$$\|v_n\|_{L^\infty(B_{R_1/4}(x_0))} \leq C[\|v_n\|_{L^2(B_{R_1/2}(x_0))} + \|f_n\|_{L^{1+\delta_0}(B_{R_1/2}(x_0))}] \leq C.$$

Here the boundedness of $\{v_n\}$ in $L^2(B_{R_1/2}(x_0))$ follows from lemma 5.4.

The case where $x_0 \in \Omega$ is similar. We just use lemma 5.2 in place of lemma 5.4. □

Lemma 6.2 *For any $\delta > 0$, $x_0 \in S_I \cup S_N$ if and only if x_0 is δ-irregular.*

Proof. Let x_0 be a δ-irregular point. Then by lemma 6.1, $\{v_n\}$ is bounded in $L^\infty(B_{R_1} \cap \Omega)$ for some R_1. Hence $x_0 \notin S_I \cup S_N$. Conversely suppose x_0 is a δ-irregular point. Then we have for every $R > 0$

$$\lim_{n\to\infty} \|v_n\|_{L^\infty(B_R(x_0)\cap\Omega)} = \infty.$$

Otherwise, there would be some $R_0 > 0$ and a subsequence, still denoted by $\{v_n\}$, such that

$$\|v_{1n}\|_{L^\infty(B_{R_0}(x_0)\cap\overline{\Omega})} \leq C$$

for some C independent of n. Then

$$f_n = \nu_n^{p_n-1} v_n^{p_n} \leq (\frac{M}{p_n})^{p_n-1} C^{p_n} \to 0$$

uniformly as $n \to \infty$ on $B_{R_0}(x_0) \cap \overline{\Omega}$. Here M is a uniform upper bound of u_p obtained in theorem 1.1. Then

$$\int_{B_{R_0}(x_0)\cap\overline{\Omega}} f_n \leq \epsilon_0 \leq \frac{2\alpha(x_0)}{L_0+2\delta}$$

which implies that x_0 is a δ-regular point. A contradiction. □

Back to the measure μ defined earlier in this section. Clearly we have

$$1 \geq \mu(\Gamma_1 \cup \Omega) \geq \sum_{x_0\in S_I\cup S_N} \frac{2\alpha(x_0)}{L_0+2\delta}$$

which in turn, if we let $\delta \to 0$, implies

Proposition 6.3

$$\sum_{x_0 \in S_I \cup S_N} \alpha(x_0) \leq \frac{1}{2} L_0.$$

From this proposition, with the aid of proposition 2.2 and corollary 3.5, we obtain part 2 and part 3 of theorem 1.2.

Remark 6.4 *We see that every peak P in Ω is a blow-up point of $v_p = \frac{u_p}{\nu_p}$ because by corollary 4.1 $\nu_p \to 0$ as $p \to \infty$.*

7 Further Results and Examples

In this section we shall focus on some special domains Ω where the corresponding quantities L_0 are indeed smaller than what we get in corollary 3.5. In these special cases, we can actually prove that the solutions of (1.1) possess single-peaks on the Neumann boundary of Ω. Let us first formulate a general result.

Theorem 7.1 *Let (Γ_1, Ω) be a pair such that α_0, defined in (4.4), with respect to this pair is strictly less than 1, i.e. $\gamma_0 < \frac{e}{2}$. Then for every sequence $\{u_{p_n}\}$ of solutions on Ω with the Neumann boundary Γ_1, there is a subsequence, again denoted by $\{u_{p_n}\}$, such that the interior blow-up set S_I is empty and the Γ_1-boundary blow-up set S_N contains at most one point.*

Proof. If we check the proof of lemma 6.1 carefully, we can see that we can use a refined inequality

$$\frac{\log x}{x} \leq \frac{\log y}{y}$$

if $x \leq y \leq e$ instead of (6.11). Notice that since we assume $\alpha_0 < 1$,

$$\limsup_{n \to \infty} \frac{u_n}{\nu_n^{1/p_n}} \leq \exp \frac{1 + \alpha_0}{2} < e.$$

Let

$$L_0' = \frac{\limsup_{n \to \infty} (1 + \alpha_0) p \int_\Omega u_p^p}{2 \exp[\frac{1+\alpha_0}{2}]}.$$

We still have, as proposition 6.3, with the aid of corollary 3.5,

$$\sum_{x_0 \in S_I \cup S_N} \alpha(x_0) \leq \frac{1}{2} L_0' < 2\beta. \tag{7.1}$$

If $S_I \neq \emptyset$, then, with the aid of proposition 2.2, $\alpha(x_0) = 2\pi$ for some $x_0 \in S_I$. If $\#S_N \geq 2$, then, with the aid of proposition 2.2 again, $\alpha(x_1) + \alpha(x_2) \geq 2\beta$ for two different x_1 and x_2 in S_N. In any case, we reach a contradiction to (7.1). □

Example 7.2 *Let* $\Omega = \{x \in R^2 : \ r < |x| < R\}$*,* $\Gamma_1 = \{x \in R^2 : \ |x| = r\}$ *and* $\Gamma_0 = \{x \in R^2 : \ |x| = R\}$*.*

In this case the constant α with respect to (Γ_1, Ω) is equal to π (See Example 3.3 [8]) and the constant β is clearly π. Hence $\gamma_0 = 1 < e/2$ and the condition of theorem 7.1 is satisfied. Indeed, since the two boundaries has no intersection, passing to a subsequence if necessary, $S_N = \{x_0\}$.

Example 7.3 *Let* $\Omega = \Sigma(\alpha, R)$*,* $0 \le \alpha \le \pi$*, and* Γ_1 *be the union of two sides of the sector.*

In this case $\beta = \alpha$ (see [6]). Hence $\gamma_0 = 1 \le e/2$ and the condition of theorem 7.1 is again satisfied.

Acknowledgment The authors would like to thank their thesis adviser, Professor Wei-Ming Ni, for bringing spike-like pattern formation problems to their attention.

References

[1] H. Brezis and F. Merle, Uniform estimate and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, Comm. Part. Diff. Equa. vol. 16, no. 8&9, 1991, 1223-1253.

[2] D. G. DeFigueiredo, P. L. Lions and R. D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pore. Appl. vol. 61, 1982, pp 41-63.

[3] H. Federer, Geometric Measure Theory, Springer-Verlag, Berlin Heidelberg New York, 1969.

[4] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys.(68) no.3, 1979, 209-243.

[5] D. Gilbarg and S. N. Trudinger, Elliptic Partial Differential Equations of Second Order, Second Edition, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983.

[6] P. L. Lions, F. Pacella and M. Tricarico, Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions, Indiana Univ. Math. J., vol. 37, no. 2 (1988), 301-324.

[7] J. Moser, A sharp form of an inequality by Trudinger, Indiana Univ. Math. Joun, vol 20, no. 11 (1971), 1077-1092.

[8] F. Pacella and M. Tricarico, Symmetrization for a class of elliptic equations with mixed boundary conditions, Att Sem. Mat. Fis. Univ. Modena XXXIV (1985-86), 75-94.

[9] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet. Math. Dokl., 6 (1965), pp. 1408-1411.

[10] X. Ren and J. Wei, On a two dimensional elliptic problem with large exponent in nonlinearity, Tran. A.M.S., to appear.

[11] X. Ren and J. Wei, Counting peaks of solutions to some quasilinear elliptic equations with large exponents, J. Diff. Equa., to appear.

[12] G. Talenti, Elliptic equations and rearrangements, Ann. Scuola. Norm. Pisa. Science, 3 (1976), pp 697-718.

[13] G. Talenti, Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces, Ann. Mat. Pura. Appl. IV, 120 1979, 159-184.

[14] W. Ziemer, Weakly Differentiable Functions, Springer-Verlag, New York Berlin Heidelberg London Paris Tokyo Hong Kong, 1989.