Stability of Spot and Ring Solutions of the Diblock Copolymer Equation *

Xiaofeng Ren
Department of Mathematics and Statistics
Utah State University
Logan, UT 84322-3900, USA

Juncheng Wei †
Department of Mathematics
Chinese University of Hong Kong
Shatin, Hong Kong

April 6, 2004

Abstract

The Γ-convergence theory shows that under certain conditions the diblock copolymer equation has spot and ring solutions. We determine the asymptotic properties of the critical eigenvalues of these solutions in order to understand their stability. In two dimensions a threshold exists for the stability of the spot solution. It is is stable if the sample size is small and unstable if the sample size is large. The stability of the ring solutions is reduced to a family of finite dimensional eigenvalue problems. In one study no two-interface ring solutions are found by the Γ-convergence method if the sample is small. A stable two-interface ring solution exists if the sample size is increased. It becomes unstable if the sample size is increased further.

Key words. diblock copolymer equation, spot solution, ring solution. critical eigenvalue, two-dimensional stability
2000 Mathematics Subject Classification. 35J55, 34D15, 45J05, 82D60

1 Introduction

A diblock copolymer is a soft material, characterized by fluid-like disorder on the molecular scale and a high degree of order at longer length scales. A molecule in a diblock copolymer is a linear sub-chain of A monomers grafted covalently to another sub-chain of B monomers. Because of the repulsion between the unlike monomers, the different type sub-chains tend to segregate, but as they are chemically bonded in chain molecules, segregation of sub-chains cannot lead to a macroscopic phase separation. Only a local micro-phase separation occurs: micro-domains rich in A and B emerge. These micro-domains form morphology patterns/phases in a larger scale.

The Ohta-Kawasaki [21] free energy of an incompressible diblock copolymer melt is a functional of the A monomer density field. Let $u(x)$ be the relative A monomer number density at point x in the sample D. When there is high A monomer concentration at x, $u(x)$ is close to 1; when there is

*Corresponding author: X. Ren, Phone: 1 435 797-0755, Fax 1 435 797-1822, E-mail: ren@math.usu.edu
†Supported in part by a Direct Grant from CUHK and an Earmarked Grant of RGC of Hong Kong.

high concentration of B monomers at x, $u(x)$ is close to 0. A value of $u(x)$ between 0 and 1 means that a mixture of A and B monomers occupies x. The re-scaled, dimensionless free energy of the system is

$$I(u) = \int_D \{\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{\epsilon\gamma}{2}|(-\Delta)^{-1/2}(u-a)|^2 + W(u)\}\, dx, \tag{1.1}$$

which is defined in the admissible set

$$X_a = \{u \in W^{1,2}(D) : \overline{u} = a\} \tag{1.2}$$

where $\overline{u} = \frac{1}{|D|}\int_D u\, dx$ is the average of u in D. a is a fixed constant in $(0,1)$. It is the ratio of the number of the A monomers to the number of all the monomers in a chain molecule.

In (1.1) ϵ is a small positive parameter and γ is a fixed positive constant, i.e.

$$\epsilon \to 0,\ \gamma \sim 1. \tag{1.3}$$

The term $W(u)$ is the internal energy field. Originally in Choksi and Ren [8] it is taken to be

$$W(u) = \begin{cases} u - u^2 & \text{if } u \in [0,1] \\ \infty & \text{otherwise} \end{cases}. \tag{1.4}$$

Here we change it to a smooth function so that W is a double well potential of equal depth. It has global minimum value 0 achieved at 0 and 1. We assume for simplicity that W is smooth, grows at least quadratically at $\pm\infty$, and symmetric about 1/2: $W(u) = W(1-u)$. 0 and 1 are non-degenerate: $W''(0) = W''(1) > 0$. An example of W is $W(u) = \frac{1}{4}(u^2 - u)^2$.

The other two terms in (1.1) give the entropy of the system. The peculiar nonlocal term is due to the fact that molecules in a diblock copolymer are connected long chains. It models a type of nonlocal interaction known as the Coulomb interaction, Muratov [17]. Mathematically we view $(-\Delta)^{-1}$ as a bounded positive operator from $\{\zeta \in L^2(D) : \overline{\zeta} = 0\}$ to $\{\xi \in W^{2,2}(D) : \overline{\xi} = 0\}$: $\xi = (-\Delta)^{-1}\zeta$ if

$$-\Delta\xi = \zeta \text{ in } D,\ \partial_\nu \xi = 0 \text{ on } \partial D,\ \overline{\xi} = 0.$$

Then $(-\Delta)^{-1/2}$ is the positive square root of $(-\Delta)^{-1}$.

To under stand the parameter range (1.3) we recall the physical parameters in a diblock copolymer system (cf. [8]).

1. The polymerization index N that is the number of all the monomers in a chain molecule. We consider the ideal situation where this N is the same in all molecules;

2. The Kuhn statistical length l measuring the average distance between two adjacent monomers in a chain molecule, which is the same regardless the monomer types;

3. The Flory-Huggins parameter χ that measures the repulsion between unlike monomers and is inversely proportional to the absolute temperature;

4. Relative A monomer ratio a mentioned earlier;

5. The volume V of the sample.

They are related to the mathematical dimensionless parameters ϵ and γ by

$$\epsilon^2 = \frac{\pi^{2/3} l^2}{12a(1-a)\chi V^{2/3}}, \quad \gamma = \frac{18\sqrt{3}V}{\pi a^{3/2}(1-a)^{3/2}\chi^{1/2}N^2 l^3}. \tag{1.5}$$

Among the physical parameters a and χ are dimensionless and of order 1. So we focus on l, V and N. N is necessarily large in a polymer system. By taking ϵ small we have assumed that the sample is large compared to l. On the other hand having $\gamma \sim 1$ means that $V \sim l^3 N^2$. After we find spot and ring solutions of a finite number of micro-domains separated by interfaces whose width is of order ϵ in the parameter range (1.3), we conclude that the size of a micro-domain is of order $l^3 N^2$ and the thickness of the interfaces is of order l, facts very well matched by experiments [21].

Another choice of γ was used in Müller [16], Nishiura and Ohnishi [19], and Ren and Wei [26]: $\gamma \sim \epsilon^{-1}$, i.e. $V \sim l^3 N^3$. In this larger sample one finds that the number of the micro-domains is of order ϵ^{-1}. Then again the size of a micro-domain is of order $l^3 N^2$.

The diblock copolymer equation

$$-\epsilon^2 \Delta u + f(u) + \epsilon\gamma(-\Delta)^{-1}(u-a) = \eta \text{ in } D, \quad \partial_\nu u = 0 \text{ on } \partial D, \quad \overline{u} = a \tag{1.6}$$

is the Euler-Lagrange equation of (1.1) where $f = W'$. For the example of W, $f(u) = u(u-1/2)(u-1)$. The unknown constant η is a Lagrange multiplier due to the constraint $\overline{u} = a$. If we integrate (1.6) over D, then

$$\eta = \overline{f(u)}. \tag{1.7}$$

Many morphology patterns are observed in diblock copolymers. See Bates and Fredrickson [4], Hamley [11], and the references therein. The most popular ones are the spherical, cylindrical, and lamellar phases, Figure 1. The existence of the lamellar phase was shown in Ren and Wei [24], and its stability in three dimensions was studied in Ren and Wei [27]. Surprisingly we found that the lamellar phase is only *marginally* stable. Physicists believe that defects should appear commonly in the lamellar phase, Tsori *et al* [36] and [17].

One type of defect is the wriggled lamellar pattern studied in Ren and Wei [33], where interfaces separating micro-domains oscillate like the sinusoidal curve. Here we study another type of defect: spot and ring like micro-domains, Figure 2. We consider (1.6) in the unit disc $D = \{x \in R^2 : |x| < 1\}$. Let $v = (-\Delta)^{-1}(u-a)$. If u and v are radially symmetric, then (1.6) may be written in the radial coordinates, $r = |x|$, as

$$\begin{cases} -\epsilon^2 u_{rr} - \frac{\epsilon^2}{r} u_r + f(u) + \epsilon\gamma v = \eta \\ -v_{rr} - \frac{1}{r} v_r = u - a \\ u_r(0) = u_r(1) = v_r(0) = v_r(1) = 0 \\ \overline{u} - a = \overline{v} = 0 \end{cases}. \tag{1.8}$$

The average now becomes $\overline{u} = 2\int_0^1 u(r) r dr$. We are interested in radial solutions of (1.6) that show the phenomenon of micro-phase separation. They are close to 0 or 1 in most of D but change between 0 and 1 in small regions. These small transition regions are called the interfaces. For a radial solution u an interface may be identified by a number r_j where $u(r_j) = 1/2$. The following theorem was proved in Ren and Wei [25] using the Γ-convergence theory (cf. De Giorgi [9], Modica [15], and Kohn and Sternberg [14]).

Theorem 1.1 (Ren and Wei [25]) *For any $\gamma > 0$, there exist two radial solutions of (1.6) on the unit disk with one circular interface when ϵ is small. If $K \geq 2$ and γ is large enough there exist two radial solutions with K circular interfaces when ϵ is small.*

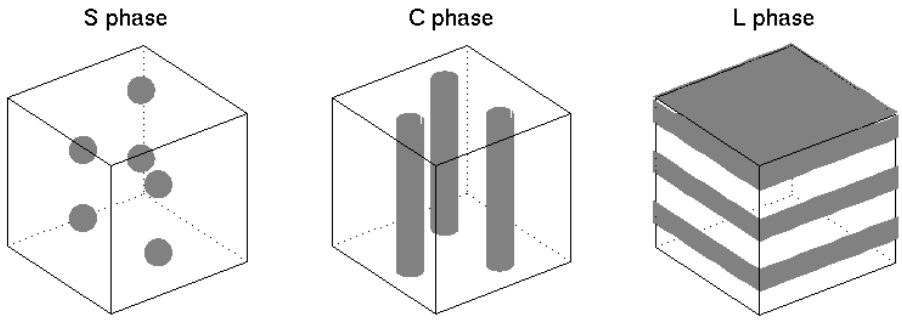

Figure 1: The spherical, cylindrical, and lamellar morphology phases commonly observed in diblock copolymer melts. The white color indicates the concentration of type A monomer, and the dark color indicates the concentration of type B monomer.

For each K one of the solutions, which we simply denote by u, in Theorem 1.1 is close to 0 near the origin and the other one is close to 1 near the origin, which we denote by $\tilde{u}$. However the two solutions are related. If we change a to $1-a$ in (1.1) and (1.2), then $1-\tilde{u}$ is a solution of the new problem which is close to 0 near the origin, and $1-u$ is a solution of the new problem which is close to 1 near the origin. So it suffices to study u. u is a spot solution if $K=1$, and a ring solution if $K \geq 2$, Figure 2. Throughout this paper $v = (-\Delta)^{-1}(u-a)$.

The spot solution is also useful in the study of the cylindrical phase, Figure 1 (2). A cross section of the cylindrical phase has a pattern of many spots. It is believed that these spots pack in a hexagonal way [4]. A good understanding of a single spot is essential before one can mathematically prove the existence of the cylindrical phase.

In this paper we derive a criterion for the stability of the spot and ring solutions by obtaining detailed information on the eigenvalues and eigenfunctions of the linearized problem:

$$L\varphi := -\epsilon^2\Delta\varphi + f'(u)\varphi - \overline{f'(u)\varphi} + \epsilon\gamma(-\Delta)^{-1}\varphi = \lambda\phi \text{ in } D,\ \partial_\nu\varphi = 0 \text{ on } \partial D,\ \overline{\phi} = 0. \tag{1.9}$$

It is easy to see, Lemma 2.4, that $\liminf_{\epsilon\to 0}\lambda \geq 0$. To determine the stability we need to study the λ's that tend to 0 as $\epsilon \to 0$. These λ's are called the critical eigenvalues. They are found in Theorems 3.1 and 4.1. Consequently we show in Theorem 5.1 that the spot solution is stable if γ is small and unstable if γ is large. The threshold of γ is denoted by $\hat{\gamma}$. It is calculated numerically for various a and the nonlinearity f.

To better appreciate this theorem let us recall the stationary Cahn-Hilliard equation [5], which is (1.6) with $\gamma = 0$, the local counterpart. It is known that the Cahn-Hilliard equation on the unit disc has an unstable spot solution. Once the nonlocal term with a small γ, which encourages oscillation, is added, the spot solution becomes stable. The abrupt change of stability here is discussed after the proof of Theorem 5.1. If γ is further increased, more oscillation is required and the spot solution, which only has one interface, becomes unstable.

The second change of stability has a simple physical explanation. According to (1.5) γ is proportional to the size of the sample. When the sample is sufficiently large, one big spot is unstable in two dimensions. It should break into multiple spots to form a cylindrical phase, Figure 1 (2).

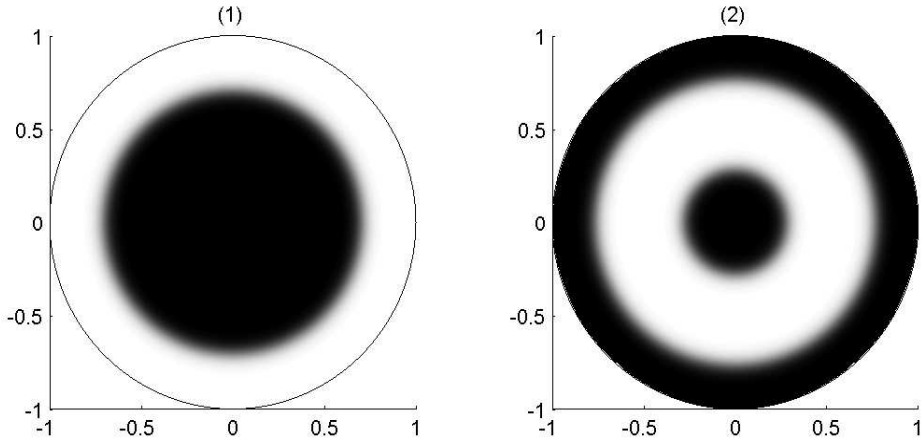

Figure 2: (1) A spot solution. (2) A $K = 2$ ring solution. In both cases $a = 1/2$ and $\gamma = 25$.

The value V corresponding to $\hat{\gamma}$ in $(1.5)_2$ suggests a scale for a cell with one spot in a multi-spot cylindrical phase.

For the ring solution ($K \geq 2$), we will use Theorems 3.1 and 4.1 to numerically study a case of $K = 2$. When γ is small, we can not find a ring solution by the Γ-convergence method. When γ is increased, there exists a ring solution that is stable in two dimensions. When γ is further increased over $\hat{\gamma}$, a ring solution exists but is no longer stable.

This change of stability of the ring solution and the second change of stability of the spot solution lead to a bifurcation phenomenon near $\hat{\gamma}$. Following [33] one should be able to find bifurcation solutions. They are depicted in Figure 3. Based on our experience in [33] we suspect that most of them are stable.

More information on the model (1.1) and its extension to triblock copolymers may be found in Nakazawa and Ohta [18], and Ren and Wei [29]. The mathematical study of stable domain structures with multiple sharp interfaces started rather recently. On the block copolymer problem the literature includes Ohnishi *et al* [20], Ren and Wei [30], Choksi [7], Fife and Hilhorst [10], Henry [13], and Teramoto and Nishiura [35]. Elsewhere Ren and Truskinovsky [23] studies the phenomenon in elastic bars, Ren and Wei [32, 28, 31] in the Seul-Andelman membrane, charged monolayers, and smectic liquid crystal films, respectively. Taniguchi [34] and Chen and Taniguchi [6] study spot and ring patterns in a free boundary problem.

The paper is organized as follows. In Section 2 we review the construction of the spot and ring solutions u, give some properties of u, and explain the classification into λ_m, where $m = 0, 1, 2, 3...$ of the eigenvalues of the linearized operator at u. The properties of λ_m are given in Theorems 3.1 and 4.1 in Sections 3 and 4 respectively. In Section 5 we show the stability property of the spot solution, calculate the second threshold $\hat{\gamma}$, and use Theorems 3.1 and 4.1 to study a $K = 2$ ring solution. This section also includes some remarks. The appendix contains the proof of a technical lemma.

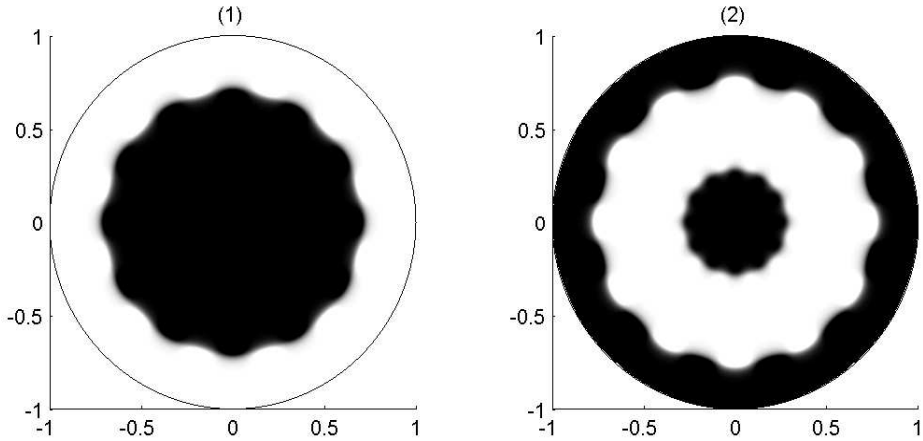

Figure 3: Bifurcation solutions for $K = 1$ and $K = 2$.

2 Preliminaries

To make the paper more readable a quantity's dependence on ϵ is usually not reflected in its notation but implied in the context. On the other hand a quantity's independence of ϵ is often emphasized with a superscript 0. For instance the spot or ring solution u is not denoted by u_ϵ, while the $L^2(D)$-limit of u as $\epsilon \to 0$ is denoted by u^0.

Throughout the paper, the L^∞ norm of a function is denoted simply by $\|\cdot\|$. Other norms are more explicitly written, like $\|\cdot\|_2$.

We define some frequently used quantities. H is the heteroclinic solution of

$$-H'' + f(H) = 0,\ H(-\infty) = 0,\ H(\infty) = 1,\ H(0) = 1/2. \tag{2.1}$$

Our assumption that $W(u) = W(1-u)$ implies that $H(t) = 1 - H(-t)$. The interface tension τ is a constant defined by

$$\tau := \int_R (H'(t))^2\, dt. \tag{2.2}$$

In the special case $W(u) = \frac{1}{4}(u^2 - u)^2$, $\tau = \frac{\sqrt{2}}{12}$.

Theorem 1.1 was proved in [25] by locally minimizing I in the radial class

$$X_a^R = \{u \in W^{1,2}(D) : u(x) = u(|x|),\ \overline{u} = a\}. \tag{2.3}$$

To do so we used the Γ-convergence theory in the perturbation variational analysis. $(\epsilon\pi)^{-1}I$ converges in a particular sense to a singular limit J. J is defined in the class $\mathcal{A}$ which may be decomposed to

$$\mathcal{A} = \cup_{K=1}^{\infty}(\mathcal{A}_K \cup \tilde{\mathcal{A}}_K). \tag{2.4}$$

A function $\mathcal{U}$ is in $\mathcal{A}_K$ if $\overline{\mathcal{U}} = a$ and there exist $q_1, q_2, ..., q_K$, satisfying $0 < q_1 < q_2 < ... < q_K < 1$, such that $\mathcal{U}(r) = 0$ if $r \in (0, q_1)$, $= 1$ if $r \in (q_1, q_2)$, $= 0$ if $r \in (q_2, q_3)$.... Similarly a function $\tilde{\mathcal{U}} \in \tilde{\mathcal{A}}_K$

if $\overline{\tilde{\mathcal{U}}} = a$ and there exist $q_1, q_2, ..., q_K$, satisfying $0 < q_1 < q_2 < ... < q_K < 1$, such that $\tilde{\mathcal{U}}(r) = 1$ if $r \in (0, q_1)$, $= 0$ if $r \in (q_1, q_2)$, $= 1$ if $r \in (q_2, q_3)$.... By the remark after Theorem 1.1 we will not consider J in $\tilde{\mathcal{A}}$. In each $\mathcal{A}_K$ the function J depends on $\mathbf{q} = (q_1, q_2, ..., q_K)$ only:

$$J(\mathbf{q}) = 2\tau(q_1 + q_2 + ... + q_K) + \gamma \int_0^1 \mathcal{V}'(r)^2 r dr. \tag{2.5}$$

In (2.5) $\mathbf{q}$ determines $\mathcal{U} \in \mathcal{A}_K$. We emphasize that $\mathcal{U}$ depends on all q_j. We sometimes use the notation $\mathcal{U} = \mathcal{U}(r; \mathbf{q})$. Let $\mathcal{V}$ be the solution of

$$-\mathcal{V}'' - \frac{\mathcal{V}'}{r} = \mathcal{U} - a, \ \mathcal{V}'(1) = 0, \ \overline{\mathcal{V}} = 0. \tag{2.6}$$

We define G_0 to be the solution operator of (2.6) so that $\mathcal{V} = G_0[\mathcal{U} - a]$. Again we may write $\mathcal{V} = \mathcal{V}(r; \mathbf{q})$. The constraint $\overline{\mathcal{U}} = a$ becomes a constraint on $\mathbf{q}$:

$$S(\mathbf{q}) := -q_1^2 + q_2^2 - q_3^2 + ... + (-1)^K q_K^2 + \frac{1 - (-1)^K}{2} = a. \tag{2.7}$$

To incorporate the constraint (2.7) we define $F := J + \nu S$ where ν is the Lagrange multiplier in accordance to the constraint.

Using ideas from [15] and [14] following result in [25].

Lemma 2.1 *If J has a strict local minimizer $\mathcal{U}(\cdot; \mathbf{r}^0) \in \mathcal{A}_K$, then there exists $\hat{\epsilon} > 0$ such that for all $\epsilon \in (0, \hat{\epsilon})$ (1.6) has a solution u with the properties $\lim_{\epsilon \to 0} \|u - \mathcal{U}(\cdot; \mathbf{r}^0)\|_2 = 0$ and $\lim_{\epsilon \to 0} \epsilon^{-1} I(u) = J(\mathcal{U}(\cdot; \mathbf{r}^0))$.*

Lemma 2.1 reduces I to J which is finite dimensional in each $\mathcal{A}_K$ and $\tilde{\mathcal{A}}_K$. To study J we define from the operator G_0 the Green function

$$G_0(r, s) = G_0[\delta(\cdot - s) - 2s](r) \tag{2.8}$$

where $2s$ is the average of $\delta(\cdot - s)$. More explicitly

$$G_0(r, s) = \begin{cases} \dfrac{sr^2}{2} - \dfrac{3s - 2s^3}{4} - s \log s & \text{if} \quad r < s \\ \\ \dfrac{sr^2}{2} - s \log r - \dfrac{3s - 2s^3}{4} & \text{if} \quad r \geq s \end{cases}. \tag{2.9}$$

Note that $G_0(r, s)$ is not symmetric in r and s, although $rG_0(r, s)$ is. Also note $\overline{\delta(\cdot - s)} = 2s$. Then we may write

$$\mathcal{V}(r) = \int_0^1 G_0(r, s)(\mathcal{U}(s) - a)\, ds = \int_0^1 G_0(r, s)\mathcal{U}(s)\, ds.$$

We calculate the derivatives of J and F. J may be rewritten as

$$J(\mathbf{q}) = 2\tau \sum_{j=1}^K q_j + \gamma \int_0^1 \mathcal{U}(r)\mathcal{V}(r) r\, dr.$$

Then

$$\begin{aligned}\frac{\partial J}{\partial q_j} &= 2\tau + \gamma\frac{\partial}{\partial q_j}[\int_{q_1}^{q_2}\mathcal{V}(r)r\,dr + \int_{q_3}^{q_4}\mathcal{V}(r)r\,dr + ...] \\ &= 2\tau + (-1)^j\gamma q_j\mathcal{V}(q_j) + \gamma\int_0^1\mathcal{U}(r)\frac{\partial}{\partial q_j}\mathcal{V}(r)r\,dr.\end{aligned}$$

Note that

$$\frac{\partial}{\partial q_j}\mathcal{V}(r) = \frac{\partial}{\partial q_j}[\int_{q_1}^{q_2}G_0(r,s)\,ds + \int_{q_3}^{q_4}G_0(r,s)\,ds + ...] = (-1)^jG_0(r,q_j).$$

Hence

$$\frac{\partial J}{\partial q_j} = 2\tau + 2(-1)^j\gamma q_j\mathcal{V}(q_j),$$

and

$$\frac{\partial F}{\partial q_j} = 2\tau + 2(-1)^j\gamma q_j\mathcal{V}(q_j) + 2\nu(-1)^jq_j.$$

Let $\mathbf{r}^0 = (r_1^0, r_2^0, ..., r_K^0)$ be a solution of $\frac{\partial F}{\partial q_j} = 0$, $j = 1, 2, ..., K$, i.e.

$$2\tau + 2(-1)^j\gamma r_j^0\mathcal{V}(r_j^0) + 2\nu(-1)^jr_j^0 = 0, \quad j = 1, 2, ..., K. \tag{2.10}$$

The second derivatives of J are

$$\begin{aligned}\frac{\partial^2 J}{\partial q_j\partial q_k} &= 2(-1)^{j+k}\gamma q_jG_0(q_j,q_k), \quad \text{if } j \neq k \\ \frac{\partial^2 J}{\partial q_j^2} &= 2(-1)^j\gamma\mathcal{V}(q_j) + 2(-1)^j\gamma q_j((-1)^jG_0(q_j,q_j) + \mathcal{V}'(q_j)) \\ &= 2\gamma q_jG_0(q_j,q_j) + 2(-1)^j\gamma(\mathcal{V}(q_j) + q_j\mathcal{V}'(q_j)).\end{aligned}$$

Hence

$$\frac{\partial^2 F}{\partial q_j\partial q_k} = \begin{cases} 2\gamma q_jG_0(q_j,q_j) + 2(-1)^j\gamma(\mathcal{V}(q_j) + q_j\mathcal{V}'(q_j)) + 2\nu(-1)^j & \text{if } j = k \\ 2(-1)^{j+k}\gamma q_jG_0(q_j,q_k) & \text{if } j \neq k \end{cases}. \tag{2.11}$$

At $\mathbf{r}^0$, because of (2.10), we have

$$\frac{\partial^2 F}{\partial q_j\partial q_k}(\mathbf{r}^0) = \begin{cases} 2\gamma r_j^0G_0(r_j^0,r_j^0) + 2(-1)^j\gamma r_j^0\mathcal{V}'(r_j^0) - \dfrac{2\tau}{r_j^0} & \text{if } j = k \\ 2(-1)^{j+k}\gamma r_j^0G_0(r_j^0,r_k^0) & \text{if } j \neq k \end{cases}. \tag{2.12}$$

We emphasize that the function $\mathcal{V}$ in (2.12) is associated with $\mathbf{r}^0$, i.e. $\mathcal{V} = \mathcal{V}(\cdot;\mathbf{r}^0)$.

Whether a critical point $\mathbf{r}^0$ is a local minimum is determined by the matrix (2.12) in the subspace

$$T = \{\mathbf{b} = (b_1, b_2, ..., b_K)^T \in R^K : \sum_{j=1}^K(-1)^jb_jr_j^0 = 0\}. \tag{2.13}$$

T is the tangent space of the domain of J at $\mathbf{r}^0$. When (2.12) is positive definite in T, i.e.

$$\sum_{j,k=1}^{K} \frac{\partial^2 F}{\partial q_j \partial q_k}(\mathbf{r}^0) b_j b_k > 0, \quad \text{if } \mathbf{b} \in T \text{ and } \mathbf{b} \neq \vec{0}, \tag{2.14}$$

the critical point $\mathbf{r}^0$ is a strict local minimum.

The condition (2.14) may be rephrased as follows. Define a K by K matrix $\mathbf{M}^0$ whose kj entry is

$$M^0_{kj} = \delta_{kj}(-\frac{\tau}{(r^0_k)^2} + \gamma(-1)^k \mathcal{V}'(r^0_k)) + \gamma(-1)^{k+j} G_0(r^0_k, r^0_j), \tag{2.15}$$

where $\delta_{kj} = 1$ if $k = j$ and 0 otherwise. M^0_{kj} is not symmetric in j and k but $r^0_k M^0_{kj}$ is. Let g^0 be a non-standard inner product on R^K defined by

$$g^0(\mathbf{A}, \mathbf{B}) = \sum_{j=1}^{K} A_j B_j r^0_j, \quad \mathbf{A} = (A_1, A_2, ..., A_K)^T \ \mathbf{B} = (B_1, B_2, ..., B_K)^T. \tag{2.16}$$

With respect to g^0, the matrix $\mathbf{M}^0$ represents a symmetric linear operator on R^K. Also with respect to g^0 we choose an orthonormal basis $\mathbf{e}^0_1$, $\mathbf{e}^0_2$,..., $\mathbf{e}^0_K$ with

$$\mathbf{e}^0_1 = \frac{1}{\sqrt{r^0_1 + r^0_2 + ... + r^0_K}}(-1, 1, -1, 1, ..., (-1)^K)^T. \tag{2.17}$$

Since

$$\frac{1}{2} \sum_{j,k=1}^{K} \frac{\partial F}{\partial q_j \partial q_k}(\mathbf{r}^0) b_j b_k = \sum_{j,k=1}^{K} M^0_{kj} b_j b_k r^0_k = g^0(\mathbf{M}^0 \mathbf{b}, \mathbf{b}),$$

(2.14) is equivalent to the condition that $\mathbf{M}^0$ is positive definite in the $K-1$ dimensional subspace perpendicular to $\mathbf{e}^0_1$ with respect to g^0. This form of (2.14) is closer to the contents of Section 3.

Lemma 2.1 now implies the following theorem.

Theorem 2.2 *If J has a critical point $\mathbf{r}^0$ at which (2.12) is positive definite in T, then there exists $\hat{\epsilon} > 0$ such that for all $\epsilon \in (0, \hat{\epsilon})$ there is a solution u of (1.6) with the properties $\lim_{\epsilon \to 0} \|u - \mathcal{U}(\cdot; \mathbf{r}^0)\|_2 = 0$ and $\lim_{\epsilon \to 0} \epsilon^{-1} I(u) = J(\mathcal{U}(\cdot; \mathbf{r}^0))$.*

Only when $K = 1$, $\mathbf{r}^0 = (r^0_1)$ always exists and equals $\sqrt{1-a}$. It is regarded trivially as a strict local minimizer of J. Hence when ϵ is small, a spot solution of (1.6) exists unconditionally.

When $K \geq 2$, J may not have a strict local minimizer. Another perturbation argument can be used. Note that when γ is large, J may be viewed as a perturbation of

$$J^*(\mathbf{q}) = \gamma \int_0^1 (\mathcal{V}'(r))^2 r \, dr. \tag{2.18}$$

It was proved in [25] that J^* has a unique critical point $\mathbf{r}^* = (r^*_1, r^*_2, ...r^*_K)$. When γ is large, (2.12) is dominated by

$$\begin{cases} 2\gamma r^*_j G(r^*_j, r^*_j) + 2(-1)^j \gamma r^*_j \mathcal{V}'(r^*_j) & \text{if } j = k \\ 2(-1)^{j+k} \gamma r^*_j G(r^*_j, r^*_k) & \text{if } j \neq k \end{cases} \tag{2.19}$$

It was shown in [25] that (2.19) is positive definite in T. For large γ $\mathbf{r}^*$ perturbs to $\mathbf{r}^0$, a strict local minimizer of J. Theorem 1.1 hence is a consequence of Theorem 2.2. In this paper we assume that the condition (2.14) is satisfied and hence u exists.

We denote the function $\mathcal{U}(\cdot;\mathbf{r}^0)$ by u^0 and set $v^0 = G_0[u^0 - \overline{u^0}]$. u^0 takes values 0 and 1, and it jumps between these two values at $r_1^0, r_2^0, ..., r_K^0$. The Γ-convergence theory asserts that u converges to u^0 in $L^2(D)$. Then there exist $r_1, r_2, ..., r_K$ such that $u(r_j) = 1/2$, $j = 1, 2, ..., K$, and $\mathbf{r} = (r_1, r_2, ..., r_K)^T \to \mathbf{r}^0$ as $\epsilon \to 0$. These r_j's are called the interfaces of u. We will see that they are the only interfaces.

We also need to know the asymptotic behavior of u. First we construct an *inner* expansion. Around each r_j we introduce the scaled variable $r = r_j + \epsilon t$ so to expand

$$u(r) = u(r_j + \epsilon t) = H_j(t) + \epsilon P_j(t) + \epsilon^2 Q_j(t) + ... \tag{2.20}$$

Correspondingly

$$v(r) = v(r_j) + \epsilon t v'(r_j) + ... \tag{2.21}$$

As we insert (2.20) and (2.21) into (1.8) we find the leading term

$$H_j(t) = H(t) \text{ if } j \text{ is odd}, \ H_j(t) = H(-t) \text{ if } j \text{ is even}. \tag{2.22}$$

The next term is $P_j(t)$ defined to be the solution of

$$-P'' + f'(H_j)P - \frac{H_j'}{r_j} + \xi_j = 0, \ P(0) = 0. \tag{2.23}$$

P_j is even. The constant ξ_j is chosen so that $-\frac{H_j'}{r_j} + \xi_j$ is perpendicular to H_j' for solvability. Therefore

$$\xi_j = \frac{(-1)^{j+1}}{r_j} \int_{\mathbf{R}} (H'(t))^2 \, dt = \frac{(-1)^{j+1}\tau}{r_j}. \tag{2.24}$$

In our rigorous setting of asymptotic expansions P_j depends on ϵ because r_j and ξ_j do so. This way we avoid expanding r_j. The third term in the inner expansion is $Q_j(t)$ which is the solution of

$$-Q'' + f'(H_j)Q - \frac{P_j'}{r_j} + \frac{tH_j'}{r_j^2} + \frac{f''(H_j)P_j^2}{2} + \gamma v'(r_j)t = 0, \ Q(0) = 0. \tag{2.25}$$

Q_j is odd. Again Q_j depends on ϵ, via r_j and $v'(r_j)$. We set the inner approximation of u near r_j to be

$$z_j(r) = H_j(\frac{r - r_j}{\epsilon}) + \epsilon P_j(\frac{r - r_j}{\epsilon}) + \epsilon^2 Q_j(\frac{r - r_j}{\epsilon}). \tag{2.26}$$

The outer approximation is done in one step. It is denoted by z and defined for all r not equal to $r_1, r_2, ..., r_K$ by the equation

$$f(z) + \epsilon\gamma v(r) - \eta = 0. \tag{2.27}$$

Since $\eta = O(\epsilon)$ and $v = O(1)$, facts proved in the appendix, z is chosen to be close to 0 or 1 on each (r_j, r_{j+1}) non-ambiguously, in agreement with the shape of u, i.e. z is close to 0 on $(0, r_1)$, close to 1 on (r_1, r_2), close to 0 on (r_2, r_3), etc.

The inner approximation is used in each $(r_j - \epsilon^\alpha, r_j + \epsilon^\alpha)$ where $\alpha \in (1/2, 1)$. The outer approximation is used in $(0, 1)\backslash(\cup_{j=1}^K (r_j - 2\epsilon^\alpha, r_j + 2\epsilon^\alpha))$. The inner approximation is matched

to the outer approximation in the matching intervals $(r_j - 2\epsilon^\alpha, r_j - \epsilon^\alpha)$ and $(r_j + \epsilon^\alpha, r_j + 2\epsilon^\alpha)$, $j = 1, 2, .., K$. Let χ_j be smooth cut-off functions so that

$$\chi_j(r) = \begin{cases} 0 & \text{if } r \notin (r_j - 2\epsilon^\alpha, r_j + 2\epsilon^\alpha) \\ 1 & \text{if } r \in (r_j - \epsilon^\alpha, r_j + \epsilon^\alpha) \end{cases},$$

and moreover $(\chi_j)_r = O(\epsilon^{-\alpha})$ and $(\chi_j)_{rr} = O(\epsilon^{-2\alpha})$ in $(r_j - 2\epsilon^\alpha, r_j - \epsilon^\alpha)$ and $(r_j + \epsilon^\alpha, r_j + 2\epsilon^\alpha)$.

We then glue the two approximations to form a uniform approximation

$$w(r) = \sum_{j=1}^{K} \chi_j z_j + (1 - \sum_{j=1}^{K} \chi_j) z. \tag{2.28}$$

Lemma 2.3 $w - u = o(\epsilon^2)$.

According to this lemma, whose proof is left to the appendix, the uniform approximation w is accurate up to order ϵ^2. This lemma also implies that the r_j's are the only interfaces of u.

To understand the stability of a spot or a ring solution in two dimensions we need to find the spectrum, which only contains eigenvalues, of the linearized operator L defined in (1.9). We separate variables in the polar coordinates to let

$$\varphi(x) = \varphi(r\cos\theta, r\sin\theta) = \sum_{m=0}^{\infty} \phi_m(r)(A_m \cos(m\theta) + B_m \sin(m\theta)). \tag{2.29}$$

After substituting (2.29) into (1.9), we deduce that $\varphi(x)$ is a linear combination of $\phi_m(r)\cos(m\theta)$ and $\phi_m(r)\sin(m\theta)$ for some nonnegative integer m. The corresponding eigenvalue λ is thus classified into $\lambda = \lambda_m$, $m = 0, 1, 2, ...$ The pair (λ_m, ϕ_m) satisfies the following equations.

1. If $m = 0$,

$$L_0\phi_0 := -\epsilon^2\phi_0'' - \frac{\epsilon^2}{r}\phi_0' + f'(u)\phi_0 - \overline{f'(u)\phi_0} + \epsilon\gamma G_0[\phi_0] = \lambda_0\phi_0,\ \phi'(0) = \phi'(1) = 0,\ \overline{\phi_0} = 0; \tag{2.30}$$

2. if $m \geq 1$,

$$L_m\phi_m := -\epsilon^2\phi_m'' - \frac{\epsilon^2}{r}\phi_m' + \frac{\epsilon^2 m^2}{r^2}\phi_m + f'(u)\phi_m + \epsilon\gamma G_m[\phi_m] = \lambda_m\phi_m,\ \phi_m(0) = \phi_m'(1) = 0. \tag{2.31}$$

The operator G_0 is defined in (2.6), and when $m \geq 1$ G_m is the inverse of the differential operator $-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{m^2}{r^2}$ with the Neumann boundary condition at $r = 1$ and the Dirichlet boundary condition at $r = 0$.

Lemma 2.4 *Let λ be an eigenvalue of L. Then* $\liminf_{\epsilon\to 0} \lambda \geq 0$.

Proof. Suppose that the lemma is false. We may assume that $\lim_{\epsilon\to 0} \lambda = \lambda^0 < 0$. Since λ is classified into λ_m, $m = 0, 1, 2, ...$, we consider the case that λ is one of λ_0. The case $m \geq 1$ may be handled similarly and we omit the proof.

Let ϕ be an eigenfunction of (2.30) associated with λ. Without the loss of generality we assume that $\|\phi\| = \phi(r_*) = 1$. First we claim that there is a r_j whose distance to r_* is of order $O(\epsilon)$. Otherwise $-\epsilon^2\phi''(r_*) \geq 0$ since r_* is a maximum; $-\frac{\epsilon^2}{r}\phi'(r_*) = 0$ whether or not r_* is on the boundary; $f'(u(r_*))\phi(r_*) > 0$ since $f'(u(r_*)) > 0$ outside any ϵ-neighborhood of r_j; $\overline{f'(u)\phi} = \overline{(f'(u) - f'(0))\phi} = O(\epsilon)$ by the uniform estimate of u in Lemma 2.3; $\epsilon\gamma G_0[\phi](r_*) = O(\epsilon)$, and $\lambda\phi(r_*) = \lambda < 0$. Then

$$-\epsilon^2\phi''(r_*) - \frac{\epsilon^2}{r}\phi'(r_*) + f'(u(r_*))\phi(r_*) - \overline{f'(u)\phi} + \epsilon\gamma G_0[\phi](r_*) > \lambda\phi(r_*),$$

and (2.30) is not satisfied at r_*.

If r_* is in a size $O(\epsilon)$ neighborhood of r_j, then $\phi(r_j + \epsilon t) \to \Phi \not\equiv 0$ in $C^2_{loc}(R)$, and Φ satisfies $-\Phi'' + f'(H_j)\Phi = \lambda^0\Phi$. However this equation has no nonzero, bounded solution when $\lambda^0 < 0$, since H_j is a minimizer of $E(U) := \int_R (\frac{1}{2}(U')^2 + W(U))\,dt$. Here U is in the class $W^{1,2}_{loc}(R)$ and $\lim_{t\to\pm\infty}(U(t) - H_j(t)) = 0$. $\square$

Hence to understand the stability of u we must analyze all the eigenvalues that tend to 0 as $\epsilon \to 0$. They are called the critical eigenvalues.

3 The critical eigenvalues λ_0

Recall $\mathbf{M}^0$ and g^0 defined in (2.15) and (2.16), respectively, and $\mathbf{e}^0_j$ with $\mathbf{e}^0_1$ defined in (2.17).

Theorem 3.1 *When ϵ is sufficiently small, there exist exactly K eigenpairs (λ_0, ϕ_0) of (2.30) with $\lambda_0 = o(1)$. One λ_0 is positive and of order ϵ. This λ_0 and its eigenfunction expand like*

$$\lambda_0 = \frac{2f'(0)\sum_{k=1}^K r^0_k}{\tau}\epsilon + o(\epsilon), \quad \phi_0 = \sum_{j=1}^K c_j(H'_j - \overline{H'_j}) + O(\epsilon|\mathbf{c}|),$$

where $\mathbf{c} = (c_1, c_2, ..., c_K)^T \to \mathbf{c}^0$ as ϵ tends to 0. $\mathbf{c}^0$ is a nonzero scaler multiple of $\mathbf{e}^0_1$.

The remaining $K-1$ λ_0's are positive and of order ϵ^2. Each of them and its corresponding eigenfunction expand like

$$\lambda_0 = \mu^0_0\epsilon^2 + o(\epsilon^2), \quad \phi_0 = \sum_{j=1}^K c_j(H'_j - \overline{H'_j} + \epsilon(P'_j - \overline{P'_j})) + O(\epsilon^2|\mathbf{c}|).$$

Let $\mathbf{c}^0 = \lim_{\epsilon\to 0}\mathbf{c}$. Then $\mathbf{c}^0 = \sum_{n=2}^K \tilde{c}^0_n\mathbf{e}^0_n$, and μ^0_0 and $(\tilde{c}^0_2, \tilde{c}^0_3, ..., \tilde{c}^0_K)^T$ form an eigenpair of the $K-1$ dimensional eigenvalue problem

$$\sum_{m=2}^K \tilde{c}^0_m g^0(\mathbf{M}^0\mathbf{e}^0_m, \mathbf{e}^0_n) = \mu^0_0\tau\tilde{c}^0_n, \quad n = 2, 3, ..., K.$$

We expect that the eigenfunctions associated with small eigenvalues may be approximately by combinations of

$$H'_j - \overline{H'_j} + \epsilon(P'_j - \overline{P'_j}). \tag{3.1}$$

Here H_j' is the derivative of $H_j = H_j(t)$ with respect to t evaluated at $t = \frac{r-r_j}{\epsilon}$. In this section we write (λ, ϕ) for an eigenpair (λ_0, ϕ_0). We decompose

$$\phi = \sum_{j=1}^{K} c_j(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})) + \phi^{\perp} \tag{3.2}$$

in the $L^2(D)$ space where $H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'}) \perp \phi^{\perp}$ for $j = 1, 2, ..., K$.

First we estimate

$$L_0(H_j' - \overline{H_j'}) \;=\; -\epsilon^2 (H_j')_{rr} - \frac{\epsilon^2}{r}(H_j')_r + f'(u)(H_j' - \overline{H_j'}) - \overline{f'(u)(H_j' - \overline{H_j'})} + \epsilon\gamma G_0[H_j' - \overline{H_j'}],$$

in which

$$\begin{aligned}\overline{f'(u)H_j'} &= 2\int_0^1 (f'(H_j) + \epsilon P_j f''(H_j))H_j' r\, dr + O(\epsilon^3)\\ &= 2\epsilon \int_R [f'(H_j)H_j' r_j + \epsilon t f'(H_j)H_j' + \epsilon P_j f''(H_j)H_j' r_j]\, dt + O(\epsilon^3) \;=\; O(\epsilon^3)\end{aligned} \tag{3.3}$$

since $\int_R f'(H_j)H_j'\, dt = \int_R t f'(H_j)H_j'\, dt = \int_R P_j f''(H_j)H_j'\, dt = 0$, ($t f'(H_j)H_j'$ and $P_j f''(H_j)H_j'$ are odd). Then

$$\begin{aligned}L_0(H_j' - \overline{H_j'}) &= (f'(u) - f'(H_j))H_j' - \frac{\epsilon}{r}H_j'' + (\overline{f'(u)} - f'(u))\overline{H_j'} + \epsilon^2\gamma(-1)^{j+1}G_0(r, r_j) + O(\epsilon^3)\\ &= \epsilon f''(H_j)P_j H_j' + \epsilon^2(f''(H_j)Q_j + \frac{f'''(H_j)P_j^2}{2})H_j' - \frac{\epsilon}{r}H_j''\\ &\quad + \epsilon^2\gamma(-1)^{j+1}G_0(r, r_j) + (\overline{f'(u)} - f'(u))\overline{H_j'} + O(\epsilon^3).\end{aligned}$$

By differentiating (2.23) we have

$$-P_j''' + f'(H_j)P_j' + f''(H_j)H_j'P_j - \frac{H_j''}{r_j} = 0.$$

Then

$$\begin{aligned}L_0(P_j' - \overline{P_j'}) &= -\epsilon^2(P_j')_{rr} - \frac{\epsilon^2}{r}(P_j')_r + f'(u)(P_j' - \overline{P_j'}) - \overline{f'(u)(P_j' - \overline{P_j'})} + \epsilon\gamma G_0[P_j' - \overline{P_j'}]\\ &= (f'(u) - f'(H_j))P_j' - f''(H_j)H_j'P_j + \frac{H_j''}{r_j} - \frac{\epsilon}{r}P_j'' + (\overline{f'(u)} - f'(u))\overline{P_j'} + O(\epsilon^2)\\ &= \epsilon f''(H_j)P_jP_j' - f''(H_j)H_j'P_j + \frac{H_j''}{r_j} - \frac{\epsilon}{r}P_j'' + (\overline{f'(u)} - f'(u))\overline{P_j'} + O(\epsilon^2),\end{aligned}$$

where we have used the fact

$$\overline{f'(u)P_j'} = 2\int_0^1 f'(u)P_j' r\, dr = 2\epsilon\int_R f'(H_j)P_j' r_j\, dt + O(\epsilon^2) = O(\epsilon^2) \tag{3.4}$$

since $f'(H_j)P_j'$ is odd. Therefore

$$\begin{aligned}
&L_0(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})) \\
&= \epsilon^2[(f''(H_j)Q_j + \frac{f'''(H_j)P_j^2}{2})H_j' + f''(H_j)P_jP_j' + (\frac{1}{\epsilon r_j} - \frac{1}{\epsilon r})H_j'' - \frac{P_j''}{r} + \gamma(-1)^{j+1}G_0(r, r_j)] \\
&\quad +(\overline{f'(u)} - f'(u))\overline{H_j' + \epsilon P_j'} + O(\epsilon^3).
\end{aligned}$$

On the other hand

$$\overline{H_j'} = 2\int_0^1 H_j' r\, dr = 2\epsilon \int_R H_j'(t)(r_j + \epsilon t)\, dt = 2\epsilon r_j \int_R H_j'(t)\, dt + 2\epsilon^2 \int_R H_j'(t) t\, dt = 2\epsilon(-1)^{j+1} r_j$$

since $H_j'(t)t$ is odd, and

$$\overline{P_j'} = 2\int_0^1 P_j' r\, dr = 2\epsilon r_j \int_R P_j'\, dt + O(\epsilon^2) = O(\epsilon^2)$$

since P_j' is odd. We find

$$\overline{H_j' + \epsilon P_j'} = 2\epsilon(-1)^{j+1} r_j + O(\epsilon^3). \tag{3.5}$$

Hence we deduce that

$$\begin{aligned}
&L_0(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})) \\
&= \epsilon^2[(f''(H_j)Q_j + \frac{f'''(H_j)P_j^2}{2})H_j' + f''(H_j)P_jP_j' + (\frac{1}{\epsilon r_j} - \frac{1}{\epsilon r})H_j'' - \frac{P_j''}{r} + \gamma(-1)^{j+1}G_0(r, r_j)] \\
&\quad +2\epsilon(-1)^{j+1} r_j(\overline{f'(u)} - f'(u)) + O(\epsilon^3).
\end{aligned} \tag{3.6}$$

Note that in (3.6)

$$(\frac{1}{\epsilon r_j} - \frac{1}{\epsilon r})H_j'' = \frac{tH''(t)}{r_j r} = O(1).$$

Rewrite the equation $L_0\phi = \lambda\phi$ as

$$\sum_{j=1}^K c_j L_0(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})) + L_0\phi^\perp = \lambda(\sum_{j=1}^K c_j(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})) + \phi^\perp). \tag{3.7}$$

Then $\phi^\perp$ satisfies

$$L_0\phi^\perp = O(\epsilon|\sum_{j=1}^K (-1)^j r_j c_j|) + O(\epsilon^2)|\mathbf{c}| + O(|\lambda|)(|\mathbf{c}| + \|\phi^\perp\|). \tag{3.8}$$

Here $\|\phi^\perp\|$ is the L^∞ norm of $\phi^\perp$ on $(0, 1)$. The following lemma estimates $\phi^\perp$.

Lemma 3.2 *There exists $C > 0$ independent of ϵ such that for all ψ in the domain of L_0 and $\psi \perp H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})$, $j = 1, 2, ..., K$, $\|\psi\| \le C\|L_0\psi\|$.*

Proof. Suppose that the lemma is false. There exist ψ and some r_* such that $\|\psi\| = \psi(r_*) = 1$, $\psi \perp H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})$, $j = 1, 2, ..., K$, and $L_0\psi = o(1)$. Then r_* must lie in a neighborhood of r_j for some j. The size of this neighborhood must be of order ϵ. Otherwise we argue as in the proof of Lemma 2.4: $-\epsilon^2\psi''(r_*) \geq 0$; $-\frac{\epsilon^2}{r}\psi'(r_*) = 0$; $\epsilon\gamma G_m[\psi](r_*) = O(\epsilon)$; $\overline{f'(u)\psi} = \overline{(f'(u) - f'(0))\psi} = O(\epsilon)$; and $f'(u)\psi(r_*)$ is positive and bounded away from 0 independent of ϵ. Then the equation $L_0\psi = o(1)$ is not satisfied at r_*.

So let us assume that r_* is in a neighborhood, of size ϵ, of r_j. Then $\psi(r_j + \epsilon t) \to \Psi_0(t)$ in $C^2_{loc}(R)$ as ϵ tends to 0. Ψ_0 satisfies $-\Psi_0'' + f'(H_j)\Psi_0 = 0$. Therefore $\Psi_0 = cH_j'$ for some constant $c \neq 0$. On the other hand if we denote the inner product in $L^2(D)$ by $\langle \cdot, \cdot \rangle$, then $\psi \perp H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})$ implies

$$0 = \langle \psi, H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'}) \rangle = 2\pi\epsilon c r_j \int_R (H')^2 \, dt + o(\epsilon),$$

which is possible only if $c = 0$. $\square$

We obtain by Lemma 3.2 that

$$\phi^\perp = O(\epsilon|\sum_{j=1}^K (-1)^j r_j c_j|) + O(\epsilon^2)|\mathbf{c}| + O(|\lambda|)(|\mathbf{c}| + \|\phi^\perp\|)$$

which implies, since $\lambda = o(1)$,

$$\phi^\perp = O(\epsilon|\sum_{j=1}^K (-1)^j r_j c_j|) + O(\epsilon^2)|\mathbf{c}| + O(|\lambda|)|\mathbf{c}|. \tag{3.9}$$

We multiply (3.7) by $H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'})$ and integrate with respect to $2\pi r \, dr$ over $(0, 1)$ to find the equations

$$\sum_{j=1}^K \langle c_j L_0(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})), H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'}) \rangle + \langle \phi^\perp, L_0(H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'})) \rangle$$

$$= \lambda \sum_{j=1}^K c_j \langle H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'}), H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'}) \rangle.$$

In these equations

$$\langle \phi^\perp, L_0(H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'})) \rangle = O(\|\phi^\perp\| \cdot \|L_0(H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'}))\|_1),$$

where $\| \cdot \|_1$ denotes the $L^1(D)$ norm. By (3.6) we find

$$\|L_0(H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'}))\|_1 = O(\epsilon^2).$$

Then by (3.9) we deduce the equations

$$\sum_{j=1}^K c_j \langle L_0(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})), H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'}) \rangle + O(\epsilon^3|\sum_{j=1}^K (-1)^j r_j c_j|) + O(\epsilon^4)|\mathbf{c}| + O(\epsilon^2|\lambda|)|\mathbf{c}|$$

$$= \lambda \sum_{j=1}^{K} c_j \langle H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'}), H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'})\rangle, \tag{3.10}$$

for $k = 1, 2, ..., K$. The inner products in (3.10) are given in the next lemma.

Lemma 3.3 *In the equations (3.10)*

1. $\langle H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'}), H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'})\rangle = 2\pi\epsilon r_k \tau \delta_{jk} + O(\epsilon^2)$;
2. $\langle L_0(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})), H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'})\rangle$

$$= 4\pi\epsilon^2(-1)^{k+j} r_j r_k \overline{f'(u)} + 2\pi\epsilon^3 r_k \{\delta_{jk}[-\frac{\tau}{r_k^2} + (-1)^k \gamma v'(r_k)] + \gamma(-1)^{k+j} G_0(r_k, r_j)\} + O(\epsilon^4).$$

Proof. 1. is obvious. To prove 2. we note that P' decays exponentially fast. Then (3.6) implies that

$$\begin{aligned}
&\langle L_0(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})), H_k' - \overline{H_k'} + \epsilon(P_k' - \overline{P_k'})\rangle \\
&= \langle L_0(H_j' - \overline{H_j'} + \epsilon(P_j' - \overline{P_j'})), H_k' + \epsilon P_k'\rangle \\
&= \epsilon^2 \langle (f''(H_j)Q_j + \frac{f'''(H_j)P_j^2}{2})H_j' + f''(H_j)P_jP_j' + \frac{tH_j''}{r_j r} - \frac{P_j''}{r} + \gamma(-1)^{j+1}G_0(r, r_j), H_k' + \epsilon P_k'\rangle \\
&\quad + 2\epsilon(-1)^{j+1} r_j \langle \overline{f'(u)} - f'(u), H_k' + \epsilon P_k'\rangle + O(\epsilon^4) \\
&= \epsilon^2 \langle (f''(H_j)Q_j + \frac{f'''(H)P_j^2}{2})H_j' + f''(H)P_jP_j' + \frac{t}{r_j r}H_j'' - \frac{P_j''}{r} + \gamma(-1)^{j+1}G_0(r, r_j), H_k'\rangle \\
&\quad + 2\epsilon\pi(-1)^{j+1} r_j \overline{f'(u)}\ \overline{H_k' + \epsilon P_k'} + O(\epsilon^4) \qquad (3.11) \\
&= 2\pi\epsilon^3 r_k \{\delta_{jk} \int_R [(f''(H_k)Q_k + \frac{f'''(H_k)P_k^2}{2})H_k' + f''(H_k)P_kP_k' + \frac{tH_k''}{r_k^2} - \frac{P_k''}{r_k}]H_k'\, dt \\
&\quad + \gamma(-1)^{k+j}G_0(r_k, r_j)\} + 4\epsilon^2\pi(-1)^{k+j} r_j r_k\ \overline{f'(u)} + O(\epsilon^4). \qquad (3.12)
\end{aligned}$$

Note that we have again used (3.3) and (3.4) to reach (3.11), and used (3.5) to reach (3.12). To find the integral in (3.12), we differentiate (2.25) to obtain

$$-Q_k''' + f'(H_k)Q_k' + f''(H_k)H'Q_k - \frac{P_k''}{r_k} + \frac{H_k' + tH_k''}{r_k^2} + \frac{f'''(H_k)H_k'P_k^2}{2} + f''(H_k)P_kP_k' + \gamma v'(r_k) = 0.$$

Multiplying by H_k' and integrating over $(-\infty, \infty)$ yield

$$\int_R [f''(H_k)Q_k(H_k')^2 - \frac{P_k''H_k'}{r_k} + \frac{(H_k')^2 + tH_k''H_k'}{r_k^2} + \frac{f'''(H_k)P_k^2(H_k')^2}{2} + f''(H_k)P_kP_k'H_k']\, dt$$
$$+ (-1)^{k+1}\gamma v'(r_k) = 0.$$

The integral in (3.12) now becomes

$$-\frac{1}{r_k^2}\int_R (H')^2\, dt + (-1)^k \gamma v'(r_k). \quad \square$$

With Lemma 3.3 we will write (3.10) in the vector form. We view $\mathbf{c} = (c_1, c_2, ..., c_K)^T$ as a column vector in R^K. Let $\mathbf{R}$ be a K by K rank one matrix:

$$\mathbf{R} = 2\overline{f'(u)} \begin{bmatrix} r_1 & -r_2 & r_3 & -r_4 & ... & (-1)^{1+K} r_K \\ -r_1 & r_2 & -r_3 & r_4 & ... & (-1)^{2+K} r_K \\ r_1 & -r_2 & r_3 & -r_4 & ... & (-1)^{3+K} r_K \\ ... & & & & & \\ (-1)^{K+1} r_1 & (-1)^{K+2} r_2 & (-1)^{K+3} r_3 & (-1)^{K+4} r_4 & ... & r_K \end{bmatrix}, \tag{3.13}$$

and $\mathbf{M}$ be a K by K matrix whose kj entry is

$$M_{kj} = \delta_{jk}(-\frac{\tau}{r_k^2} + (-1)^k \gamma v'(r_k)) + \gamma(-1)^{k+j} G_0(r_k, r_j).$$

In R^K we define a non-standard inner product g by

$$g(\mathbf{A}, \mathbf{B}) = \sum_{j=1}^{K} A_j B_j r_j, \quad \mathbf{A} = (A_1, A_2, ..., A_K)^T, \ \mathbf{B} = (B_1, B_2, ..., B_K)^T. \tag{3.14}$$

The matrices $\mathbf{R}$ and $\mathbf{M}$ represent symmetric linear operators on R^K with respect to this inner product. The symmetry of $\mathbf{M}$ under g is a consequence of the fact that $r_k G_0(r_k, r_j) = r_j G_0(r_j, r_k)$. Let $\{\mathbf{e}_n\}$ be an orthonormal basis under g in which

$$\mathbf{e}_1 = \sqrt{r_1 + r_2 + ... + r_K}\ (-1, 1, -1, 1, ..., 1)^T. \tag{3.15}$$

$\mathbf{e}_1$ is an eigenvector vector of $\mathbf{R}$ with eigenvalue $2\overline{f'(u)}(r_1 + r_2 + ... + r_K)$. $\mathbf{e}_2, \mathbf{e_3}, ..., \mathbf{e}_K$ span the eigenspace of the eigenvalue 0, which has multiplicity $K - 1$.

Now we rewrite (3.10) as

$$\epsilon^2 \mathbf{R}\mathbf{c} + \epsilon^3 \mathbf{M}\mathbf{c} + O(\epsilon^3 |g(\mathbf{c}, \mathbf{e}_1)|) + O(\epsilon^4 |\mathbf{c}|) + O(\epsilon^2 |\lambda|\ |\mathbf{c}|) = \epsilon\tau\lambda\mathbf{c}. \tag{3.16}$$

In (3.16) $|\mathbf{c}|$, the norm of $\mathbf{c}$, may be understood as either the norm under the standard inner product or the norm under g, because the two norms are equivalent uniformly in ϵ.

We must consider two cases:

$$1.\ \ g(\frac{\mathbf{c}}{|\mathbf{c}|}, \mathbf{e}_1) \not\to 0; \qquad 2.\ \ g(\frac{\mathbf{c}}{|\mathbf{c}|}, \mathbf{e}_1) = o(1).$$

Of course when $K = 1$, the second case does not occur.

In the first case we use a rough form of (3.16):

$$\epsilon^2 \mathbf{R}\mathbf{c} + O(\epsilon^3 |\mathbf{c}|) + O(\epsilon^2 |\lambda|\ \mathbf{c}|) = \epsilon\tau\lambda\mathbf{c}. \tag{3.17}$$

Take the g-inner product of (3.17) and $\mathbf{e}_1$:

$$2\epsilon^2 \overline{f'(u)} (\sum_{j=1}^{K} r_j) g(\mathbf{c}, \mathbf{e}_1) + O(\epsilon^3 |\mathbf{c}|) + O(\epsilon^2 |\lambda|\ \mathbf{c}|) = \epsilon\tau\lambda g(\mathbf{c}, \mathbf{e}_1). \tag{3.18}$$

Since $g(\frac{\mathbf{c}}{|\mathbf{c}|}, \mathbf{e}_1) \not\to 0$, (3.18) implies that

$$\lambda = \frac{\epsilon}{\tau}(\sum_{k=1}^{K} 2r_k \overline{f'(u)}) + O(\epsilon^2). \tag{3.19}$$

This eigenvalue is positive for small ϵ and of order ϵ. Consequently (3.9) implies that

$$\phi^{\perp} = O(\epsilon|\mathbf{c}|). \tag{3.20}$$

If we take the g-inner product of (3.17) and $\mathbf{e}_n$, $n \geq 2$, then

$$g(\mathbf{c}, \mathbf{e}_n) = O(\epsilon|\mathbf{c}|), \quad n \geq 2. \tag{3.21}$$

The asymptotic properties of λ and ϕ in the first case follows from (3.19), (3.20), and (3.21).

In the second case we take the g-inner product of (3.16) and $\mathbf{e}_n$, $n \geq 2$, to deduce

$$\epsilon^3 g(\mathbf{M}\mathbf{c}, \mathbf{e}_n) + O(\epsilon^3|g(\mathbf{c}, \mathbf{e}_1)|) + O(\epsilon^4|\mathbf{c}|) + O(\epsilon^2|\lambda|\ |\mathbf{c}|) = \epsilon\tau\lambda g(\mathbf{c}, \mathbf{e}_n), \quad n = 2, 3, ..., K. \tag{3.22}$$

Note that $g(\frac{\mathbf{c}}{|\mathbf{c}|}, \mathbf{e}_1) = o(1)$ and (3.22) imply that $\lambda = O(\epsilon^2)$.

Then we take the g-inner product of (3.16) and $\mathbf{e}_1$:

$$2\epsilon^2 \overline{f'(u)}(\sum_{j=1}^{K} r_j) g(\mathbf{c}, \mathbf{e}_1) + O(\epsilon^3|\mathbf{c}|) + O(\epsilon^3|g(\mathbf{c}, \mathbf{e}_1)|) + O(\epsilon^2|\lambda|\ |\mathbf{c}|) = \epsilon\tau\lambda g(\mathbf{c}, \mathbf{e}_1). \tag{3.23}$$

(3.23) and $\lambda = O(\epsilon^2)$ imply that

$$g(\mathbf{c}, \mathbf{e}_1) = O(\epsilon|\mathbf{c}|), \tag{3.24}$$

which turns (3.9) to

$$\phi^{\perp} = O(\epsilon^2|\mathbf{c}|), \tag{3.25}$$

and (3.22) is simplified to

$$\epsilon^3 g(\mathbf{M}\mathbf{c}, \mathbf{e}_n) + O(\epsilon^4|\mathbf{c}|) = \epsilon\tau\lambda g(\mathbf{c}, \mathbf{e}_n), \quad n = 2, 3, ..., K. \tag{3.26}$$

We pass limit in (3.26) and (3.24). Let $\mathbf{M}^0 = \lim_{\epsilon\to 0} \mathbf{M}$, $\mathbf{R}^0 = \lim_{\epsilon\to 0} \mathbf{R}$, $g^0 = \lim_{\epsilon\to 0} g$, $\mathbf{e}_j^0 = \lim_{\epsilon\to 0} \mathbf{e}_j$, and $\mu_0^0 = \lim_{\epsilon\to 0} \frac{\lambda}{\epsilon^2}$, and $\mathbf{c}^0 = \lim_{\epsilon\to 0} \mathbf{c}$, where $|\mathbf{c}^0| \neq 0$. Then

$$g^0(\mathbf{M}^0\mathbf{c}^0, \mathbf{e}_n^0) = \mu_0^0 \tau g^0(\mathbf{c}^0, \mathbf{e}_n^0), \quad (n = 2, 3, ..., K), \quad g^0(\mathbf{c}^0, \mathbf{e}_1^0) = 0. \tag{3.27}$$

The second equation implies that we can decompose $\mathbf{c}^0$ as

$$\mathbf{c}^0 = \sum_{n=2}^{K} \tilde{c}_n^0 \mathbf{e}_n^0. \tag{3.28}$$

The first equation in (3.27) becomes

$$\sum_{m=2}^{K} \tilde{c}_m^0 g^0(\mathbf{M}^0\mathbf{e}_m^0, \mathbf{e}_n^0) = \mu_0^0 \tau \tilde{c}_n^0, \quad n = 2, 3, ..., K. \tag{3.29}$$

Here (3.29) is a $K-1$ dimensional eigenvalue problem from which we find $K-1$ pairs of μ_0^0 and $(\tilde{c}_2^0, \tilde{c}_3^0, ..., \tilde{c}_K^0)$. This proves the asymptotic properties of λ and ϕ in the second case.

As we have explained in Section 2 that the construction of u via the Γ-convergence theory assumes that (2.12) is positive definite in T. The paragraph after (2.12) shows that this condition, (2.14), is equivalent to the condition that μ_0^0 in (3.29) are all positive. Hence λ_0 are all positive when ϵ is sufficiently small.

In summary we have proved that if (λ_0, ϕ_0) is an eigenpair of (2.30) with the property $\lambda_0 = o(1)$ then λ_0 and ϕ_0 must possess the asymptotic properties described in Theorem 3.1. We still need to show that there indeed exist exactly K eigenpairs of (2.30) with the properties. The proof of this fact uses some ideas from the linear perturbation theory. Not to prolong this section we omit the proof. Instead we will give a full proof in the next section for the $m \geq 1$ case, which is similar to the one for the $m = 0$ case.

4 The critical eigenvalues λ_m

Theorem 4.1 *When ϵ is sufficiently small, there exist exactly K eigenpairs (λ_m, ϕ_m) of (2.31) with $\lambda_m = o(1)$. Each λ_m and ϕ_m have the asymptotic expansion*

$$\lambda_m = \epsilon^2 \mu_m^0 + o(\epsilon^2), \;\; \phi_m = \sum_{j=1}^{K} c_j (H_j' + \epsilon P_j') + O(\epsilon^2 |\mathbf{c}|).$$

μ_m^0 and the limit $\mathbf{c}^0 = \lim_{\epsilon \to 0}(c_1, c_2, ..., c_K)$ form an eigenpair of the K-dimensional eigenvalue problem

$$\{\frac{(m^2-1)\tau}{(r_k^0)^2} + (-1)^k \gamma (v^0)'(r_k^0)\} c_k^0 + \gamma \sum_{j=1}^{K} (-1)^{k+j} G_m(r_k^0, r_j^0) c_j^0 = \mu_m^0 \tau c_k^0, \quad k = 1, 2, ..., K.$$

G_m is defined after (2.31): $G_m(r,s) = G_m[\delta(\cdot - s)](r)$. More explicitly

$$G_m(r,s) = \begin{cases} (\frac{s^{1-m}}{2m} + \frac{s^{1+m}}{2m}) r^m & \text{if} \quad r < s \\ \frac{s^{1+m}}{2m}(r^m + r^{-m}) & \text{if} \quad r \geq s \end{cases}. \tag{4.1}$$

Note that $G_m(r,s)$ is not symmetric in r and s, although $rG_m(r,s)$ is. So with respect to g^0 the matrix in the K dimensional eigenvalue problem represents a symmetric operator.

In the proof of Theorem 4.1 we write (λ, ϕ) for (λ_m, ϕ_m) for simplicity. We decompose in $L^2(D)$

$$\phi(r) = \sum_{j=1}^{K} c_j (H_j' + \epsilon P_j') + \phi^{\perp}, \quad \text{where } \phi^{\perp} \perp H_j' + \epsilon P_j' \; (j = 1, 2, ..., K). \tag{4.2}$$

First we compute

$$L_m H_j' \;=\; -\epsilon^2 (H_j')_{rr} - \frac{\epsilon^2}{r}(H_j')_r + \frac{\epsilon^2 m^2}{r^2} H_j' + f'(u) H_j' + \epsilon \gamma G_m[H_j']$$

$$
\begin{aligned}
&= (f'(u)-f'(H_j))H_j' - \frac{\epsilon}{r}H_j'' + \frac{\epsilon^2 m^2}{r^2}H_j' + \epsilon^2\gamma(-1)^{j+1}G_m(r,r_j) + O(\epsilon^3) \\
&= \epsilon f''(H_j)P_jH_j' + \epsilon^2(f''(H_j)Q_j + \frac{f'''(H_j)P_j^2}{2})H_j' - \frac{\epsilon}{r}H_j'' + \frac{\epsilon^2 m^2}{r^2}H_j' \\
&\quad +\epsilon^2\gamma(-1)^{j+1}G_m(r,r_j) + O(\epsilon^3).
\end{aligned}
$$

By differentiating (2.23) we have

$$
-P_j''' + f'(H_j)P_j' + f''(H_j)H_j'P_j - \frac{H_j''}{r_j} = 0.
$$

Then

$$
\begin{aligned}
L_mP_j' &= -\epsilon^2(P_j')_{rr} - \frac{\epsilon^2}{r}(P_j')_r + \frac{\epsilon^2 m^2}{r^2}P_j' + f'(u)P_j' + \epsilon\gamma G_m[P_j'] \\
&= (f'(u)-f'(H_j))P_j' - f''(H_j)H_j'P_j + \frac{H_j''}{r_j} - \frac{\epsilon}{r}P_j'' + O(\epsilon^2) \\
&= \epsilon f''(H_j)P_jP_j' - f''(H_j)H_j'P_j + \frac{H_j''}{r_j} - \frac{\epsilon}{r}P_j'' + O(\epsilon^2).
\end{aligned}
$$

Therefore

$$
L_m(H_j' + \epsilon P_j') =
$$

$$
\epsilon^2[(f''(H_j)Q_j + \frac{f'''(H_j)P_j^2}{2})H_j' + f''(H_j)P_jP_j' + \frac{tH_j''}{r_jr} + \frac{m^2H_j'}{r^2} - \frac{P_j''}{r} + \gamma(-1)^{j+1}G_m(r,r_j)] + O(\epsilon^3). \tag{4.3}
$$

In particular

$$
L_m(H_j' + \epsilon P_j') = O(\epsilon^2). \tag{4.4}
$$

Rewrite the equation $L_m\phi = \lambda\phi$ as

$$
\sum_{j=1}^{K} c_jL_m(H_j' + \epsilon P_j') + L_m\phi^\perp = \lambda(\sum_{j=1}^{K} c_j(H_j' + P_j') + \phi^\perp). \tag{4.5}
$$

Then $\phi^\perp$ satisfies

$$
L_m\phi^\perp = O(\epsilon^2)|\mathbf{c}| + O(|\lambda|)(|\mathbf{c}| + \|\phi^\perp\|).
$$

Lemma 4.2 *There exists $C > 0$ independent of ϵ such that for all $\psi \perp H_j' + \epsilon P_j'$, $j = 1, 2, ..., K$, $\|\psi\| \le C\|L_m\psi\|$.*

The proof of this lemma is similar to that of Lemma 3.2, so we omit it. We obtain by Lemma 4.2 that

$$
\phi^\perp = O(\epsilon^2)|\mathbf{c}| + O(|\lambda|)(|\mathbf{c}| + \|\phi^\perp\|)
$$

which implies, since $\lambda = o(1)$, that

$$
\phi^\perp = O(\epsilon^2)|\mathbf{c}| + O(|\lambda|)|\mathbf{c}|. \tag{4.6}
$$

We multiply (4.5) by $H_k' + \epsilon P_k'$ and integrate with respect to $2\pi r\, dr$ over $(0,1)$. Then

$$\sum_{j=1}^{K} \langle c_j L_m(H_j' + \epsilon P_j'), H_k' + \epsilon P_k' \rangle + \langle \phi^{\perp}, L_m(H_k' + \epsilon P_k') \rangle = \lambda \sum_{j=1}^{K} c_j \langle H_j' + \epsilon P_j', H_k' + \epsilon P_k' \rangle,$$

which, by (4.6) and (4.4), may be written as

$$\sum_{j=1}^{K} c_j \langle L_m(H_j' + \epsilon P_j'), H_k' + \epsilon P_k' \rangle + O(\epsilon^4)|\mathbf{c}| + O(\epsilon^2|\lambda|)|\mathbf{c}| = \lambda \sum_{j=1}^{K} c_j \langle H_j' + \epsilon P_j', H_k' + \epsilon P_k' \rangle \tag{4.7}$$

for $k = 1, 2, ..., K$.

Lemma 4.3 *In the equations (4.7)*

1. $\langle H_j' + \epsilon P_j', H_k' + \epsilon P_k' \rangle = 2\pi\epsilon r_k \tau \delta_{jk} + O(\epsilon^2)$,
2. $\langle L_m(H_j' + \epsilon P_j'), H_k' + \epsilon P_k' \rangle$

$$= 2\pi\epsilon^3 r_k \{\delta_{jk}[\frac{(m^2-1)\tau}{r_k^2} + (-1)^k \gamma v'(r_k)] + \gamma(-1)^{k+j} G_m(r_k, r_j)\} + O(\epsilon^4).$$

Proof. 1. is obvious. To prove 2. we note that P' decays exponentially fast. Then (4.3) implies that

$$\begin{aligned} \langle L_m(H_j' + \epsilon P_j'), H_k' + \epsilon P_k' \rangle &= \langle L_m(H_j' + \epsilon P_j'), H_k' \rangle + O(\epsilon^4) \\ &= 2\pi\epsilon^3 r_k \{\delta_{jk} \int_R [(f''(H_k)Q_k + \frac{f'''(H_k)P_k^2}{2})H_k' + f''(H_k)P_k P_k' + \frac{tH_k''}{r_k^2} + \frac{m^2 H_k'}{r_k^2} - \frac{P_k''}{r_k}]H_k'\, dt \\ &\quad + \gamma(-1)^{k+j} G_m(r_k, r_j)\} + O(\epsilon^4). \end{aligned}$$

To find the integral in the last line we follow the argument used in the proof of Lemma 3.3. □

This lemma simplifies (4.7) to

$$(\frac{(m^2-1)\tau}{r_k^2} + (-1)^k \gamma v'(r_k))c_k + \gamma \sum_{j=1}^{K} (-1)^{k+j} G_m(r_k, r_j) c_j + O(\epsilon|\mathbf{c}|) + O(\frac{|\lambda|\ |\mathbf{c}|}{\epsilon}) = \frac{\tau\lambda c_k}{\epsilon^2}. \tag{4.8}$$

Hence λ is of order ϵ^2. (4.6) now becomes

$$\phi^{\perp} = O(\epsilon^2|\mathbf{c}|). \tag{4.9}$$

After passing limit in (4.8) we deduce the asymptotic properties in Theorem 4.1 for λ and ϕ.

We have proved that if (λ_m, ϕ_m) is an eigenpair associated with m with $\lambda = o(1)$, then it must have the asymptotic behavior described in Theorem 4.1. To complete the proof of the theorem we proceed to show that there exist exactly K simple eigenpairs of (2.31) with the properties.

Let F be the linear subspace spanned by critical eigenfunction. It is defined unambiguously by $F = \text{span}\{\phi \in L^2(0,1) : L_m(\phi) = \lambda\phi,\ |\lambda| < \epsilon^{1/2}\}$. Since the critical eigenvalues of L_m are of order ϵ^2, F includes all the critical eigenfunctions.

First dim F, the dimension of F, is at most K. Suppose that this is not the case. There exist two distinct eigenpairs (λ, ϕ) and (λ', ϕ') with the same asymptotic behavior. That is

$$\lambda = \epsilon^2\eta + o(\epsilon^2),\ \lambda' = \epsilon^2\eta + o(\epsilon^2),$$

$$\phi = \sum_j c_j(H'_j + \epsilon P'_j) + \psi,\ \phi' = \sum_j c'_j(H'_j + \epsilon P'_j) + \psi',$$

$$\lim_{\epsilon\to 0} c_j = \lim_{\epsilon\to 0} c'_j = c^0_j.$$

But the two eigenfunctions must be orthogonal, so

$$0 = \langle \phi, \phi' \rangle = 2\epsilon\pi g^0(\mathbf{c}^0, \mathbf{c}^0)\int_{-\infty}^{\infty} (H'(t))^2\, dt + o(\epsilon)|\mathbf{c}^0|^2.$$

This is obviously impossible when ϵ is sufficiently small.

Next dim F is at least K. Suppose otherwise that dim $F < K$. Define a subspace of $L^2(0,1)$: $S = \text{span}\{\sum_j c^0_j(H'_j+\epsilon P'_j)\}$ where c^0_j are the K eigenvectors of the K-dimensional eigenvalue problem in the statement of the theorem. We use a perturbation argument. The asymmetric distance between the closed subspaces S and F is

$$d(S, F) = \sup\{d(\varphi, F) : \varphi \in S, \|\varphi\|_2 = 1\}$$

where $d(x, F) = \inf\{\|x - y\|_2 : y \in F\}$. Since dim $F <$ dim S, there exists $\sum_j b^0_j(H'_j + \epsilon P'_j) \in S$ such that for every eigenvector in F which may be written as $\sum_j c_j(H'_j + \epsilon P'_j) + \psi$ with $\|\psi\| = O(\epsilon^2|\mathbf{c}|)$ and $g^0(\frac{\mathbf{c}}{|\mathbf{c}|}, \frac{\mathbf{b}^0}{|\mathbf{b}^0|}) = o(1)$. Then straight calculations show that

$$\langle \frac{\sum_j c_j(H'_j + \epsilon P'_j) + \psi}{\|\sum_j c_j(H'_j + \epsilon P'_j) + \psi\|_2}, \frac{\sum_j b^0_j(H'_j + \epsilon P'_j)}{\|\sum_j b^0_j(H'_j + \epsilon P'_j)\|_2} \rangle = o(1).$$

So if we use $\varphi = \frac{\sum_j b^0_j(H'_j+\epsilon P'_j)}{\|\sum_j b^0_j(H'_j+\epsilon P'_j)\|_2}$, $d(\varphi, F) = 1 - o(1)$ and $d(S, F) = 1 - o(1)$. The following lemma due to Helffer and Sjöstrand [12] will give us a contradiction.

Lemma 4.4 *Let L be a self-adjoint operator on a Hilbert space H, Q a compact interval in $(-\infty, \infty)$ and $e_1, e_2, ..., e_K$ normalized linearly independent elements in the domain of L. Assume that the following are true.*

1. *$L(e_k) = p_k e_k + r_k$, $\|r_k\|_H \le \epsilon'$ and $p_k \in Q$, $k = 1, 2, ..., K$.*
2. *There is $\omega > 0$ so that Q is ω-isolated in the spectrum of L, i.e. $(\sigma(L)\backslash Q) \cap (Q + (-\omega, \omega)) = \emptyset$.*

Then $d(S, F) \le \dfrac{K^{1/2}\epsilon'}{\omega\kappa^{1/2}}$ where $S = span\{e_1, ..., e_K\}$, $F =$ the closed subspace associated to $\sigma(L) \cap Q$, and $\kappa =$ the smallest eigenvalue of the matrix $[\langle e_j, e_k\rangle]$.

a	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\hat{m}$	19	9	6	4	3, 4	3	2	2	2
$\hat{\gamma}$	2468.56	356.23	123.86	64.69	42.67	30.38	27.76	28.23	56.61

Table 1: The value of $\hat{\gamma}$ for various a and the corresponding mode $\hat{m}$ of the principal eigenvalue $\lambda_{\hat{m}}$ which vanishes up to order ϵ^2. Here $\tau = \sqrt{2}/12$.

Here we take $L = L_m$, each e_k is normalized and proportional to $\sum_j c_j^0(H_j' + \epsilon P_j')$ for each one of the K vectors c^0, and S, F as before. ω and κ are positive and bounded away from 0 as $\epsilon \to 0$. Set $p_k = \eta\epsilon^2$ and $Q = [-\epsilon^{1/2}, \epsilon^{1/2}]$. From (4.3) we find

$$L_m(\sum_j c_j^0(H_j' + \epsilon P_j')) - p_k \sum_j c_j^0(H_j' + \epsilon P_j') = O(\epsilon^2|\mathbf{c}^0|),$$

and on the other hand $\|\sum_j c_j^0(H_j' + \epsilon P_j')\|_2 \sim \epsilon^{1/2}|\mathbf{c}^0|$. Therefore $\|r_k\|_2 = O(\epsilon^{3/2})$. Consequently $d(S, F) = o(1)$, a contradiction.

5 The cases of $K = 1$ and $K = 2$

We know from Theorem 1.1 that the spot solution ($K = 1$) exists for all γ. However the stability of the solution in two dimensions depends on γ. For small ϵ, the spot solution is stable if γ is small and unstable if γ is large. More precisely we have

Theorem 5.1 *Let $K = 1$. There exists $\hat{\gamma} > 0$ such that when $\gamma \in (0, \hat{\gamma})$, there exists $\hat{\epsilon}$ such that for every $\epsilon \in (0, \hat{\epsilon})$ all $\lambda_m > 0$, i.e. the spot solution u is stable. On the other hand if $\gamma > \hat{\gamma}$, there exist $\tilde{\epsilon} > 0$ and $m \geq 2$ such that for all $\epsilon \in (0, \tilde{\epsilon})$, $\lambda_m < 0$, i.e. u is unstable.*

Proof. Theorem 3.1 shows that when $K = 1$, there is only one λ_0 with the property $\lambda = o(1)$. This λ_0 is positive and of order ϵ for all γ if ϵ is sufficiently small.

When $m = 1$, in Theorem 4.1:

$$\gamma\{-(v^0)'(r_1^0) + G_1(r_1^0, r_1^0)\} = \mu_1^0\tau. \tag{5.1}$$

According to (4.1), $G_1(r_1^0, r_1^0) = ((r_1^0)^3 + r_1^0)/2$. When $K = 1$, $a = 1 - (r_1^0)^2$ by (2.7) and $(v^0)'(r_1^0) = (r_1^0 - (r_1^0)^3)/2$ by solving the equation

$$-(v^0)'' - \frac{1}{r}(v^0)' = u^0 - a, \ (v^0)'(0) = (v^0)'(1) = 0.$$

Therefore $\mu_1^0 = \gamma(r_1^0)^3/\tau > 0$ and $\lambda_1 > 0$.

When $m \geq 2$, let $K = 1$ in Theorem 4.1:

$$\frac{(m^2 - 1)\tau}{(r_1^0)^2} + \gamma\{\frac{(r_1^0)^3 - r_1^0}{2} + \frac{(r_1^0)^{2m+1} + r_1^0}{2m}\} = \mu_m^0\tau. \tag{5.2}$$

Clearly when γ is small, the first term on the left side dominates and μ_m^0 is positive for all $m \geq 2$. On the other hand we find that the quantity in the braces is negative if m is sufficiently large. Fixing such m and taking γ large enough, we find that the entire left side of (5.2) becomes negative. □

The borderline value $\hat{\gamma}$ for γ can be calculated easily from (5.2) in two steps.

μ_0^0	μ_1^0	μ_2^0	μ_3^0	μ_4^0	μ_5^0	μ_6^0	μ_7^0	μ_8^0	μ_9^0	μ_{10}^0
14.90	8.15	27.80	16.73	19.11	29.36	45.07	65.30	89.59	117.70	149.52
	107.71	39.65	94.79	179.58	290.33	426.53	587.96	774.51	986.13	1222.77

Table 2: μ_m^0 when $\gamma = 25$. Here $\mathbf{r}^0 = (0.2832, 0.7616)$.

1. For each integer $m \geq 2$ find $\hat{\gamma}_m$ by setting the right side of (5.2) to be 0 and solving the equation for γ. If the resulting $\hat{\gamma}_m$ is less than or equal to 0, this mode m dose not yield a zero eigenvalue. Discard such $\hat{\gamma}_m$.

2. Minimize the $\hat{\gamma}_m$'s from the last step with respect to $m \geq 2$. The minimum is $\hat{\gamma}$, achieved at $m = \hat{m}$ where $\lambda_{\hat{m}}$, the principal eigenvalue, vanishes up to order ϵ^2.

The values $\hat{\gamma}$ for several a are reported in Table 1. Curiously when $a = 1/2$ the borderline $\hat{\gamma}$ occurs at two modes $\hat{m} = 3$ and $\hat{m} = 4$. In this case if $\gamma = \hat{\gamma}$ both λ_3 and λ_4 are of order $o(\epsilon^2)$ while the other λ_m's ($m \geq 2$) are positive and $\sim \epsilon^2$.

One gains more insight into the diblock copolymer equation by comparing with the Cahn-Hilliard equation, which is (1.6) with $\gamma = 0$. The Cahn-Hilliard equation also has a spot solution. Its critical eigenvalues are again classified into λ_m for non-negative integers m. If we formally set $\gamma = 0$ in Theorem 3.1 and (5.2) it appears that for the Cahn-Hilliard equation λ_0 is positive and of order ϵ, and λ_m with $m \geq 2$ is also positive and of order ϵ^2. From (5.1) with $\gamma = 0$, one thinks that up to order ϵ^2, λ_1 vanishes. These statements are actually all correct, although the exact value of λ_1 is actually negative, and the spot solution is unstable in the Cahn-Hilliard problem. Therefore Theorem 5.1 does not cover the Cahn-Hilliard equation. Nevertheless the distance between λ_1 and 0 is exponentially small there and is not visible in (5.1). The smallness of λ_1 is related to the phenomenon of the slow motion of a bubble profile in a general domain (see Alikakos and Fusco [2, 3], Ward [37], and Alikakos, Bronsard and Fusco [1]). One may feel uneasy about the abrupt change from negative λ_1 to positive λ_1 as we add a nonlocal term with a small γ. This is a result of our setting of fixing γ while taking ϵ small. To find the threshold where $\lambda_1 = 0$ one must take γ to vary with ϵ. We suspect that a borderline lies where γ is exponentially small compared to ϵ.

When we further increase γ, we reach the second threshold where one of λ_m with $m \geq 2$ becomes 0. Beyond this critical γ value the spot solution is unstable. It no longer has enough oscillation demanded by the stronger nonlocal term now. Note that the first stability threshold occurs because of λ_1 which is related to the translation of the spot, while the second threshold occurs because of some λ_m with $m \geq 2$ which is related to the oscillation of the boundary of the spot.

The situation is more complex when $K \geq 2$, because the existence of u is conditional. According to Theorem 2.2, we have u if (2.12) is positive definite in T. This condition requires two things. First (2.10) must have a solution $\mathbf{r}^0$. From this $\mathbf{r}^0$ we construct $\mathcal{U}(\cdot; \mathbf{r}^0)$, $\mathcal{V}(\cdot; \mathbf{r}^0)$, g^0, $\mathbf{e}_j^0$, and finally the matrix $\mathbf{M}^0$. The second requirement is that the eigenvalues of the $K-1$ by $K-1$ matrix $g^0(\mathbf{M}^0\mathbf{e}_n^0, \mathbf{e}_m^0)$, $n, m = 2, 3, ..., K$, in Theorem 3.1 must all be positive. When these two requirements are met, u exists and its stability in two dimensions is determined by the eigenvalues λ_m, $m \geq 1$. Their leading order approximations μ_m^0 are calculated from the K by K matrix in Theorem 4.1.

The determination of $\mathbf{r}^0$ and the analysis of the matrices have to be done numerically. As an example we consider $K = 2$. Let $a = 1/2$, $\tau = \sqrt{2}/12$, and try various values of γ. Instead of

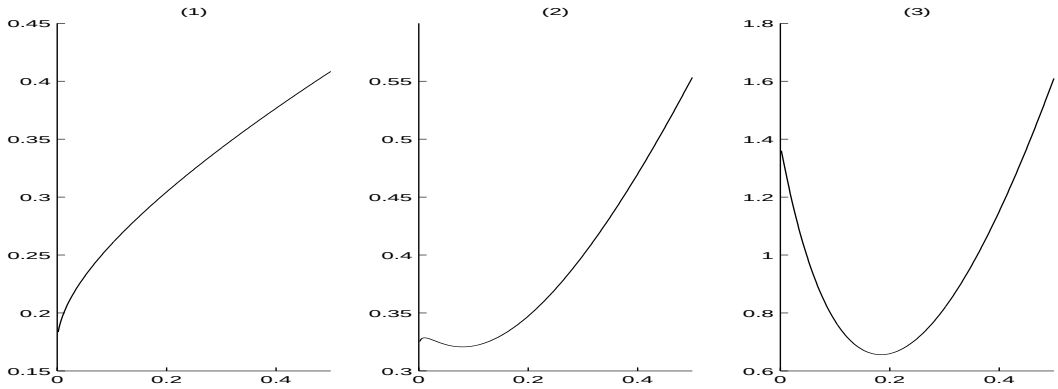

Figure 4: (1) When $\gamma = 1$, $J(y)$ is increasing in y. No $\mathbf{r}^0$ exists. (2) When $\gamma = 25$, a local minimum of $J(y)$ appears and $\mathbf{r}^0$ exists. The $K = 2$ ring solution is stable. (3) When $\gamma = 200$, $\mathbf{r}^0$ still exists, but the $K = 2$ ring solution is unstable. In all three cases $a = 1/2$ and $\tau = \sqrt{2}/12$.

μ_0^0	μ_1^0	μ_2^0	μ_3^0	μ_4^0	μ_5^0	μ_6^0	μ_7^0	μ_8^0	μ_9^0	μ_{10}^0
135.39	48.34	-5.03	-21.81	-10.89	19.86	18.00	15.40	21.97	35.43	54.42
	1220.57	384.95	163.82	75.22	35.73	68.85	130.74	205.72	293.01	392.18

Table 3: μ_m^0 when $\gamma = 200$. Here $\mathbf{r}^0 = (0.4290, 0.8271)$.

considering q_1 and q_2 under the constraint $-q_1^2 + q_2^2 = a$, we let $y = q_1^2$ and $q_2 = \sqrt{y+a}$. Then as done in [25] J may be treated as a function of y without constraint: $J(y) = J(q_1(y), q_2(y))$. According to Section 2 for given y we find q_1 and q_2, $\mathcal{U}(\cdot; q_1, q_2)$, $\mathcal{V}(\cdot; q_1, q_2)$, and $J(y)$. When γ is small, e.g. $\gamma = 1$, J is increasing in y, Figure 4 (1), and (2.10) has no solution.

When γ is increased to 25, J has a critical point at $y = 0.0802$, Figure 4 (2), i.e. (2.10) has a solution $\mathbf{r}^0 = (0.2832, 0.7616)$. We calculate $g^0(\mathbf{M}^0\mathbf{e}_2^0, \mathbf{e}_2^0)$ which turns out to be positive. Hence μ_0^0 is positive, $y = 0.0802$ is a local minimum of J, and a solution u exists. Then we compute the eigenvalues μ_m^0 of the matrix in Theorem 4.1. They are all positive, Table 2. So u is a stable solution in two dimensions.

When γ is further increased to 200, J has a critical point at $y = 0.1841$, Figure 4 (3), corresponding to $\mathbf{r}^0 = (0.4290, 0.8271)$. $g^0(\mathbf{M}^0\mathbf{e}_2^0, \mathbf{e}_2^0)$ is positive, so a solution u exists. However some μ_m^0 $(m \geq 1)$ are negative, Table 3. Hence u is unstable in two dimensions.

There is something interesting in Figure 4 (2) and (3). If we blow them up near $y = 0$, Figure 5, then in each case we find a local *maximum* near $y = 0$. This is because that $J(y)$ is increasing in y near $y = 0$ and near $y = 1-a$. So whenever there is a local minimum, there must be a local maximum before the local minimum. This local maximum gives rise to a solution $\hat{\mathbf{r}}^0$ of (2.10). However we can not use the Γ-convergence theory to find a solution of (1.6) near $\mathcal{U}(\cdot; \hat{\mathbf{r}}^0)$. We conjecture that such a solution exists.

When the critical eigenvalues of a spot or a ring solution, determined from Theorems 3.1 and 4.1, are non-zero, we may expect to have a similar solution of (1.6) on a slightly perturbed domain.

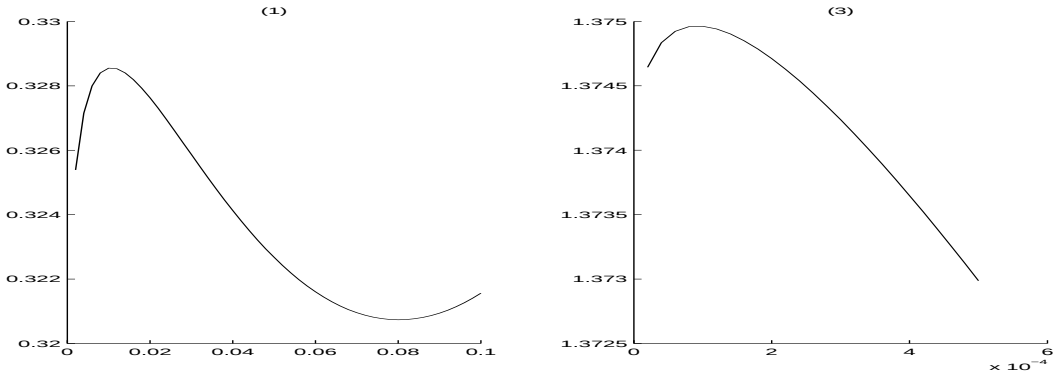

Figure 5: (1) The enlarged Figure 4 (2) near $y = 0$. (2) The enlarged Figure 4 (3) near $y = 0$.

However finding solutions of (1.6) on a general domain $\Omega \subset R^N$ is rather difficult. It was noted in [19] that (1.6) has a singular limit as $\epsilon \to 0$. One looks for a function $u^0 \in BV(\Omega)$ defined such that for a.e. $x \in \Omega$ $u^0(x) = 0$ or $u^0(x) = 1$ and $\overline{u^0} = a$. Let S be the union of the hyper-surfaces that separate the regions $u^0 = 0$ from the regions $u^0 = 1$, and $v^0 = (-\Delta)^{-1}(u^0 - a)$. Then one requires that at every $x \in S$

$$\tau\kappa(x) + \gamma v^0(x) = \eta \tag{5.3}$$

where $\kappa(x)$ is the mean curvature of S at x viewed from the $u^0 = 1$ side, and η is a Lagrange multiplier to be determined. If the free boundary problem (5.3) admits an isolated stable solution u^0, then near u^0, in the $L^2(\Omega)$ sense, there exists a local minimizer solution u of (1.6) by the Γ-convergence theory. However (5.3) is a challenging nonlocal geometric problem. Even though Figure 1 (2) and (3) suggest we look for solutions with multiple spots, (5.3) implies that for such a solution the curvature of the boundary of a spot is in general not constant (there is the impact of v^0), i.e. the spots are not exactly round, unless we deal with the one spot or the ring solutions in a disc as in this paper. Nevertheless if we consider the situation where a is close to 0 (or 1), then v^0 is near constant throughout Ω and hence κ becomes close to a constant and the spots are approximately round. The cylindrical and spherical phases in Figure 1 are thus heuristically explained. Note that in the singular limit of the Cahn-Hilliard equation, which is (5.3) without the $\gamma v^0(x)$ term, κ is constant.

A Proof of Lemma 2.3

Since $\eta = \overline{f(u)}$, we obtain a rough estimate for η:

$$|\eta| = |\overline{f(u)}| \le C(\int_D W(u)\,dx)^{1/2} = O(\epsilon^{1/2}), \tag{A.1}$$

since $I(u) = O(\epsilon)$. $\|u\|_2 = O(1)$ implies that $\|v\|_{2,2} = O(1)$ and in particular $v = O(1)$. A maximum principle argument shows that

$$-O(\epsilon^{1/2}) = -(O(\epsilon) + O(|\eta|)) \le u \le 1 + O(\epsilon) + O(|\eta|) = 1 + O(\epsilon^{1/2}). \tag{A.2}$$

In the Γ-convergence theory u satisfies $u \to u^0$ in $L^2(D)$ and $(\epsilon\pi)^{-1}I(u) \to J(u^0)$ [25]. The fact $u \to u^0$ in $L^2(D)$ implies the existence of r_j where $u(r_j) = 1/2$ and that $r_j \to r_j^0$ for $j = 1, 2, ..., K$. We construct a preliminary approximation h of u:

$$h(r) = H(\frac{r - r_1}{\epsilon}) + [H(-\frac{r - r_2}{\epsilon}) - 1] + H(\frac{r - r_3}{\epsilon}) + [H(-\frac{r - r_4}{\epsilon}) - 1] + ..., \quad r \in (r_1, 1),$$

and let $d = u - h$.

If we consider h on $(r_1, 1)$, the argument in Proposition 8.2 [26] shows that $d = o(1)$ on $[r_1, 1]$. Next we improve (A.1) to

$$\eta = O(\epsilon). \tag{A.3}$$

and show that

$$d = O(\epsilon) \text{ in } [r_1, 1]. \tag{A.4}$$

Note that $d = u - h$ satisfies the equation

$$-\epsilon^2 d_{rr} + f'(h)d + O(\|d\|^2) + O(\epsilon) = \eta, \quad d(r_j) = 0, \ (j = 1, 2, ..., K), \ d'(1) = 0$$

on $(r_1, 1)$. Then $d = O(\epsilon + |\eta|)$ in $[r_1, 1]$. Now we use an idea of Pohozaev [22]. Multiply the first equation of (1.8) by $r^2 u_r$ and integrate with respect to dr on $(0, 1)$. Then

$$\int_0^1 [-\epsilon^2 (ru_r)_r (ru_r) + r^2 f(u)u_r + \epsilon\gamma r^2 v u_r]\, dr = \eta \int_0^1 r^2 u_r\, dr.$$

The first term on the left side becomes 0 after integration. Applying integration by parts to the second and third terms on the left side and the right side shows that

$$r^2 W(u)|_{r=0}^{r=1} - 2\int_0^1 W(u) r\, dr + \epsilon\gamma r^2 v u|_{r=0}^{r=1} - \epsilon\gamma \int_0^1 u(r^2 v)_r\, dr = \eta(r^2 u|_{r=0}^{r=1} - 2\int_0^1 u\ r\, dr),$$

which is simplified to

$$W(u(1)) - \frac{1}{\pi}\int_D W(u)\, dx + O(\epsilon) = \eta(u(1) - a)$$

since $\epsilon\gamma v(1)u(1) = O(\epsilon)$ and $\epsilon\gamma \int_0^1 u(r^2 v)_r\, dr = O(\epsilon)$. The integral in the last equation is of order $O(\epsilon)$ since it is a part of $I(u)$ and $I(u) = O(\epsilon)$ by $(\epsilon\pi)^{-1}I(u) \to J(u^0)$. Moreover $u(1) \to 0$ or 1 to which a is not equal, so the last equation reads

$$\eta = O(\epsilon) + O(W(u(1))).$$

However $d = O(\epsilon + |\eta|)$ on $[r_1, 1]$ proved earlier implies that $u(1) = O(\epsilon + |\eta|)$ or $u(1) = 1 + O(\epsilon + |\eta|)$. Then $W(u(1)) = W(O(\epsilon + |\eta|)) = O((\epsilon + |\eta|)^2)$, or $W(u(1)) = W(1 + O(\epsilon + |\eta|)) = O((\epsilon + |\eta|)^2)$. Hence we derive

$$\eta = O(\epsilon) + O((\epsilon + |\eta|)^2), \quad \text{i.e. } \eta = O(\epsilon).$$

Consequently $d = O(\epsilon)$ in $[r_1, 1]$.

Now we consider u, h, and d on $(0, r_1)$. We proceed to show that $d = o(1)$ on $(0, r_1)$. Suppose that this is false. Then there exist a small $\delta > 0$, independent of ϵ, and $r_* \in [0, r_1)$ such that $|d(r_*)| = \delta$ and $|d(r)| < \delta$ if $r \in (r_*, r_1)$. δ is so small that 0 is the only critical point of W in $(-\delta, \delta)$.

Since $u(\epsilon t + r_1) \to H(t)$ in $C^2_{loc}(R)$, $(r_1 - r_*)/\epsilon \to \infty$. Moreover the argument in Proposition 8.2 [26] shows that $r_* = o(1)$. There are two cases left: 1. $r_*/\epsilon \to \infty$ and $r_* = o(1)$, and 2. $r_* = O(\epsilon)$.

In the first case we multiply the first equation of (1.8) by u_r and integrate with respect to dr:

$$-\int_0^1 \frac{\epsilon^2}{r} u_r^2 \, dr + W(u(1)) - W(u(0)) + \epsilon\gamma \int_0^1 v u_r \, dr = \eta(u(1) - u(0)).$$

Here $W(u(1))$ is of order $O(\epsilon^2)$ by (A.4). The right side is of order $O(\epsilon)$ by (A.3). $\epsilon\gamma \int_0^1 v u_r \, dr$ is of order $O(\epsilon)$ after integration by parts. Hence

$$\int_0^1 \frac{\epsilon^2}{r} u_r^2 \, dr + W(u(0)) = O(\epsilon).$$

Since $W(u(0)) \geq 0$,

$$\int_0^1 \frac{\epsilon^2}{r} u_r^2 \, dr = O(\epsilon). \tag{A.5}$$

On the other hand if we scale u at r_0 so that $U(t) := u(r_* + \epsilon t) \to H(t)$ locally in C^2, then

$$\int_0^1 \frac{\epsilon^2}{r} u_r^2 \, dr = \frac{\epsilon}{r_*} \int_{-r_*/\epsilon}^{(1-r_*)/\epsilon} \frac{1}{1 + (\epsilon t/r_*)} (U')^2 \, dt \geq \frac{\epsilon}{r_*} \Big(\int_R (H')^2 \, dt + o(1) \Big). \tag{A.6}$$

However (A.5) and (A.6) are inconsistent if $r_* = o(1)$.

In the second case we scale u so that $U(t) := u(\epsilon t) \to U^0(t)$ locally in C^2 and

$$-U^0_{tt} + \frac{U^0_t}{t} + f(U^0) = 0 \text{ in } R, \; U^0(\infty) = 1, \; \|U^0\| \leq 1.$$

Moreover $U(r_*/\epsilon) \to \delta$. We multiply the equation for U^0 by U^0_t and integrate with respect to dt over $(0, \infty)$. Then

$$-W(U^0(0)) - \int_0^\infty \frac{(U^0_t)^2}{t} \, dt = 0,$$

which implies that $U^0 \equiv 0$ or $U^0 \equiv 1$. Neither case is consistent with $U(r_*/\epsilon) \to \delta \in (0, 1)$.

We have shown that $d = u - h = o(1)$ on $(0, 1)$. In particular we know that there are exactly K interfaces $r_1, r_2, ..., r_K$. Now we consider the more accurate approximation w of u defined in Section 2. We call $(r_j - \epsilon^\alpha, r_j + \epsilon^\alpha)$ an inner region, $(0, 1) \backslash (\cup_{j=1}^K (r_j - 2\epsilon^\alpha, r_j + 2\epsilon^\alpha))$ the outer region, and $(r_j - 2\epsilon^\alpha, r_j - \epsilon^\alpha)$ and $(r_j + \epsilon^\alpha, r_j + 2\epsilon^\alpha)$ matching regions. Recall that $\alpha \in (1/2, 1)$.

In the inner and matching regions, using (2.22), (2.23) and (2.25) we find that

$$\begin{aligned}
-\epsilon^2 \Delta z_j + f(z_j) &= -\epsilon^2 \Delta(H_j + \epsilon P_j + \epsilon^2 Q_j) + f(H_j + \epsilon P_j + \epsilon^2 Q_j) \\
&= -[f(H_j) + \epsilon(\frac{H_j'}{r} + f'(H_j) P_j - \frac{H_j'}{r_j} + \xi_j) \\
&\quad + \epsilon^2 (\frac{P_j'}{r} + f'(H_j) Q_j - \frac{P_j'}{r_j} + \frac{t H_j'}{r_j^2} + \frac{f''(H_j) P_j^2}{2} + \gamma v'(r_j) t)] \\
&\quad + f(H_j) + \epsilon f'(H_j) P_j + \epsilon^2 (\frac{f''(H_j) P_j^2}{2} + f'(H_j) Q_j) + O(\epsilon^3)
\end{aligned}$$

$$
\begin{aligned}
&= \epsilon\xi_j - \epsilon^2\gamma v'(r_j)t + \frac{\epsilon^2 t}{r_j}(\frac{1}{r} - \frac{1}{r_j})H_j' + \epsilon^3\frac{tP_j'}{r_j r} + O(\epsilon^3). \\
&= \epsilon\xi_j - \epsilon^2\gamma v'(r_j)t + O(\epsilon^3).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
-\epsilon^2\Delta z_j + f(z_j) + \epsilon\gamma v - \eta &= \epsilon\xi_j + \epsilon\gamma v(r_j) - \eta + O(\epsilon^3 t^2) + O(\epsilon^3) \\
&= \sigma_j + O(\epsilon^{1+2\alpha})
\end{aligned} \tag{A.7}
$$

where we have defined

$$
\sigma_j = \epsilon\xi_j + \epsilon\gamma v(r_j) - \eta. \tag{A.8}
$$

By (A.3) implicit differentiation of (2.27) and $v'' = O(1)$ yield that

$$
-\epsilon^2\Delta z + f(z) + \epsilon\gamma v - \eta = -\epsilon^2\Delta z = O(\epsilon^3), \tag{A.9}
$$

which is valid on $(0,1)\backslash\{r_1, r_2, ..., r_K\}$.

We now estimate the difference of z_j and z on a matching region. First using (2.23) and (2.25) we find

$$
\epsilon^2\Delta z_j = O(\epsilon^3).
$$

Then (A.7) implies that

$$
f(z_j) + \epsilon\gamma v - \eta = \sigma_j + O(\epsilon^{1+2\alpha}).
$$

Comparing this to (2.27) we deduce that

$$
z_j - z = O(|\sigma_j|) + O(\epsilon^{1+2\alpha}) \tag{A.10}
$$

on the matching regions $(r_j - 2\epsilon^\alpha, r_j - \epsilon^\alpha)$ and $(r_j + \epsilon^\alpha, r_j + 2\epsilon^\alpha)$. Then we consider w in the matching region. Here by (A.10)

$$
\begin{aligned}
-\epsilon^2\Delta w + f(w) + \epsilon\gamma v - \eta &= -\epsilon^2\Delta w + f(z) + \epsilon\gamma v - \eta + O(\|z_j - z\|) \\
&= -\epsilon^2\Delta w + O(|\sigma_j|) + O(\epsilon^{1+2\alpha}) \\
&= -\epsilon^2\Delta z - \epsilon^2\Delta(\chi_j(z_j - z)) + O(|\sigma_j|) + O(\epsilon^{1+2\alpha}) \\
&= -\epsilon^2((\chi_j)_{rr}(z_j - z) + 2(\chi_j)_r(z_j - z)_r + \chi_j(z_j - z)_{rr}) - \frac{\epsilon^2}{r}((\chi_j)_r(z_j - z) + \chi_j(z_j - z)_r) \\
&\quad + O(|\sigma_j|) + O(\epsilon^{1+2\alpha}) \\
&= O(|\sigma_j|) + O(\epsilon^{1+2\alpha}) + O(\epsilon^{3-\alpha}).
\end{aligned} \tag{A.11}
$$

If we let $g = u - w$, then (A.7), (A.9) and (A.11) imply that

$$
-\epsilon^2\Delta g + f'(w)g + O(\|g\|^2) = \begin{cases} -\sigma_j + O(\epsilon^{1+2\alpha}) & \text{in an inner region} \\ O(|\sigma_j|) + O(\epsilon^{1+2\alpha}) + O(\epsilon^{3-\alpha}) & \text{in a matching region} \\ O(\epsilon^3) & \text{in the outer region} \end{cases}. \tag{A.12}
$$

We deduce from (A.12) and $g(r_j) = 0$ that

$$
g = O(|\sigma_j|) + O(\epsilon^{1+2\alpha}) + O(\epsilon^{3-\alpha}). \tag{A.13}
$$

On the other hand if we multiply (A.12) by H_j' and integrate with respect to $r\,dr$ on $(0,1)$, then

$$\int_0^1 [-\epsilon^2 (rg_r)_r H_j' + f'(w) g H_j' r]\,dr + O(\epsilon\|g\|^2) = (-1)^j \epsilon \sigma_j r_j + O(\epsilon^2|\sigma_j|) + O(\epsilon^{2+2\alpha}).$$

But the integral on the left side after integration by parts becomes

$$\int_0^1 [-\epsilon g H_j'' + (f'(w) - f'(H_j)) g H_j' r]\,dr = O(\epsilon^2\|g\|),$$

from which we conclude that

$$\sigma_j = O(\epsilon\|g\|) + O(\|g\|^2) + O(\epsilon^{1+2\alpha}). \tag{A.14}$$

Inserting (A.14) to (A.13) we find that

$$g = O(\epsilon^{1+2\alpha}) + O(\epsilon^{3-\alpha}); \tag{A.15}$$

substituting (A.15) to (A.14) we deduce that

$$\sigma_j = O(\epsilon^{1+2\alpha}). \tag{A.16}$$

Since $\alpha \in (1/2, 1)$, (A.15) implies that $g = o(\epsilon^2)$.

References

[1] N. Alikakos, L. Bronsard, and G. Fusco. Slow motion in the gradient theory of phase transitions via energy and spectrum. *Calc. Var. Partial Differential Equations*, 6(1):39–66, 1998.

[2] N. Alikakos and G. Fusco. Slow dynamics for the cahn-hilliard equation in higher space dimension, part I: Spectral estimates. *Comm. Partial Differential Equations*, 19(9-10):1397–1447, 1994.

[3] N. Alikakos and G. Fusco. Slow dynamics for the cahn-hilliard equation in higher space dimension: The motion of bubbles. *Arch. Rational Mech. Anal.*, 141(1):1–61, 1998.

[4] F.S. Bates and G.H. Fredrickson. Block copolymers - designer soft materials. *Physics Today*, 52(2):32–38, 1999.

[5] J.W. Cahn and J.E. Hilliard. Free energy of a nonuniform system. I. Interfacial free energy. *J. Chem. Phys.*, 28(2):258–267, 1958.

[6] X. Chen and M. Taniguchi. Instability of spherical interfaces in a nonlinear free boundary problem. *Adv. Differential Equations*, 5(4-6):747–772, 2000.

[7] R. Choksi. Scaling laws in microphase separation of diblock copolymers. *J. Nonlinear Sci.*, 11(3):223–236, 2001.

[8] R. Choksi and X. Ren. On the derivation of a density functional theory for microphase separation of diblock copolymers. *J. Statist. Phys.*, 113(1-2):151–176, 2003.

[9] E. De Giorgi. Sulla convergenza di alcune successioni di integrali del tipo della'area. *Rendiconti di Matematica*, 8:277–294, 1975.

[10] P.C. Fife and D. Hilhorst. The Nishiura-Ohnishi free boundary problem in the 1D case. *SIAM J. Math. Anal.*, 33(3):589–606, 2001.

[11] I.W. Hamley. *The Physics of Block Copolymers.* Oxford Science Publications, 1998.

[12] B. Helffer and J. Sjöstrand. Multiple wells in the semiclassical limit. I. *Comm. Partial Differential Equations*, 9(4):337–408, 1984.

[13] M. Henry. Singular limit of a fourth order problem arising in the micro-phase separation of diblock copolymers. *Adv. Differential Equations*, 6(9):1049–1114, 2001.

[14] R. Kohn and P. Sternberg. Local minimisers and singular perturbations. *Proc. Roy. Soc. Edinburgh Sect. A*, 111(1-2):69–84, 1989.

[15] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rat. Mech. Anal.*, 98(2):123–142, 1987.

[16] S. Müller. Singular perturbations as a selection criterion for periodic minimizing sequences. *Calc. Var. Partial Differential Equations*, 1(2):169–204, 1993.

[17] C.B. Muratov. Theory of domain patterns in systems with long-range interactions of Coulomb type. *Phys. Rev. E*, 66:066108, 2002.

[18] H. Nakazawa and T. Ohta. Microphase separation of ABC-type triblock copolymers. *Macromolecules*, 26(20):5503–5511, 1993.

[19] Y. Nishiura and I. Ohnishi. Some mathematical aspects of the microphase separation in diblock copolymers. *Physica D*, 84:31–39, 1995.

[20] I. Ohnishi, Y. Nishiura, M. Imai, and Y. Matsushita. Analytical solutions describing the phase separation driven by a free energy functional containing a long-range interaction term. *Chaos*, 9(2):329–341, 1999.

[21] T. Ohta and K. Kawasaki. Equilibrium morphology of block copolymer melts. *Macromolecules*, 19(10):2621–2632, 1986.

[22] S. Pohozaev. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Soviet Math. Dokl.*, 6:1408–1411, 1965.

[23] X. Ren and L. Truskinovsky. Finite scale microstructures in nonlocal elasticity. In recognition of the sixtieth birthday of Roger L. Fosdick (Blacksburg, VA, 1999). *J. Elasticity*, 59(1-3):319–355, 2000.

[24] X. Ren and J. Wei. On the multiplicity of solutions of two nonlocal variational problems. *SIAM J. Math. Anal.*, 31(4):909–924, 2000.

[25] X. Ren and J. Wei. Concentrically layered energy equilibria of the di-block copolymer problem. *European J. Appl. Math.*, 13(5):479–496, 2002.

[26] X. Ren and J. Wei. On energy minimizers of the di-block copolymer problem. *Interfaces Free Bound.*, 5(2):193–238, 2003.

[27] X. Ren and J. Wei. On the spectra of 3-D lamellar solutions of the diblock copolymer problem. *SIAM J. Math. Anal.*, 35(1):1–32, 2003.

[28] X. Ren and J. Wei. Soliton-stripe patterns in charged Langmuir monolayers. *J. Nonlinear Sci.*, 13(6):603–624, 2003.

[29] X. Ren and J. Wei. Triblock copolymer theory: Free energy, disordered phase and weak segregation. *Physica D*, 178(1-2):103–117, 2003.

[30] X. Ren and J. Wei. Triblock copolymer theory: Ordered ABC lamellar phase. *J. Nonlinear Sci.*, 13(2):175–208, 2003.

[31] X. Ren and J. Wei. Chiral symmetry breaking and the soliton-stripe pattern in Langmuir monolayers and smectic films. *Nonlinearity*, 17(2):617–632, 2004.

[32] X. Ren and J. Wei. The soliton-stripe pattern in the Seul-Andelman membrane. *Physica D*, 188(3-4):277–291, 2004.

[33] X. Ren and J. Wei. Wriggled lamellar solutions and their stability in the diblock copolymer problem. preprint.

[34] M. Taniguchi. Multiple existence and linear stability of equilibrium balls in a nonlinear free boundary problem. *Quart. Appl. Math.*, 58(2):283–302, 2000.

[35] T. Teramoto and Y. Nishiura. Double gyroid morphology in a gradient system with nonlocal effects. *Journal of the Physical Society of Japan*, 71(7):1611–1614, 2002.

[36] Y. Tsori, D. Andelman, and M. Schick. Defects in lamellar diblock copolymers: Chevron- and Ω-shaped boundaries. *Phys. Rev. E*, 61(3):2848–2858, 2000.

[37] M. Ward. Metastable bubble solutions for the Allen-Cahn equation with mass conservation. *SIAM J. Appl. Math.*, 56(5):1247–1279, 1996.