Asymmetric and symmetric double bubbles in a ternary inhibitory system

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Abstract

A ternary inhibitory system contains two terms in its free energy: the interface energy that favors micro-domain growth and the longer ranging confinement energy that prevents unlimited spreading. In a parameter regime where two constituents are small in size compared to the third constituent and the longer ranging energy does not dominate, there is a double-bubble-like stable stationary point of the energy functional. The two minority constituents occupy the two bubbles of the double bubble, respectively, and the majority constituent fills the background. A special way of perturbing an exact double bubble leads to a restricted class of perturbed double bubbles that can be described by internal variables which are elements in a Hilbert space. The exact double bubble is non-degenerate in this class and nearby there is a perturbed double bubble that locally minimizes the free energy within the restricted class. This perturbed double bubble satisfies three of the four equations for stationary points of the free energy, namely, the three equations involving the curvature and the inhibitor variables on its three boundary curves. However it does not satisfy the 120 degree angle condition at its triple points. By translating and rotating the entire restricted class of perturbed double bubbles, one finds a particular direction and location in the domain of the problem where the locally minimizing perturbed double bubble in this specific restricted class also satisfies the 120 degree condition. This approach can handle both asymmetric and symmetric double bubbles.

Key words. asymmetric, symmetric, double bubble, ternary, inhibitory system

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1 Introduction

Growth and inhibition are two central properties in pattern forming multi-constituent physical and biological systems. In such a system a deviation from homogeneity has a strong positive feedback on its further increase. In the meantime a longer ranging confinement mechanism prevents unlimited spreading. Together they lead to a locally self-enhancing and self-organizing process.

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An archetype of inhibitory systems, the block copolymer is a soft material characterized by fluid-like disorder on the molecular scale and a high degree of order at a longer length scale. A molecule in a block copolymer is a linear sub-chain of monomers of one type grafted covalently to another or more sub-chains of monomers of different types. Because of the repulsion between the unlike monomers, different type sub-chains tend to segregate, but as they are chemically bonded in chain molecules, segregation of sub-chains cannot lead to a macroscopic phase separation. Only a local micro-phase separation occurs: micro-domains rich in monomers of different types emerge as a result. These micro-domains form patterns that are known as morphology phases [4].

We consider a ternary system originally derived by the authors in [27] from Nakazawa and Ohta's density functional formulation for triblock copolymers [20]. Let D be a bounded and connected open set of $\mathbb{R}^2$ with smooth boundary, and ω_1 and ω_2 be two positive numbers such that $\omega_1 + \omega_2 < 1$. For two measurable subsets Ω_1 and Ω_2 of D satisfying $|\Omega_1| = \omega_1|D|$, $|\Omega_2| = \omega_2|D|$, and $|\Omega_1 \cap \Omega_2| = 0$, set $\Omega_3 = D\backslash(\Omega_1 \cup \Omega_2)$ and $\Omega = (\Omega_1, \Omega_2)$. Here $|\Omega_1|$, $|\Omega_2|$, and $|\Omega_1 \cap \Omega_2|$ stand for the area (or the Lebesgue measure) of Ω_1, Ω_2, and $\Omega_1 \cap \Omega_2$, respectively. The free energy of the system is

$$\mathcal{J}(\Omega) = \frac{1}{2}\sum_{i=1}^{3}\mathcal{P}_D(\Omega_i) + \sum_{i,j=1}^{2}\int_D \frac{\gamma_{ij}}{2}\Big((-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i)\Big)\Big((-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j)\Big)\,dx. \tag{1.1}$$

The first term in (1.1) is responsible for growth. It is the total length of the interfaces separating the three domains Ω_1, Ω_2, and Ω_3. Three types of interfaces exist: $\partial\Omega_1\backslash\partial\Omega_2$, the interfaces separating Ω_1 from Ω_3; $\partial\Omega_2\backslash\partial\Omega_1$, the interfaces separating Ω_2 from Ω_3; $\partial\Omega_1 \cap \partial\Omega_2$, the interfaces separating Ω_1 from Ω_2. One can write the total size of the interfaces of all three types as $\frac{1}{2}(\mathcal{P}_D(\Omega_1) + \mathcal{P}_D(\Omega_2) + \mathcal{P}_D(\Omega_3))$. Here $\mathcal{P}_D(\Omega_i)$ is the perimeter of Ω_i in D. For a set Ω_i with a piecewise C^1 boundary, this is simply the length of $\partial\Omega_i \cap D$. For a general Lebesgue measurable subset Ω_i of D,

$$\mathcal{P}_D(\Omega_i) = \sup\Big\{\int_{\Omega_i} \operatorname{div} g(x)\,dx : \ g \in C_0^1(D, \mathbb{R}^2),\ |g(x)| \le 1\ \forall x \in D\Big\},$$

where $\operatorname{div} g$ is the divergence of the C^1 vector field g on D with compact support and $|g(x)|$ stands for the Euclidean norm of the vector $g(x) \in \mathbb{R}^2$; see, for instance, [8]. In $\mathcal{P}_D(\Omega_1) + \mathcal{P}_D(\Omega_2) + \mathcal{P}_D(\Omega_3)$, each of $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1 \cap \partial\Omega_2$ is counted twice. The half is put here to avoid double counting. To make this term small, the Ω_i's like to form large regions separated by curves as short as possible.

The second term in (1.1) provides an inhibition mechanism. The operator $(-\Delta)^{-1/2}$ is the positive square root of the inverse of the $-\Delta$ operator (see (1.6)); χ_{Ω_i} is the characteristic function of Ω_i ($\chi_{\Omega_i}(x) = 1$ if $x \in \Omega_i$ and 0 otherwise). The matrix $\gamma = [\gamma_{ij}]$ is symmetric and positive definite (eigenvalues of γ are positive) or positive semi-definite (eigenvalues of γ are non-negative). For the second term to be small, the functions χ_{Ω_i}, the characteristic functions of the sets Ω_i, must have frequent fluctuations.

Since the perimeter is a more local property and $(-\Delta)^{-1/2}$ is more nonlocal in nature, growth is more prevalent at smaller scale while inhibition is more dominant at larger scale. This combination prevents the χ_{Ω_i}'s from occupying large regions. It introduces a saturation effect that forces χ_{Ω_i} to develop an oscillation over a characteristic distance, and gives the system a self-organizing property.

Pattern formation driven by competing short range and long range free energy interactions is ubiquitous in nature. The system (1.1) is minimalist and attractive because it captures the most essential properties of growth and inhibition. Starting from this, one can build more complex models that may include features like mixing of constituents, interface thickness, dynamics and fluidity, fluctuation, etc. There are examples in biological systems that, although lacking an obvious free energy, contain inhibitor variables that create a mechanism which function exactly like the long range term in the free energy of (1.1). The appendix of [33] explains this story with the Gierer-Meinhardt system [10].

Although experimentally an almost unlimited number of architectures can be synthetically accessed in ternary systems like triblock copolymers [4, Figure 5 and the magazine's cover], mathematical study of $\mathcal{J}$

Figure 1: On the left is the $ABC...ABC$-lamellar pattern found in triblock copolymers; on the right is the $ABAB...ABAC$-pattern found in homopolymer/diblock copolymer blends.

is still in an early stage due to the complexity of $\mathcal{J}$. Found by the authors in [28] and depicted in the left plot of Figure 1 is a one dimensional solution of the Euler-Lagrange equations of $\mathcal{J}$, consisting of alternating A, B, and C micro-domains. The functional $\mathcal{J}$ is posed on the unit interval with the periodic boundary condition. Cyclic patterns of $3k$, $k \in \mathbb{N}$, micro-domains are all local minimizers of $\mathcal{J}$. All the type A domains (depicted in blue color) have the same length, and the same property holds for B and C domains.

Another one dimensional solution to the Euler-Lagrange equations, again an energy local minimizer, was found by Choksi and Ren in [6]. It models a diblock copolymer/homopolymer blend. Such a blend is a mixture of an AB diblock copolymer with a homopolymer of monomer species C, where the species C is thermodynamically incompatible with both the A and B monomer species. By a homopolymer of species C we mean a polymer chain consisting purely of the monomer species C. When such a mixture contains a sufficient concentration of the C homopolymers, the result in the melt phase is a *macroscopic phase separation* into homopolymer-rich and copolymer-rich domains followed by *micro-phase separation* within the copolymer-rich domains into A-rich and B-rich subdomains. See the right plot in Figure 1 for the $ABAB...ABAC$ phase pattern.

The same model (1.1) is used to study both triblock copolymer in [28] and polymer blends in [6]. In the latter case the free energy functional is derived from Ohta and Ito's work on polymer blends [22]. For a triblock copolymer the nonlocal interaction matrix γ is positive definite; namely, the two eigenvalues of γ are both positive [28, Lemma 3.4]. For a homopolymer/diblock copolymer blend one eigenvalue of γ is positive but the other one is zero [6, (4.36)].

The most interesting phenomenon in a ternary system in higher dimensions is arguably a triple junction. In two dimensions a triple junction appears at points where Ω_1, Ω_2, and Ω_3 all come to meet. A double bubble is a typical structure of this property. It is a pair of two adjacent sets bounded by three circular arcs of radii r_i; see Figure 2. In this picture the radius of the left arc is r_1, the radius of the right arc is r_2, and the radius of the middle arc is r_0. The radii r_i satisfy a relation $\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{r_0}$. The three arcs meet at two points, called triple junction points or triple points, and they meet at 120 degree angles.

There is a special symmetric double bubble when the radius r_1 an r_2 are equal. Then the middle arc becomes a straight line, i.e. an arc of infinite radius; see Figure 3.

The double bubble arises as the optimal configuration of the two component isoperimetric problem. Let $m_1 > 0$ and $m_2 > 0$. Find two disjoint sets E_1 and E_2 in $\mathbb{R}^n$ such that $|E_1| = m_1$, $|E_2| = m_2$, and the size of $\partial E_1 \cup \partial E_2$, i.e. $\frac{1}{2}(\mathcal{P}(E_1) + \mathcal{P}(E_2) + \mathcal{P}(E_3))$, where $E_3 = \mathbb{R}^n \backslash (E_1 \cup E_2)$ and $\mathcal{P}(E_i)$ is the perimeter of E_i in $\mathbb{R}^n$, is minimum. The double bubble described here (or its higher dimensional analogy) is the unique solution to this isoperimetric problem by the works of Almgren [3], Taylor [38], Foisy *et al* [9], Hass and Schlafly [12], Hutchings *et al* [13], and Reichardt [25]. Compared to the first modern proof of the standard isoperimetric problem of one component by Schwarz [36] in 1884, these results on the two component isoperimetric problem are very recent, a manifestation of the great difficulties associated with a triple junction.

A stationary point $\Omega = (\Omega_1, \Omega_2)$ of $\mathcal{J}$ is a solution to the following equations:

$$\kappa_1 + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2} = \lambda_1 \quad \text{on } \partial\Omega_1 \backslash \partial\Omega_2, \tag{1.2}$$

$$\kappa_2 + \gamma_{12} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} = \lambda_2 \quad \text{on } \partial\Omega_2 \backslash \partial\Omega_1, \tag{1.3}$$

$$\kappa_0 + (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{12} - \gamma_{22}) I_{\Omega_2} = \lambda_1 - \lambda_2 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2, \tag{1.4}$$

$$\nu_1 + \nu_2 + \nu_0 = \vec{0} \quad \text{at } \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_3. \tag{1.5}$$

Here we assume that Ω_1 and Ω_2 do not touch the boundaries of D. Otherwise we need to add another condition that the boundary of Ω_1 (or Ω_2) meets the boundary of D perpendicularly.

In (1.2)-(1.4) κ_1, κ_2, and κ_0 are the curvatures of the curves $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1 \cap \partial\Omega_2$, respectively. These are signed curvatures defined with respect to a choice of normal vectors. On $\partial\Omega_1\backslash\partial\Omega_2$ the normal vector points inward into Ω_1. On $\partial\Omega_2\backslash\partial\Omega_1$, the normal vector points inward into Ω_2. On $\partial\Omega_1 \cap \partial\Omega_2$, the normal vector points from Ω_2 towards Ω_1, i.e. inward with respect to Ω_1 and outward with respect to Ω_2. If a curve bends in the direction of the normal vector, then the curvature is positive.

Also in (1.2)-(1.4), I_{Ω_1} and I_{Ω_2} are two functions on D determined from Ω_1 and Ω_2, respectively. The function I_{Ω_i}, called an inhibitor, is the solution of Poisson's equation

$$-\Delta I_{\Omega_i} = \chi_{\Omega_i} - \omega_i \ \text{ in } D, \ \ \partial_n I_{\Omega_i} = 0 \ \text{ on } \partial D, \ \ \int_D I_{\Omega_i}(x)\,dx = 0, \tag{1.6}$$

where $\partial_n I_{\Omega_i}$ stands for the outward normal derivative of I_{Ω_i} on ∂D. Note that the constraint $|\Omega_i| = \omega_i|D|$ implies that the integral of the right side of the PDE in (1.6) is zero, so the PDE together with the boundary condition is solvable. The solution is unique up to an additive constant. The last condition $\int_D I_{\Omega_i}(x)\,dx = 0$ fixes this constant and selects a particular solution. One also writes $I_{\Omega_i} = (-\Delta)^{-1}(\chi_{\Omega_i} - \omega_i)$ as the outcome of the operator $(-\Delta)^{-1}$ on $\chi_{\Omega_i} - \omega_i$. The operator $(-\Delta)^{-1/2}$ in (1.1) is the positive square root of $(-\Delta)^{-1}$.

The constants λ_1 and λ_2 are Lagrange multipliers corresponding to the constraints $|\Omega_1| = \omega_1|D|$ and $|\Omega_2| = \omega_2|D|$. They are unknown and are to be found with Ω_1 and Ω_2.

In the last equation, (1.5), ν_1, ν_2, and ν_0 are the inward pointing, unit tangent vectors of the curves $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1 \cap \partial\Omega_2$ at triple points. The requirement that the three unit vectors sum to zero is equivalent to the condition the three curves meet at 120 degree angles.

We will find a double-bubble-like solution to (1.2)-(1.5), when ω_1, ω_2, and γ are in a particular parameter regime, where the system is biased towards the third constituent and the first and the second constituents are more or less comparable in size. In other words, ω_1 and ω_2 are small, and $\frac{\omega_1}{\omega_2}$ stays away from 0 and ∞. The matrix γ can be large to some extent, but it must be positive definite with comparable eigenvalues.

To make these conditions more precise we introduce a fixed number $m \in (0,1)$ and a small ϵ so that $\omega_1|D| = \epsilon^2 m$ and $\omega_2|D| = \epsilon^2(1-m)$. The area constraints $|\Omega_1| = \omega_1|D|$ and $|\Omega_2| = \omega_2|D|$ now take the form

$$|\Omega_1| = \epsilon^2 m \ \text{ and } \ |\Omega_2| = \epsilon^2(1-m). \tag{1.7}$$

Instead of ω_1 and ω_2, ϵ becomes one parameter of our problem.

The other parameter is the matrix γ. It must be positive definite and satisfy a uniform positivity condition. Namely, there exists $\iota > 0$ so that

$$\iota\,\overline{\overline{\lambda}}(\gamma) \le \overline{\lambda}(\gamma), \tag{1.8}$$

where $\overline{\lambda}(\gamma)$ and $\overline{\overline{\lambda}}(\gamma)$ are the two eigenvalues of γ such that $0 < \overline{\lambda}(\gamma) \le \overline{\overline{\lambda}}(\gamma)$. The matrix γ must also have an upper bound, namely, that $|\gamma|\epsilon^3$ is small. Any of the equivalent norms of γ may be used for $|\gamma|$. We take it to be the operator norm for definiteness.

The main result in this paper is the following existence theorem.

Theorem 1.1 *Let $m \in (0,1)$ and $\iota \in (0,1]$. There exist $\delta > 0$ and $\sigma > 0$ depending on the domain D, m, and ι only, such that if $\epsilon < \delta$, $|\gamma|\epsilon^3 < \sigma$, and $\iota\,\overline{\overline{\lambda}}(\gamma) \le \overline{\lambda}(\gamma)$, then a perturbed double bubble exists as a stable solution to the problem (1.2)-(1.5) satisfying the constraints (1.7). Each of the two perturbed bubbles is bounded by a continuous curve that is C^∞ except at the two triple junction points.*

The authors proved this theorem in [34] for the symmetric case where the two bubbles have the same area, i.e. $m = \frac{1}{2}$. For various reasons the symmetric case is known to be simpler in the study of double

bubbles. For instance the two volume isoperimetric problem in three dimensions was first proved in the symmetric case by Hass and Schafly [12] and later in the general case by Hutchings *et al.* [13].

In this work, since the double-bubble solution is a perturbed double bubble, not an exact double bubble, one must find an effective way to describe perturbations from an exact double bubble. It is easier to find a mathematical description of a perturbation in the symmetric case, as was done in our earlier work [34]. Unfortunately the method used in [34] depends too much on the symmetry and cannot be generalized to the asymmetric case where $m \neq \frac{1}{2}$.

In this paper we present a new approach that does not require the symmetry. This breakthrough is achieved by dividing a perturbation process into two steps. The first step is a new idea. It changes an exact double bubble to two sets still bounded by three circular arcs. More precisely, the two triple points of an exact double bubble are moved vertically by the same distance in opposite directions. The three circular arcs are changed to three new circular arcs connecting the new triple points. One requires that the areas of the regions bounded by the new arcs remain the same. Another requirement is that the three new arcs continue to satisfy the radii relation. On the other hand the three arcs no longer meet at 120 degree angles. The new shape is characterized by one number only, the height of a new triple point, which we denote by η. The height of the corresponding triple point of the original exact double bubble is denoted h.

The end result of the first step serves as a skeleton from which the second step of perturbation is carried out. In the second step one perturbs the shape of the new circular arcs so that the radius of each arc becomes a function $u_i(t)$, where $t \in (-1, 1)$ and $i = 1,\ 2,\ 0$ refers to left, right, and center curves, respectively. As we perturbed the circular arcs to curves, the triple points stay fixed and the areas of the two sets bounded by the new curves remain unchanged. Next replace u_i by three new variables ϕ_i. The requirement that the triple points are not changed in the second step implies that $\phi_i(\pm 1) = 0$. Moreover the area constraints are linear integral constraints on ϕ_i; see (3.14).

Then we can use ϕ_i and η, termed internal variables because they do not have obvious geometric meanings but can yield all geometric variables through transformations, to characterize a perturbation. The quadruple $(\phi_1, \phi_2, \phi_0, \eta)$ is an element of a Hilbert space, and we recast our problem as a variation problem on this space.

The kind of perturbations described here are called restricted perturbations, because the triple points can only move vertically and only by the same distance in the opposite directions. It is a key idea in this work that one singles out this class of perturbations. In this class we will prove an important non-degeneracy property; see Lemma 5.4.

Below is an outline of the proof of Theorem 1.1. In Section 2, a detailed description of an exact double bubble E is given. Then for small ϵ, ξ in a slightly smaller subset of D, and $\theta \in \mathbb{S}^1$, where $\mathbb{S}^1$ is the unit circle or the interval $[0, 2\pi]$ of identified end points, we take a transform $T_{\epsilon,\xi,\theta}$ that maps the double bubble E to $T_{\epsilon,\xi,\theta}(E)$ inside D. This image is a scaled down exact double bubble centered at ξ of the direction θ. Lemma 2.1 gives an estimate of $\mathcal{J}(T_{\epsilon,\xi,\theta}(E))$, the energy of the exact double bubble $T_{\epsilon,\xi,\theta}(E)$.

The crucial idea in this work is the construction of restricted perturbations of the exact double bubble $T_{\epsilon,\xi,\theta}(E)$ presented in Section 3. As discussed above, two steps of perturbation lead to internal variables $(\phi_1, \phi_2, \phi_0, \eta)$ by which the problem is recast as a variational problem on a Hilbert space.

In section 4 one calculates the first variation of the energy functional and obtains a nonlinear operator $\mathcal{S}$ so that a locally minimizing perturbed double bubble in the restricted class is a solution of $\mathcal{S}(\phi, \eta) = 0$, where ϕ stands for the triple (ϕ_1, ϕ_2, ϕ_0).

This equation is solved by a fixed point argument near the exact double bubble $T_{\epsilon,\xi,\theta}(E)$, which in terms of the internal variables is represented by $(0, h)$. In Section 5 one studies the second variation of $\mathcal{J}$ in the restricted class or, in other words, the Fréchet derivative $\mathcal{S}'(0, h)$ of $\mathcal{S}$, at the exact double bubble.

This linear operator turns out to be invertible. In Section 6 one finds a solution (ϕ^*, η^*) as a locally minimizing fixed point in the restricted class. It is also shown that (ϕ^*, η^*) satisfies (1.2)-(1.4), but not necessarily (1.5).

To find a perturbed double bubble that solves all the equations (1.2)-(1.5), one investigates the dependence on ξ and θ, the center and the direction of the restricted class. Denote (ϕ^*, η^*) by $(\phi^*(\cdot, \xi, \theta), \eta^*(\xi, \theta))$ and

consider $\mathcal{J}(\phi^*(\cdot,\xi,\theta),\eta^*(\xi,\theta))$ as a function of (ξ,θ). In Section 7, it is proved that this function attains a minimum at a point $(\xi^*,\theta^*)\in D\times\mathbb{S}^1$, and at this (ξ^*,θ^*) the perturbed double bubble represented by $(\phi^*(\cdot,\xi^*,\theta^*),\eta^*(\xi^*,\theta^*))$ solves all the equations (1.2)-(1.5).

This approach is presented in detail for the asymmetric case, i.e. $m\neq\frac{1}{2}$. For the symmetric case $m=\frac{1}{2}$ one needs to make some small adjustments. These modifications are given in Section 8. We point out that even for the symmetric case, the approach presented in Section 8 based on our current method is more elegant than the one in [34].

In this work all estimates indicate their dependencies on ϵ and γ. For instance if something is bounded by $C|\gamma|\epsilon^3$, then this C may at most depend on D, m, and ι, but must be independent of ϵ and γ. If a quantity is of order $O(|\gamma|\epsilon^4)$, then there is $C>0$ independent of ϵ and γ such that the quantity is bounded by $C|\gamma|\epsilon^4$.

Since we work in two dimensions, it is convenient to adopt the complex notation. For instance we opt to write $\rho e^{\mathrm{i}\alpha t}+\beta$, where $\rho,\alpha,\beta\in\mathbb{R}$, instead of $(\rho\cos(\alpha t),\rho\sin(\alpha t))+(\beta,0)$.

Finally we mention that the functional $\mathcal{J}$ has a simpler counterpart in a binary system. Let $\omega\in(0,1)$ and $\gamma>0$. For $\Omega\subset D$ with the fixed area, $|\Omega|=\omega|D|$, the binary energy of Ω is

$$\mathcal{J}_B(\Omega)=\mathcal{P}_D(\Omega)+\frac{\gamma}{2}\int_D|(-\Delta)^{-1/2}(\chi_\Omega-\omega)|^2\,dx. \tag{1.9}$$

A stationary point of this functional satisfies the equation

$$\kappa+\gamma I_\Omega=\lambda \tag{1.10}$$

on $\partial\Omega$. Equation (1.10) or the functional (1.9) may be derived from the Ohta-Kawasaki theory [23] for diblock copolymers; see [21, 26]. The equation can also be derived from the Gierer-Meinhardt system [33]. This binary problem has been studied intensively in recent years. All solutions to (1.10) in one dimension are known to be local minimizers of $\mathcal{J}_B$ [26]. Many solutions in two and three dimensions have been found that match the morphological phases in diblock copolymers [24, 30, 29, 31, 32, 15, 16, 33, 35, 39]. Global minimizers of $\mathcal{J}_B$ are studied in [2, 37, 19, 5, 18, 17, 11] for various parameter ranges. Applications of the second variation of $\mathcal{J}_B$ and its connections to minimality and Gamma-convergence are found in [7, 1, 14].

2 The exact double bubble

Recall that an exact double bubble, depicted in Figure 2, is a pair of two adjacent sets E_1 and E_2, denoted by $E=(E_1,E_2)$. The set E_1 is bounded by two circular arcs of radii r_1 and r_0. One arc, whose radius is r_0, is also on the boundary of E_2. The rest of the boundary of E_2 is another circular arc whose radius is r_2.

We consider the asymmetric case $r_1\neq r_2$ until Section 7. In Section 8 we will deal with the symmetric case. Without loss of generality assume that

$$r_1<r_2, \tag{2.1}$$

so E_1 is smaller than E_2, i.e. $0<m<\frac{1}{2}$.

The three radii satisfy the condition

$$\frac{1}{r_1}-\frac{1}{r_2}=\frac{1}{r_0}. \tag{2.2}$$

The two points where the three arcs meet are termed triple junction points, or just triple points. The three arcs meet at the triple points at 120 degree angle. Denote by a_1, a_2, and a_0 the angles associated with the three arcs; see Figure 2. The 120 degree angle condition and (2.1) imply that

$$a_1=\frac{2\pi}{3}-a_0,\quad a_2=\frac{2\pi}{3}+a_0,\quad a_0\in\Big(0,\frac{\pi}{3}\Big). \tag{2.3}$$

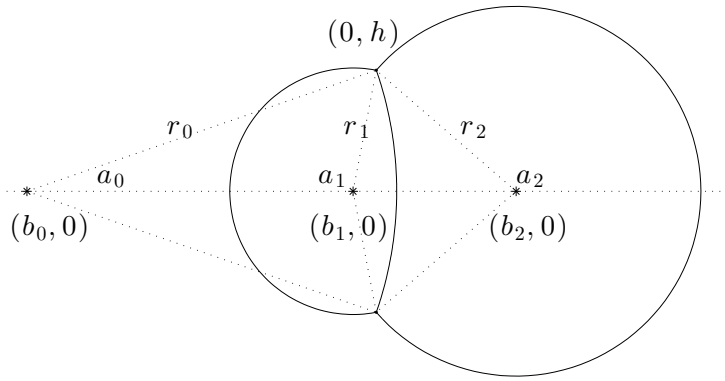

Figure 2: An asymmetric exact double bubble with angles a_i, radii r_i, and centers $(b_i, 0)$. One of the two triple points is $(0, h)$.

In this paper we assume that the area of E_1 is fixed at m and the area of E_2 is $1-m$, where m is given before (1.7). These constraints, $|E_1| = m$ and $|E_2| = 1-m$, can be expressed as

$$r_1^2(a_1 - \cos a_1 \sin a_1) + r_0^2(a_0 - \cos a_0 \sin a_0) = m, \tag{2.4}$$

$$r_2^2(a_2 - \cos a_2 \sin a_2) - r_0^2(a_0 - \cos a_0 \sin a_0) = 1-m. \tag{2.5}$$

Note that

$$m \in \left(0, \frac{1}{2}\right) \tag{2.6}$$

by the assumption (2.1). Place the exact double bubble $E = (E_1, E_2)$ in $\mathbb{R}^2$ so that the triple points are $(0, h)$ and $(0, -h)$, where

$$h = r_i \sin a_i, \quad i = 1, 2, 0, \tag{2.7}$$

is positive. Moreover the centers of the three arcs are denoted $(b_i, 0)$, $i = 1, 2, 0$, respectively.

Scale the exact double bubble E down by a factor ϵ and put it inside the domain D. The middle point of the two triple points is ξ and the angle of the line connecting the three centers is θ. Here $\xi \in \overline{D_{\overline{\delta}}}$ and $\theta \in \mathbb{S}^1$. The set $\overline{D_{\overline{\delta}}}$ is the closure of the set

$$D_{\overline{\delta}} = \{x \in D : \mathrm{dist}(x, \partial D) > \overline{\delta}\} \tag{2.8}$$

which is a proper subset of D, and the set $\mathbb{S}^1$ is the unit circle synonymous with the interval $[0, 2\pi]$ of identified end points. The scaling factor ϵ is bounded by δ:

$$0 < \epsilon < \delta. \tag{2.9}$$

To describe $\overline{\delta}$ and δ more precisely, recall the Green's function $G(x, y)$ of $-\Delta$ on D with the Neumann boundary condition. It satisfies

$$-\Delta G(\cdot, y) = \delta(\cdot - y) - \frac{1}{|D|} \text{ in } D, \quad \partial_n G(\cdot, y) = 0 \text{ on } \partial D, \quad \int_D G(x, y)\, dx = 0 \tag{2.10}$$

for every $y \in D$. Here $\delta(\cdot - y)$ is the delta measure centered at y and $\partial_n G$ stands for the outward normal derivative at ∂D of G with respect to its first argument x. One can write

$$G(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + R(x, y), \tag{2.11}$$

where R is the regular part of G, a smooth function on $D \times D$. It is known that

$$R(z,z) \to \infty \quad \text{as } z \to \partial D. \tag{2.12}$$

We choose $\overline{\delta}$ small enough so that

$$\min_{z\in D} R(z,z) < \min_{z \in D\backslash D_{\overline{\delta}}} R(z,z). \tag{2.13}$$

This $\overline{\delta}$ is fixed throughout the paper. Next take δ such that

$$0 < 2\max\{r_1, r_2\}\delta < \overline{\delta}. \tag{2.14}$$

For the moment we only assume that δ satisfies (2.14). Later more conditions on δ will be imposed.

With ϵ bounded by δ and ξ in $D_{\overline{\delta}}$, define a transformation $T_{\epsilon,\xi,\theta}$ by

$$T_{\epsilon,\xi,\theta} : \hat{x} \to \epsilon e^{\mathrm{i}\theta}\hat{x} + \xi. \tag{2.15}$$

Then the scaled down double bubble is $T_{\epsilon,\xi,\theta}(E)$:

$$T_{\epsilon,\xi,\theta}(E) = (T_{\epsilon,\xi,\theta}(E_1), T_{\epsilon,\xi,\theta}(E_2)), \quad \text{where } T_{\epsilon,\xi,\theta}(E_i) = \{\epsilon e^{\mathrm{i}\theta}\hat{x} + \xi : \ \hat{x} \in E_i\}. \tag{2.16}$$

Our choice of $\overline{\delta}$ and δ ensures that $T_{\epsilon,\xi,\theta}(E_i) \subset D$.

The next lemma estimates the energy of $T_{\epsilon,\xi,\theta}(E)$. Let

$$\overline{\overline{\delta}} = \overline{\delta} - 2\max\{r_1, r_2\}\delta > 0 \tag{2.17}$$

and

$$D_{\overline{\overline{\delta}}} = \{x \in D : \ \mathrm{dist}(x, \partial D) > \overline{\overline{\delta}}\}. \tag{2.18}$$

If a double bubble $T_{\epsilon,\xi,\theta}(E)$ satisfies (2.9) and $\xi \in \overline{D_{\overline{\delta}}}$, then $T_{\epsilon,\xi,\theta}(E_i) \subset \overline{D_{\overline{\overline{\delta}}}}$. Actually $T_{\epsilon,\xi,\theta}(E_i)$ has some distance from $\partial D_{\overline{\overline{\delta}}}$, so a small perturbation of $T_{\epsilon,\xi,\theta}(E_i)$ will remain in $\overline{D_{\overline{\overline{\delta}}}}$, a property needed in later sections.

Lemma 2.1 *The energy $\mathcal{J}(T_{\epsilon,\xi,\theta}(E))$ of the scaled down exact double bubble $T_{\epsilon,\xi,\theta}(E)$ is estimated as follows:*

$$\Big|\mathcal{J}(T_{\epsilon,\xi,\theta}(E)) - \Big\{2\epsilon\sum_{i=0}^{2} a_i r_i + \sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\Big[\frac{\epsilon^4}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)m_i m_j + \epsilon^4\int_{E_i}\int_{E_j}\frac{1}{2\pi}\log\frac{1}{|\hat{x}-\hat{y}|}\,d\hat{x}d\hat{y} + \epsilon^4 m_i m_j R(\xi,\xi)\Big]\Big\}\Big|$$

$$\leq 2\max\{r_1, r_2\}\max_{x,y\in\overline{D_{\overline{\delta}}}}|\nabla R(x,y)|\Big(\sum_{i,j=1}^{2}|\gamma_{ij}|m_i m_j\Big)\epsilon^5,$$

where $m_1 = m$, $m_2 = 1-m$, and ∇R denotes the gradient of $R(x,y)$ with respect to its first variable x.

Proof. In this proof the transformation $T_{\epsilon,\xi,\theta}$ is written simply as T. Clearly the first term of $\mathcal{J}(T(E))$ is

$$\frac{1}{2}(\mathcal{P}_D(T(E_1)) + \mathcal{P}_D(T(E_2)) + \mathcal{P}_D(D\backslash(T(E_1)\cup T(E_2)))) = 2\epsilon\sum_{i=0}^{2} a_i r_i. \tag{2.19}$$

To estimate the second term of $\mathcal{J}(T(E))$ note that, with the help of the Green's function G,

$$\int_D\int_D\Big((-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i)\Big)\Big((-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j)\Big)\,dx = \int_{\Omega_i}\int_{\Omega_j} G(x,y)\,dxdy. \tag{2.20}$$

Therefore

$$\int_{T(E_i)}\int_{T(E_j)} G(x,y)\,dxdy \ = \int_{T(E_i)}\int_{T(E_j)}\Big(\frac{1}{2\pi}\log\frac{1}{|x-y|} + R(x,y)\Big)\,dxdy \tag{2.21}$$

$$= \ \frac{\epsilon^4}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)m_i m_j + \epsilon^4\int_{E_i}\int_{E_j}\frac{1}{2\pi}\log\frac{1}{|\hat{x}-\hat{y}|}\,d\hat{x}d\hat{y} + \epsilon^4\int_{E_i}\int_{E_j} R(\epsilon e^{\mathrm{i}\theta}\hat{x}+\xi, \epsilon e^{\mathrm{i}\theta}\hat{y}+\xi)\,d\hat{x}d\hat{y}.$$

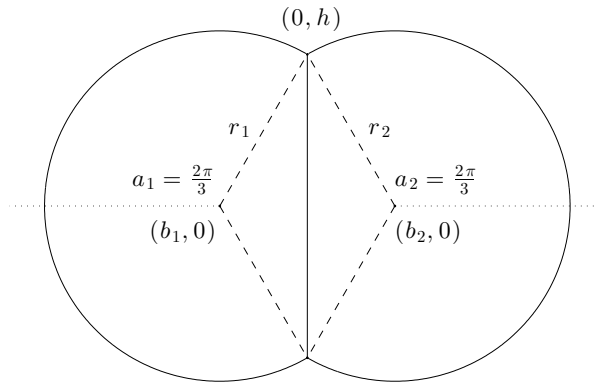

Figure 3: A symmetric exact double bubble in which $r_1 = r_2$ and $r_0 = \infty$.

For the last term note that by the symmetry $R(x,y) = R(y,x)$, there exists $\tau \in (0,1)$ such that

$$\begin{aligned} &|R(\epsilon e^{\mathrm{i}\theta}\hat{x} + \xi, \epsilon e^{\mathrm{i}\theta}\hat{y} + \xi) - R(\xi,\xi)| \\ &= |\nabla R(\tau\epsilon e^{\mathrm{i}\theta}\hat{x} + \xi, \tau\epsilon e^{\mathrm{i}\theta}\hat{y} + \xi)\cdot(\epsilon e^{\mathrm{i}\theta}\hat{x}) + \widetilde{\nabla} R(\tau\epsilon e^{\mathrm{i}\theta}\hat{x} + \xi, \tau\epsilon e^{\mathrm{i}\theta}\hat{y} + \xi)\cdot(\epsilon e^{\mathrm{i}\theta}\hat{y})| \\ &\leq \Big(\max_{x,y\in\overline{D_{\overline{\delta}}}} |\nabla R(x,y)|\Big)(|\epsilon\hat{x}| + |\epsilon\hat{y}|) \ \leq 4\epsilon\max\{r_1,r_2\} \max_{x,y\in\overline{D_{\overline{\delta}}}} |\nabla R(x,y)|, \end{aligned} \tag{2.22}$$

where $\widetilde{\nabla}$ denotes the gradient of $R(x,y)$ with respect to its second variable y. The lemma follows from (2.19), (2.21), and (2.22). $\square$

Consider a situation where the exact double bubble $T_{\epsilon,\xi,\theta}(E)$ is perturbed to a set $\Omega = (\Omega_1, \Omega_2)$. The boundaries $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1 \cap \partial\Omega_2$, are parametrized by $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, and $\mathbf{r}_0(t)$ ($t \in [-1,1]$), respectively. Here the perturbations are assumed to be sufficiently smooth so that Ω_1 and Ω_2 are disjoint, share part of their boundaries, and have two triple points. Later we will consider perturbations with more specific properties.

The two triple points correspond to $t = 1$ and $t = -1$, respectively, in each of the $\mathbf{r}_i$'s. Since the three curves $\mathbf{r}_i$ meet at these two points, the conditions

$$\mathbf{r}_1(1) = \mathbf{r}_2(1) = \mathbf{r}_0(1) \ \text{ and } \ \mathbf{r}_1(-1) = \mathbf{r}_2(-1) = \mathbf{r}_0(-1) \tag{2.23}$$

must hold.

The unit tangent vectors of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_0$ are denoted $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_0$ and given by

$$\mathbf{T}_i(t) = \frac{\mathbf{r}_i'(t)}{|\mathbf{r}_i'(t)|}. \tag{2.24}$$

The unit normal vectors to $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_0$ are $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_0$, respectively. We adopt the following direction convention: $\mathbf{N}_1$ points inward with respect to Ω_1, $\mathbf{N}_2$ points inward with respect to Ω_2, and $\mathbf{N}_0$ points from Ω_2 towards Ω_1, i.e. inward with respect to Ω_1 and outward with respect to Ω_2. The curvature of $\mathbf{r}_i$ is denoted κ_i. Here $\mathbf{N}_i$ and κ_i conform to the sign convention so that $\kappa_i\mathbf{N}_i$ is the (orientation independent) curvature vector. Under this sign convention

$$\frac{d\mathbf{T}_i}{ds} = k_i\mathbf{N}_i, \tag{2.25}$$

where $ds = |\mathbf{r}_i'(t)|dt$ is the length element.

The following two lemmas can be proved by direct computation.

Lemma 2.2 *Let $\mathbf{r}^\varepsilon(t)$ be a deformation of $\mathbf{r}(t)$ with $\mathbf{r}^0 = \mathbf{r}$. Let $\mathbf{X}$ be the infinitesimal element of the deformation $\mathbf{r}^\varepsilon$: $\mathbf{X}(t) = \frac{\partial \mathbf{r}^\varepsilon(t)}{\partial \varepsilon}|_{\varepsilon=0}$. Then*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{-1}^{1} |(\mathbf{r}^\varepsilon)'|\, dt = \mathbf{T}\cdot\mathbf{X}\Big|_{-1}^{1} - \int_{-1}^{1} \kappa \mathbf{N}\cdot\mathbf{X}\, ds,$$

where $\int_{-1}^{1} |(\mathbf{r}^\varepsilon)'|\, dt$ is the length of $\mathbf{r}^\varepsilon$.

Lemma 2.3 *Suppose that a bounded domain U is enclosed by a curve ∂U, and U^ε is a deformation of U. Let $\mathbf{X}$ be the infinitesimal element of the deformation of ∂U. Then*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{U^\varepsilon} f(x)\, dx = -\int_{\partial U} f(x)\mathbf{N}\cdot\mathbf{X}\, ds$$

where $\mathbf{N}$ is the inward unit normal vector on ∂U.

Let Ω be a perturbed double bubble. A deformation Ω^ε of Ω is a family of perturbed double bubbles parametrized by ε in a neighborhood of 0. The three curves $\partial\Omega_1^\varepsilon\backslash\partial\Omega_2^\varepsilon$, $\partial\Omega_2^\varepsilon\backslash\partial\Omega_1^\varepsilon$, and $\partial\Omega_1^\varepsilon\cup\partial\Omega_2^\varepsilon$ that enclose Ω^ε are parametrized, respectively, by $\mathbf{r}_1^\varepsilon$, $\mathbf{r}_2^\varepsilon$, and $\mathbf{r}_0^\varepsilon$. At $\varepsilon = 0$, $\mathbf{r}_i^0 = \mathbf{r}_i$; $\mathbf{r}_i^\varepsilon$ also satisfy the compatibility condition (2.23). Define

$$\mathbf{X}_i(t) = \frac{\partial \mathbf{r}_i^\varepsilon(t)}{\partial \varepsilon}\Big|_{\varepsilon=0} \tag{2.26}$$

which is the infinitesimal element of the deformation $\mathbf{r}_i^\varepsilon$.

We introduce $\mathcal{J}_s(\Omega)$ and $\mathcal{J}_l(\Omega)$, the short and the long range parts of the free energy, to denote the first and the second terms of $\mathcal{J}(\Omega)$ in (1.1), respectively.

Lemma 2.4 *Let Ω^ε be a deformation of a perturbed double bubble Ω. The three curves $\partial\Omega_1^\varepsilon\backslash\partial\Omega_2^\varepsilon$, $\partial\Omega_2^\varepsilon\backslash\partial\Omega_1^\varepsilon$ and $\partial\Omega_1^\varepsilon\cap\partial\Omega_2^\varepsilon$, are parametrized by $\mathbf{r}_1^\varepsilon(t)$, $\mathbf{r}_2^\varepsilon(t)$ and $\mathbf{r}_0^\varepsilon(t)$ respectively, which satisfy (2.23). Then*

$$\begin{aligned}
\frac{d\mathcal{J}_s(\Omega^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} &= (\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)\cdot\mathbf{X}\Big|_{-1}^{1} - \int_{\partial\Omega_1\backslash\partial\Omega_2} \kappa_1\mathbf{N}_1\cdot\mathbf{X}_1\, ds - \int_{\partial\Omega_2\backslash\partial\Omega_1} \kappa_2\mathbf{N}_2\cdot\mathbf{X}_2\, ds \\
&\quad - \int_{\partial\Omega_1\cap\partial\Omega_2} \kappa_0\mathbf{N}_0\cdot\mathbf{X}_0\, ds, && (2.27)\\
\frac{d\mathcal{J}_l(\Omega^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} &= -\int_{\partial\Omega_1\backslash\partial\Omega_2} (\gamma_{11}I_{\Omega_1}+\gamma_{12}I_{\Omega_2})\mathbf{N}_1\cdot\mathbf{X}_1\, ds - \int_{\partial\Omega_2\backslash\partial\Omega_1} (\gamma_{12}I_{\Omega_1}+\gamma_{22}I_{\Omega_2})\mathbf{N}_2\cdot\mathbf{X}_2\, ds \\
&\quad - \int_{\partial\Omega_1\cap\partial\Omega_2} ((\gamma_{11}-\gamma_{12})I_{\Omega_1}+(\gamma_{12}-\gamma_{22})I_{\Omega_2})\mathbf{N}_0\cdot\mathbf{X}_0\, ds, && (2.28)\\
\frac{d|\Omega_1^\varepsilon|}{d\varepsilon}\Big|_{\varepsilon=0} &= -\int_{\partial\Omega_1\backslash\partial\Omega_2} \mathbf{N}_1\cdot\mathbf{X}_1\, ds - \int_{\partial\Omega_1\cap\partial\Omega_2} \mathbf{N}_0\cdot\mathbf{X}_0\, ds, && (2.29)\\
\frac{d|\Omega_2^\varepsilon|}{d\varepsilon}\Big|_{\varepsilon=0} &= -\int_{\partial\Omega_2\backslash\partial\Omega_1} \mathbf{N}_2\cdot\mathbf{X}_2\, ds + \int_{\partial\Omega_1\cap\partial\Omega_2} \mathbf{N}_0\cdot\mathbf{X}_0\, ds. && (2.30)
\end{aligned}$$

In (2.27) of Lemma 2.4, $\mathbf{X}$ denotes the $\mathbf{X}_i$'s at the triple points. Since (2.23) holds for $\mathbf{r}_i^\varepsilon$, $\mathbf{X}$ is well defined.

Proof. . The first formula (2.27) follows directly from Lemma 2.2.

To show (2.28), recall I_{Ω_i} from (1.6) which can be written as

$$I_{\Omega_i}(x) = \int_{\Omega_i} G(x,y)\, dy, \quad i = 1, 2, \tag{2.31}$$

in terms of the Green's function. Then the product rule of differentiation implies that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\Omega_i^\varepsilon}\int_{\Omega_j^\varepsilon} G(x,y)\,dx\,dy = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\Omega_i^\varepsilon} I_{\Omega_j}(x)\,dx + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\Omega_j^\varepsilon} I_{\Omega_i}(x)\,dx. \tag{2.32}$$

However, Lemma 2.3 shows

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\Omega_i^\varepsilon} I_{\Omega_j}(x)\,dx = \begin{cases} -\displaystyle\int_{\partial\Omega_1\setminus\partial\Omega_2} I_{\Omega_j}\mathbf{N}_1\cdot\mathbf{X}_1\,ds - \int_{\partial\Omega_1\cap\partial\Omega_2} I_{\Omega_j}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, & i=1,\\ -\displaystyle\int_{\partial\Omega_2\setminus\partial\Omega_1} I_{\Omega_j}\mathbf{N}_2\cdot\mathbf{X}_2\,ds + \int_{\partial\Omega_1\cap\partial\Omega_2} I_{\Omega_j}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, & i=2. \end{cases} \tag{2.33}$$

Therefore

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\Omega_i^\varepsilon}\int_{\Omega_j^\varepsilon} G(x,y)\,dx\,dy \tag{2.34}$$

$$= \begin{cases} -2\displaystyle\int_{\partial\Omega_1\setminus\partial\Omega_2} I_{\Omega_1}\mathbf{N}_1\cdot\mathbf{X}_1\,ds - 2\int_{\partial\Omega_1\cap\partial\Omega_2} I_{\Omega_1}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, & i=j=1,\\ -2\displaystyle\int_{\partial\Omega_2\setminus\partial\Omega_1} I_{\Omega_2}\mathbf{N}_2\cdot\mathbf{X}_2\,ds + 2\int_{\partial\Omega_1\cap\partial\Omega_2} I_{\Omega_2}\mathbf{N}_0\cdot\mathbf{X}_0\,ds, & i=j=2,\\ -\displaystyle\int_{\partial\Omega_1\setminus\partial\Omega_2} I_{\Omega_2}\mathbf{N}_1\cdot\mathbf{X}_1\,ds - \int_{\partial\Omega_2\setminus\partial\Omega_1} I_{\Omega_1}\mathbf{N}_2\cdot\mathbf{X}_2\,ds - \int_{\partial\Omega_1\cap\partial\Omega_2} (I_{\Omega_2}-I_{\Omega_1})\mathbf{N}_0\cdot\mathbf{X}_0\,ds, & i=1, j=2. \end{cases}$$

Hence,

$$\begin{aligned} \frac{d\mathcal{J}_l(\Omega^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{\Omega_i^\varepsilon}\int_{\Omega_j^\varepsilon} G(x,y)\,dx\,dy \\ &= -\int_{\partial\Omega_1\setminus\partial\Omega_2}(\gamma_{11}I_{\Omega_1}+\gamma_{12}I_{\Omega_2})\mathbf{N}_1\cdot\mathbf{X}_1\,ds - \int_{\partial\Omega_2\setminus\partial\Omega_1}(\gamma_{12}I_{\Omega_1}+\gamma_{22}I_{\Omega_2})\mathbf{N}_2\cdot\mathbf{X}_2\,ds \\ &\quad -\int_{\partial\Omega_1\cap\partial\Omega_2}[(\gamma_{11}-\gamma_{12})I_{\Omega_1}+(\gamma_{12}-\gamma_{22})I_{\Omega_2}]\mathbf{N}_0\cdot\mathbf{X}_0\,ds. \end{aligned} \tag{2.35}$$

This proves (2.28).

The formulas (2.29) and (2.30) follow from Lemma 2.3 with $f(x)=1$. □

3 Restricted perturbations

Let E be an exact double bubble in $\mathbb{R}^2$ with two triple points at $(0,h)$ and $(0,-h)$. We perform a particular type of perturbation in two steps.

In the first step, the two triple points are moved vertically to $(0,\eta)$ and $(0,-\eta)$, respectively. The three circular arcs are perturbed to three new circular arcs whose radii are ρ_1, ρ_2, and ρ_0; the angles a_i are perturbed to α_i accordingly; see Figure 4. The ρ_i's are required to satisfy the equation $\frac{1}{\rho_1}-\frac{1}{\rho_2}=\frac{1}{\rho_0}$. The ρ_i's and the α_i's are determined from η implicitly by solving the following system of equations

$$\rho_1^2(\alpha_1-\cos\alpha_1\sin\alpha_1)+\rho_0^2(\alpha_0-\cos\alpha_0\sin\alpha_0) = m, \tag{3.1}$$

$$\rho_2^2(\alpha_2-\cos\alpha_2\sin\alpha_2)-\rho_0^2(\alpha_0-\cos\alpha_0\sin\alpha_0) = 1-m, \tag{3.2}$$

$$\rho_i\sin\alpha_i = \eta,\quad i=1,2,0, \tag{3.3}$$

$$\rho_1^{-1}-\rho_2^{-1} = \rho_0^{-1}. \tag{3.4}$$

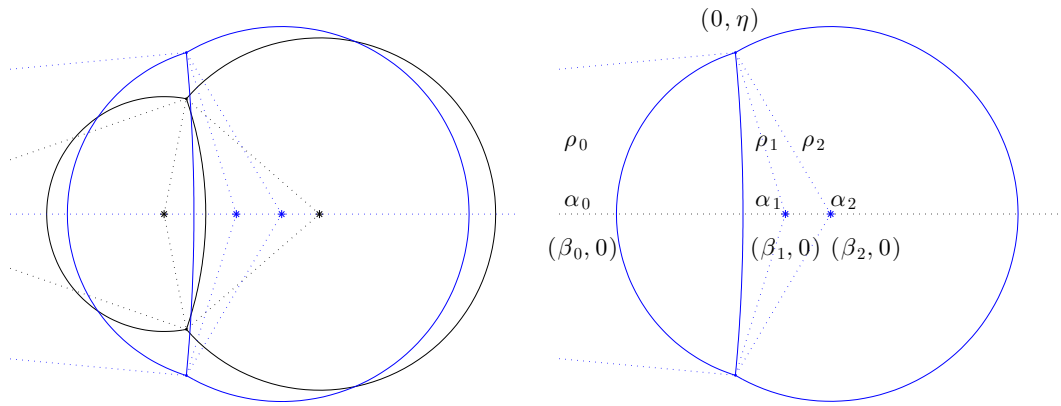

Figure 4: First step of perturbation. Left: the exact double bubble is perturbed to a pair of sets bounded by three circular arcs governed by (3.1) - (3.4). Right: the same perturbed pair without the exact double bubble. Also showing are the angles α_i, the radii ρ_i, the centers $(\beta_i, 0)$, and one triple point $(0, \eta)$.

The regions bounded by the new arcs still have the areas m and $1-m$; hence (3.1) and (3.2). The centers of the new arcs are denoted $(\beta_i, 0)$, $i = 1, 2, 0$. This step is explained in more detail and shown to be well defined when η is close to h in Appendix B.

In the second step of the perturbation we further perturb the shape of the circular arcs. Introduce three functions $u_i(t)$, $i = 1, 2, 0$, for $t \in (-1, 1)$. The circular arcs are replaced by curves parametrized by

$$u_1(t)e^{\mathrm{i}(\pi-\alpha_1 t)} + \beta_1, \quad u_2(t)e^{\mathrm{i}\alpha_2 t} + \beta_2, \quad u_0(t)e^{\mathrm{i}\alpha_0 t} + \beta_0; \tag{3.5}$$

see Figure 5. It is required that the u_i's do not change the triple points $(0, \eta)$ and $(0, -\eta)$. Therefore

$$u_i(\pm 1) = \rho_i. \tag{3.6}$$

Note that a sector perturbed by u_i has the area $\int_{-1}^{1} \frac{\alpha_i u_i^2(t)}{2}\, dt$. Since the areas of the newly perturbed regions must still be m and $1-m$, one requires that

$$\int_{-1}^{1} \frac{\alpha_1 u_1^2(t) - \rho_1^2 \cos\alpha_1 \sin\alpha_1}{2}\, dt + \int_{-1}^{1} \frac{\alpha_0 u_0^2(t) - \rho_0^2 \cos\alpha_0 \sin\alpha_0}{2}\, dt \quad = \quad m, \tag{3.7}$$

$$\int_{-1}^{1} \frac{\alpha_2 u_2^2(t) - \rho_2^2 \cos\alpha_2 \sin\alpha_2}{2}\, dt - \int_{-1}^{1} \frac{\alpha_0 u_0^2(t) - \rho_0^2 \cos\alpha_0 \sin\alpha_0}{2}\, dt \quad = \quad 1-m. \tag{3.8}$$

This perturbed double bubble is denoted $F = (F_1, F_2)$.

Similarly to the exact double bubble E and its image $T_{\epsilon,\xi,\theta}(E)$ under the transformation $T_{\epsilon,\xi,\theta}$, the perturbed double bubble F is also transformed by $T_{\epsilon,\xi,\theta}$, and the scaled down version of F is denoted Ω:

$$\Omega = T_{\epsilon,\xi,\theta}(F) = (T_{\epsilon,\xi,\theta}(F_1), T_{\epsilon,\xi,\theta}(F_2)). \tag{3.9}$$

The boundaries $\partial\Omega_1 \backslash \partial\Omega_2$, $\partial\Omega_2 \backslash \partial\Omega_1$, $\partial\Omega_1 \cap \partial\Omega_2$ of Ω are parametrized by

$$\mathbf{r}_i(t) = \begin{cases} T_{\epsilon,\xi,\theta}\big(u_1(t)e^{\mathrm{i}(\pi-\alpha_1 t)} + \beta_1\big) & \text{if} \quad i = 1, \\ T_{\epsilon,\xi,\theta}\big(u_i(t)e^{\mathrm{i}\alpha_i t} + \beta_i\big) & \text{if} \quad i = 2, 0, \end{cases} \tag{3.10}$$

respectively. Consequently

$$\mathbf{r}_i'(t) = \begin{cases} \epsilon e^{\mathrm{i}\theta}(u_1'(t)e^{\mathrm{i}(\pi-\alpha_1 t)} + \alpha_1 u_1(t)e^{\mathrm{i}(\pi-\alpha_1 t)}(-\mathrm{i})) & \text{if} \quad i = 1, \\ \epsilon e^{\mathrm{i}\theta}(u_i'(t)e^{\mathrm{i}\alpha_i t} + \alpha_i u_i(t)e^{\mathrm{i}\alpha_i t}\mathrm{i}) & \text{if} \quad i = 2, 0, \end{cases} \tag{3.11}$$

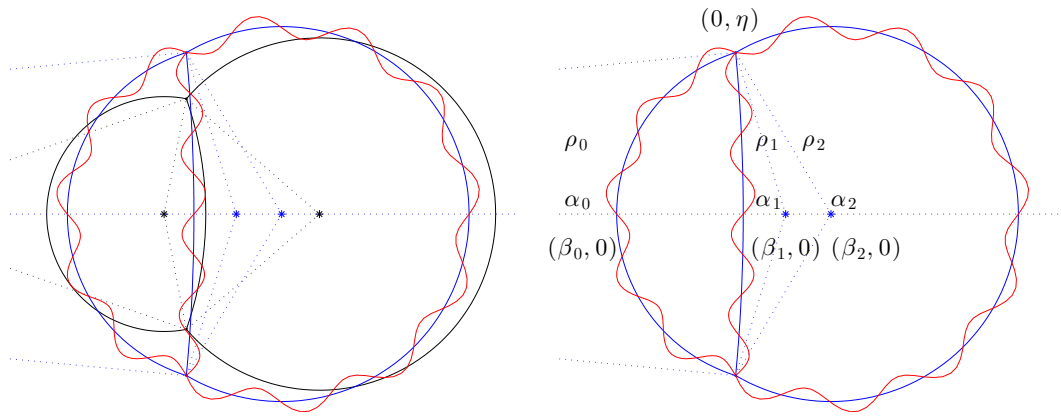

Figure 5: Second step of perturbation. Left: The circular arcs obtained in the first step are perturbed to more general curves. Right: the same perturbed double bubble without the exact double bubble showing.

and the tangent and normal vectors are given by

$$\mathbf{T}_i(t) = \frac{\mathbf{r}_i'(t)}{|\mathbf{r}_i'(t)|}, \quad \mathbf{N}_i(t) = \begin{cases} \mathbf{T}_1(t)(-\mathrm{i}) & \text{if} \quad i=1, \\ \mathbf{T}_i(t)\mathrm{i} & \text{if} \quad i=2,0. \end{cases} \tag{3.12}$$

Although the u_i's describe the shape of the perturbed double bubble well, the constraints (3.7) and (3.8) are nonlinear and hard to work with. We introduce new variables ϕ_i, $i=1,2,0$, in place of u_i,

$$\phi_i(t) = \frac{\alpha_i u_i^2(t) - \alpha_i \rho_i^2}{2}, \tag{3.13}$$

to describe the perturbed double bubble F. Write ϕ for (ϕ_1, ϕ_2, ϕ_0). F now depends on (ϕ, η), and the scaled down version Ω depends on ϵ, ξ, θ, and (ϕ, η). We call ϕ_i and η internal variables.

Because ρ_i and α_i satisfy the conditions (3.1) and (3.2), the area constraints (3.7) and (3.8) become linear constraints,

$$\int_{-1}^{1} \phi_1(t)\,dt + \int_{-1}^{1} \phi_0(t)\,dt = 0 \quad \text{and} \quad \int_{-1}^{1} \phi_2(t)\,dt - \int_{-1}^{1} \phi_0(t)\,dt = 0, \tag{3.14}$$

on the ϕ_i's. By (3.6) and (3.13) the ϕ_i's also satisfy the boundary condition

$$\phi_i(\pm 1) = 0, \quad i = 1,2,0, \tag{3.15}$$

in order for the triple points $(0, \pm\eta)$ to stay unchanged.

The length of each perturbed arc in F is

$$\int_{-1}^{1} \sqrt{(u_i'(t))^2 + \alpha_i^2 u_i^2(t)}\,dt, \quad i = 1,2,0, \tag{3.16}$$

in terms of the variable u_i. In terms of ϕ_i this becomes

$$\int_{-1}^{1} L_i(\phi_i', \phi_i, \eta)\,dt, \quad \text{where} \quad L_i(\phi_i', \phi_i, \eta) = \sqrt{\frac{(\phi_i')^2}{\alpha_i(2\phi_i + \alpha_i\rho_i^2)} + \alpha_i(2\phi_i + \alpha_i\rho_i^2)}. \tag{3.17}$$

By (3.17) and (2.20) the energy of Ω can be written as

$$\mathcal{J}(\Omega) = \epsilon \sum_{i=0}^{2} \int_{-1}^{1} L_i(\phi_i', \phi_i, \eta)\,dt + \sum_{i=1}^{2} \frac{\gamma_{ij}}{2} \int_{\Omega_i} \int_{\Omega_j} G(x,y)\,dx\,dy. \tag{3.18}$$

The first term in (3.18) is the short range energy $\mathcal{J}_s(\Omega)$ and the second term is the long range energy $\mathcal{J}_l(\Omega)$.

To specify the domain of the functional $\mathcal{J}$ in the restricted class of perturbed double bubbles, let

$$\mathcal{Y} = \{(\phi,\eta) \in H_0^1((-1,1);\mathbb{R}^3) \times \mathbb{R} : \int_{-1}^{1} (\phi_1(t) + \phi_0(t))\, dt = \int_{-1}^{1} (\phi_2(t) - \phi_0(t))\, dt = 0\}. \tag{3.19}$$

This space is equipped with the norm derived from the usual H^1 norm; see (4.23). Note that $(0,h)$ represents the exact double bubble E, where $\phi_i = 0$ and $\eta = h$.

The functional is defined on a neighborhood of $(0,h) \in \mathcal{Y}$; namely, there exists $\bar{c} > 0$ such that the domain of $\mathcal{J}$ is the open ball of radius $\bar{c}$ centered at $(0,h)$ in $\mathcal{Y}$:

$$\mathcal{D}(\mathcal{J}) = \{(\phi,\eta) \in \mathcal{Y} : \ \|(\phi,\eta - h)\|_{\mathcal{Y}} < \bar{c}\}. \tag{3.20}$$

Note that $\bar{c}$ does not depend on ϵ or γ. It only needs to be small enough so that the resulting perturbed double bubbles stay inside the subset $D_{\bar{\bar{\delta}}}$ of D, which is given in (2.18).

4 First variation

Since a perturbed double bubble Ω is described by internal variables ϕ_i and η, there is an easy way to generate deformations Ω^ε. Start with a deformation of $(\phi,\eta) \in \mathcal{D}(\mathcal{J})$ in the form

$$\phi_i \to \phi_i + \varepsilon\psi_i, \quad \eta \to \eta + \varepsilon\zeta \tag{4.1}$$

for $(\psi,\zeta) \in \mathcal{Y}$. Then (3.13) defines a deformation of u_i denoted by u_i^ε (with u_i^0 being u_i), namely, by

$$\phi_i + \varepsilon\psi_i = \frac{\alpha_i(\eta+\varepsilon\zeta)(u_i^\varepsilon)^2 - \alpha_i(\eta+\varepsilon\zeta)\rho_i^2(\eta+\varepsilon\zeta)}{2}. \tag{4.2}$$

Here α_i and ρ_i are treated as functions of η. Differentiating (4.2) with respect to ε and setting ε to be 0 yield

$$\psi_i = \alpha_i u_i \frac{\partial u_i^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0} + \frac{\alpha_i'\zeta u_i^2}{2} - \frac{\alpha_i'\zeta\rho_i^2}{2} - \alpha_i\rho_i\rho_i'\zeta. \tag{4.3}$$

Note that since α_i, ρ_i, and β_i depend on η,

$$\frac{d\alpha_i(\eta+\varepsilon\zeta)}{d\varepsilon}\Big|_{\varepsilon=0} = \alpha_i'(\eta)\zeta, \quad \frac{d\rho_i(\eta+\varepsilon\zeta)}{d\varepsilon}\Big|_{\varepsilon=0} = \rho_i'(\eta)\zeta, \quad \frac{d\beta_i(\eta+\varepsilon\zeta)}{d\varepsilon}\Big|_{\varepsilon=0} = \beta_i'(\eta)\zeta. \tag{4.4}$$

In (4.3) α_i, α_i', ρ_i, ρ_i' are all functions of η and are all evaluated at η.

Recall $\mathbf{X}_i$ from (2.26) so here

$$\mathbf{X}_i = \begin{cases} \epsilon e^{\mathrm{i}\theta}\left(\frac{\partial u_1^\varepsilon}{\partial\varepsilon}\big|_{\varepsilon=0} e^{\mathrm{i}(\pi-\alpha_1 t)} + \alpha_1'\zeta u_1 t e^{\mathrm{i}(\pi-\alpha_1 t)}(-\mathrm{i}) + \beta_1'\zeta\right) & \text{if} \quad i=1, \\ \epsilon e^{\mathrm{i}\theta}\left(\frac{\partial u_i^\varepsilon}{\partial\varepsilon}\big|_{\varepsilon=0} e^{\mathrm{i}\alpha_i t} + \alpha_i'\zeta u_i t e^{\mathrm{i}\alpha_i t}\mathrm{i} + \beta_i'\zeta\right) & \text{if} \quad i=2,0. \end{cases} \tag{4.5}$$

Lemma 4.1 *At the triple points* $\mathbf{X}_i(\pm1) = \zeta\,\mathbf{X}^S(\pm1)$, *where* $\mathbf{X}^S(\pm1) = \pm\epsilon e^{\mathrm{i}\theta}\mathrm{i}$.

Proof. By (4.3), since $\psi_i(\pm1) = 0$, $u_i(\pm1) = \rho_i$,

$$\frac{\partial u_i^\varepsilon(\pm1)}{\partial\varepsilon}\Big|_{\varepsilon=0} = \rho_i'\zeta$$

and, hence,

$$\begin{aligned} \mathbf{X}_i(\pm1) &= \begin{cases} \epsilon e^{\mathrm{i}\theta}(\rho_1'\zeta e^{\mathrm{i}(\pi\mp\alpha_1)} \pm \alpha_1'\rho_1\zeta e^{\mathrm{i}(\pi\mp\alpha_1)}(-\mathrm{i}) + \beta_1'\zeta) & \text{if} \quad i=1, \\ \epsilon e^{\mathrm{i}\theta}(\rho_i'\zeta e^{\pm\mathrm{i}\alpha_i} \pm \alpha_i'\rho_i\zeta e^{\pm\mathrm{i}\alpha_i}\mathrm{i} + \beta_i'\zeta) & \text{if} \quad i=2,0 \end{cases} \\ &= \begin{cases} \zeta\epsilon e^{\mathrm{i}\theta}\frac{d(\rho_1 e^{\mathrm{i}(\pi\mp\alpha_1)}+\beta_1)}{d\eta} & \text{if} \quad i=1, \\ \zeta\epsilon e^{\mathrm{i}\theta}\frac{d(\rho_i e^{\pm\mathrm{i}\alpha_i}+\beta_i)}{d\eta} & \text{if} \quad i=2,0 \end{cases} \\ &= \zeta\epsilon e^{\mathrm{i}\theta}\frac{d(\pm\eta\mathrm{i})}{d\eta} \ = \ \zeta(\pm\epsilon e^{\mathrm{i}\theta}\mathrm{i}). \end{aligned}$$

In this proof, ρ_i', α_i', and β_i' are derivatives of ρ_i, α_i, and β_i with respect to η evaluated at η. $\square$

The vectors $\mathbf{X}^S(\pm 1)$ suggest a deformation that stretches the triple points.

Next compute

$$-\mathbf{N}_i \cdot \mathbf{X}_i \, ds = \begin{cases} (\mathbf{r}_1' \mathrm{i}) \cdot \mathbf{X}_1 \, dt & \text{if} \quad i = 1, \\ -(\mathbf{r}_i' \mathrm{i}) \cdot \mathbf{X}_i \, dt & \text{if} \quad i = 2, 0. \end{cases} \tag{4.6}$$

It follows from (3.11), (4.3), (4.5), and (4.6) that

$$\begin{aligned} &-\mathbf{N}_i \cdot \mathbf{X}_i \, ds \\ &= \begin{cases} \epsilon^2 \Big[\psi_1 + \Big(-\dfrac{\alpha_1' u_1^2}{2} - \alpha_1' u_1 u_1' t + \dfrac{\alpha_1' \rho_1^2}{2} + \alpha_1 \rho_1 \rho_1' + \beta_1' \cdot (\alpha_1 u_1 e^{\mathrm{i}(\pi - \alpha_1 t)} - u_1' e^{\mathrm{i}(\pi - \alpha_1 t)}(-\mathrm{i})) \Big) \zeta \Big] \, dt, \\ \epsilon^2 \Big[\psi_i + \Big(-\dfrac{\alpha_i' u_i^2}{2} - \alpha_i' u_i u_i' t + \dfrac{\alpha_i' \rho_i^2}{2} + \alpha_i \rho_i \rho_i' + \beta_i' \cdot (\alpha_i u_i e^{\mathrm{i}\alpha_i t} - u_i' e^{\mathrm{i}\alpha_i t} \mathrm{i}) \Big) \zeta \Big] \, dt. \end{cases} \end{aligned} \tag{4.7}$$

Write (4.7) as

$$-\mathbf{N}_i \cdot \mathbf{X}_i \, ds = \epsilon^2 (\psi_i + \mathcal{E}_i(\phi_i, \eta) \zeta) \, dt, \tag{4.8}$$

where the $\mathcal{E}_i$'s are operators given by

$$\mathcal{E}_i(\phi_i, \eta) = \begin{cases} -\dfrac{\alpha_1' u_1^2}{2} - \alpha_1' u_1 u_1' t + \dfrac{\alpha_1' \rho_1^2}{2} + \alpha_1 \rho_1 \rho_1' + \beta_1' \cdot (\alpha_1 u_1 e^{\mathrm{i}(\pi - \alpha_1 t)} - u_1' e^{\mathrm{i}(\pi - \alpha_1 t)}(-\mathrm{i})), \\ -\dfrac{\alpha_i' u_i^2}{2} - \alpha_i' u_i u_i' t + \dfrac{\alpha_i' \rho_i^2}{2} + \alpha_i \rho_i \rho_i' + \beta_i' \cdot (\alpha_i u_i e^{\mathrm{i}\alpha_i t} - u_i' e^{\mathrm{i}\alpha_i t} \mathrm{i}), \end{cases} \tag{4.9}$$

where u_i is related to ϕ_i and η via (3.13). In (4.9) α_i', ρ_i', and β_i' are derivatives of α_i, ρ_i, and β_i with respect to η. All these functions of η, namely, α_i, α_i', ρ_i, ρ_i', β_i, and β_i', are evaluated at η. On the other hand u_i' in (4.9) is just the derivative of $u_i(t)$ with respect to t.

Define three more functions of η:

$$\mu_i = \rho_i^2 (\alpha_i - \cos \alpha_i \sin \alpha_i), \quad i = 1, 2, 0. \tag{4.10}$$

Geometrically for $i = 1, 2$, μ_i is the sum of the area of a sector and the area of a triangle, associated with the left or right arc, after the first step of restricted perturbation; see Figure 4. For $i = 0$, μ_0 is the difference of the area of a sector and the area of a triangle associated with the middle arc. By (3.1) and (3.2) the μ_i's satisfy

$$\mu_1 + \mu_0 = m, \quad \mu_2 - \mu_0 = 1 - m. \tag{4.11}$$

It is straightforward to show the following lemma.

Lemma 4.2 *The operator $\mathcal{E}_i$ satisfies the property*

$$\int_{-1}^{1} \mathcal{E}_i(\phi_i, \eta) \, dt = \mu_i'. \tag{4.12}$$

Moreover

$$\int_{-1}^{1} \mathcal{E}_1(\phi_1, \eta) \, dt + \int_{-1}^{1} \mathcal{E}_0(\phi_0, \eta) \, dt = \int_{-1}^{1} \mathcal{E}_2(\phi_2, \eta) \, dt - \int_{-1}^{1} \mathcal{E}_0(\phi_0, \eta) \, dt = 0. \tag{4.13}$$

Proof. By (4.9)

$$\begin{aligned} \int_{-1}^{1} \mathcal{E}_i(\phi_i, \eta) \, dt &= \begin{cases} -\dfrac{\alpha_1'}{2} t u_1^2 \Big|_{-1}^{1} + \alpha_1' \rho_1^2 + 2\alpha_1 \rho_1 \rho_1' - \beta_1' \cdot u_1 e^{\mathrm{i}(\pi - \alpha_1 t)}(-\mathrm{i}) \Big|_{-1}^{1} & \text{if} \quad i = 1, \\ -\dfrac{\alpha_i'}{2} t u_i^2 \Big|_{-1}^{1} + \alpha_i' \rho_i^2 + 2\alpha_i \rho_i \rho_i' - \beta_i' \cdot u_i e^{\mathrm{i}\alpha_i t} \mathrm{i} \Big|_{-1}^{1} & \text{if} \quad i = 2, 0 \end{cases} \\ &= \begin{cases} 2\alpha_1 \rho_1 \rho_1' - 2\eta \beta_1' & \text{if} \quad i = 1, \\ 2\alpha_i \rho_i \rho_i' + 2\eta \beta_i' & \text{if} \quad i = 2, 0. \end{cases} \end{aligned}$$

On the other hand,

$$\mu_i' = \begin{cases} (\alpha_1\rho_1^2 - \beta_1\eta)' \\ (\alpha_i\rho_i^2 + \beta_i\eta)' \end{cases} = \begin{cases} 2\alpha_1\rho_1\rho_1' + \alpha_1'\rho_1^2 - \beta_1 - \eta\beta_1' & \text{if} \quad i=1 \\ 2\alpha_i\rho_i\rho_i' + \alpha_i'\rho_i^2 + \beta_i + \eta\beta_i' & \text{if} \quad i=2,0 \end{cases}.$$

Hence

$$\begin{aligned} \mu_i' - \int_{-1}^{1} \mathcal{E}_i(\phi_i,\eta)\,dt &= \begin{cases} \alpha_1'\rho_1^2 - \beta_1 + \eta\beta_1' & \text{if} \quad i=1 \\ \alpha_i'\rho_i^2 + \beta_i - \eta\beta_i' & \text{if} \quad i=2,0 \end{cases} = \begin{cases} \alpha_1'\rho_1^2 - \beta_1^2(\frac{\eta}{\beta_1})' \\ \alpha_i'\rho_i^2 + \beta_i^2(\frac{\eta}{\beta_i})' \end{cases} \\ &= \alpha_i'\rho_i^2 - \beta_i^2(\tan\alpha_i)' = \alpha_i'\rho_i^2 - \beta_i^2(\sec\alpha_i)^2\alpha_i' = 0. \end{aligned}$$

This proves the first part of the lemma. The constraints (4.11) on μ_i imply that

$$\mu_1' + \mu_0' = \mu_2' - \mu_0' = 0$$

from which the second part follows. $\square$

Let $(\phi,\eta) \in \mathcal{D}(\mathcal{J})$ and $(\psi,\zeta) \in \mathcal{Y}$, and calculate

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{J}_s((\phi,\eta)+\varepsilon(\psi,\zeta)), \quad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{J}_l((\phi,\eta)+\varepsilon(\psi,\zeta)).$$

For the former if $\phi_i \in H^2(-1,1)$,

$$\begin{aligned} &\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{J}_s((\phi,\eta)+\varepsilon(\psi,\zeta)) \\ &= \epsilon\sum_{i=0}^{2}\int_{-1}^{1}\Big(\frac{\partial L_i(\phi_i',\phi_i,\eta)}{\partial\phi_i'}\psi_i' + \frac{\partial L_i(\phi_i',\phi_i,\eta)}{\partial\phi_i}\psi_i\Big)\,dt + \epsilon\Big(\sum_{i=0}^{2}\int_{-1}^{1}\frac{\partial L_i(\phi_i',\phi_i,\eta)}{\partial\eta}\,dt\Big)\zeta \\ &= \epsilon\sum_{i=0}^{2}\int_{-1}^{1}\Big(\frac{d}{dt}\Big(-\frac{\partial L_i(\phi_i',\phi_i,\eta)}{\partial\phi_i'}\Big) + \frac{\partial L_i(\phi_i',\phi_i,\eta)}{\partial\phi_i}\Big)\psi_i\,dt + \epsilon\Big(\sum_{i=0}^{2}\int_{-1}^{1}\frac{\partial L_i(\phi_i',\phi_i,\eta)}{\partial\eta}\,dt\Big)\zeta \\ &= \epsilon\int_{-1}^{1}\sum_{i=0}^{2}\mathcal{K}_i(\phi_i,\eta)\psi_i\,dt + \epsilon\widetilde{\mathcal{K}}(\phi,\eta)\zeta \\ &= \langle\epsilon(\mathcal{K}(\phi,\eta),\widetilde{\mathcal{K}}(\phi,\eta)),(\psi,\zeta)\rangle. \end{aligned} \tag{4.14}$$

In (4.14) the three operators $\mathcal{K}_i$, $i=0,1,2$, and the functional $\widetilde{\mathcal{K}}$ are given by

$$\mathcal{K}_i(\phi_i,\eta) = \frac{d}{dt}\Big(-\frac{\partial L_i(\phi_i',\phi_i,\eta)}{\partial\phi_i'}\Big) + \frac{\partial L_i(\phi_i',\phi_i,\eta)}{\partial\phi_i}, \ i=0,1,2, \tag{4.15}$$

$$\widetilde{\mathcal{K}}(\phi,\eta) = \sum_{i=0}^{2}\int_{-1}^{1}\frac{\partial L_i(\phi_i',\phi_i,\eta)}{\partial\eta}\,dt \tag{4.16}$$

and we write $\mathcal{K}$ for $(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_0)$.

In (4.14) the inner product $\langle\cdot,\cdot\rangle$ comes from the Hilbert space $L^2((-1,1);\mathbb{R}^3)\times\mathbb{R}$:

$$\langle(\phi,\eta),(\tilde\phi,\tilde\eta)\rangle = \sum_{i=0}^{2}\int_{-1}^{1}\phi_i(t)\tilde\phi_i(t)\,dt + \eta\tilde\eta. \tag{4.17}$$

Comparing (4.14) with (2.27) of Lemma 2.4 and using (4.8) one finds, with the help of Lemma 4.1,

$$\mathcal{K}_i = \epsilon\kappa_i, \quad i=1,2,0, \quad \text{and } \widetilde{\mathcal{K}} = \epsilon^{-1}\Big(\sum_{i=0}^{2}\mathbf{T}_i\Big)\cdot\mathbf{X}^S\Big|_{-1}^{1} + \sum_{i=0}^{2}\int_{-1}^{1}\mathcal{K}_i\mathcal{E}_i\,dt. \tag{4.18}$$

Moreover, by (4.8), (2.28) of Lemma 2.4 implies

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{J}_l((\phi,\eta)+\varepsilon(\psi,\zeta)) = \left\langle \epsilon^2 \begin{pmatrix} \gamma_{11}I_{\Omega_1}+\gamma_{12}I_{\Omega_2} \\ \gamma_{12}I_{\Omega_1}+\gamma_{22}I_{\Omega_2} \\ (\gamma_{11}-\gamma_{12})I_{\Omega_1}+(\gamma_{12}-\gamma_{22})I_{\Omega_2} \\ \mathcal{Q}(\phi,\eta) \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_0 \\ \zeta, \end{pmatrix} \right\rangle. \tag{4.19}$$

In (4.19) the functional $\mathcal{Q}$ is given by

$$\begin{aligned} \mathcal{Q}(\phi,\eta) &= \int_{-1}^{1} \big((\gamma_{11}I_{\Omega_1}+\gamma_{12}I_{\Omega_2})\mathcal{E}_1(\phi_1,\eta)+(\gamma_{12}I_{\Omega_1}+\gamma_{22}I_{\Omega_2})\mathcal{E}_2(\phi_2,\eta) \\ &\quad +((\gamma_{11}-\gamma_{12})I_{\Omega_1}+(\gamma_{12}-\gamma_{22})I_{\Omega_2})\mathcal{E}_0(\phi_0,\eta)\big)\,dt. \end{aligned} \tag{4.20}$$

Two more spaces are needed in this work:

$$\mathcal{X} = \{(\phi,\eta)\in\mathcal{Y} : \phi\in H^2((-1,1);\mathbb{R}^3)\}, \tag{4.21}$$

$$\mathcal{Z} = \{(\phi,\eta) : \phi\in L^2((-1,1);\mathbb{R}^3),\ \eta\in\mathbb{R},\ \int_{-1}^{1}(\phi_1+\phi_0)\,dt = \int_{-1}^{1}(\phi_2-\phi_0)\,dt = 0\}. \tag{4.22}$$

Clearly $\mathcal{X}\subset\mathcal{Y}\subset\mathcal{Z}\subset L^2((-1,1);\mathbb{R}^3)\times\mathbb{R}$. The norms of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are given by

$$\|(\phi,\eta)\|_{\mathcal{X}}^2 = \sum_{i=0}^{2}\|\phi_i\|_{H^2}^2+\eta^2,\quad \|(\phi,\eta)\|_{\mathcal{Y}}^2 = \sum_{i=0}^{2}\|\phi_i\|_{H^1}^2+\eta^2,\quad \|(\phi,\eta)\|_{\mathcal{Z}}^2 = \sum_{i=0}^{2}\|\phi_i\|_{L^2}^2+\eta^2, \tag{4.23}$$

where $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^2}$ are the usual H^1 and H^2 norms of Sobolev spaces, and $\|\cdot\|_{L^2}$ is the usual L^2 norm. Denote the orthogonal projection from $L^2((-1,1);\mathbb{R}^3)\times\mathbb{R}$ to $\mathcal{Z}$ by Π; namely,

$$\Pi(\psi,\zeta) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_0 \\ \zeta \end{pmatrix} - \Big[\int_{-1}^{1}\Big(\frac{\psi_1}{3}+\frac{\psi_2}{6}+\frac{\psi_0}{6}\Big)\,dt\Big]\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \Big[\int_{-1}^{1}\Big(\frac{\psi_1}{6}+\frac{\psi_2}{3}-\frac{\psi_0}{6}\Big)\,dt\Big]\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}. \tag{4.24}$$

The gradient of $\mathcal{J}_s$ is an operator $\mathcal{S}_s$ from a neighborhood of $(0,h)$ in $\mathcal{X}$ to $\mathcal{Z}$ such that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{J}_s((\phi,\eta)+\varepsilon(\psi,\zeta)) = \langle \mathcal{S}_s(\phi,\eta),(\psi,\zeta)\rangle \tag{4.25}$$

for all $(\psi,\zeta)\in\mathcal{X}$. From (4.14) one sees that

$$\mathcal{S}_s(\phi,\eta) = \Pi\epsilon \begin{pmatrix} \mathcal{K}_1(\phi_1,\eta) \\ \mathcal{K}_2(\phi_2,\eta) \\ \mathcal{K}_0(\phi_0,\eta) \\ \widetilde{\mathcal{K}}(\phi,\eta) \end{pmatrix}. \tag{4.26}$$

The gradient of $\mathcal{J}_l$ is

$$\mathcal{S}_l(\phi,\eta) = \Pi\epsilon^2 \begin{pmatrix} \gamma_{11}I_{\Omega_1}+\gamma_{12}I_{\Omega_2} \\ \gamma_{12}I_{\Omega_1}+\gamma_{22}I_{\Omega_2} \\ (\gamma_{11}-\gamma_{12})I_{\Omega_1}+(\gamma_{12}-\gamma_{22})I_{\Omega_2} \\ \mathcal{Q}(\phi,\eta) \end{pmatrix}. \tag{4.27}$$

A remark regarding the I_{Ω_i}'s in (4.27) is in order. Recall that each I_{Ω_i}, $i=1,2$, is a function on D given in (1.6), and the set Ω_i is determined by the internal variables ϕ_i, ϕ_0, and η for $i=1,2$. The I_{Ω_i}'s ($i=1,2$) in the first three components on the right side of (4.27) are now considered as outcomes of the operators

$$\mathcal{I}_{ij}:\ (\phi_i,\phi_0,\eta)\to I_{\Omega_i}(\mathbf{r}_j(t)),\quad i=1,2,\ j=1,2,0, \tag{4.28}$$

where $j = 1, 2, 0$ corresponds to the first, second, and third components in (4.27) respectively.

The gradient of $\mathcal{J}$ is

$$\mathcal{S} = \mathcal{S}_s + \mathcal{S}_l \tag{4.29}$$

Therefore

$$\mathcal{S}(\phi, \eta) = \Pi \begin{pmatrix} \epsilon\mathcal{K}_1(\phi_1, \eta) + \epsilon^2(\gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2}) \\ \epsilon\mathcal{K}_2(\phi_2, \eta) + \epsilon^2(\gamma_{12} I_{\Omega_1} + \gamma_{22} I_{\Omega_2}) \\ \epsilon\mathcal{K}_0(\phi_0, \eta) + \epsilon^2(\gamma_{11} - \gamma_{12}) I_{\Omega_1} + \epsilon^2(\gamma_{12} - \gamma_{22}) I_{\Omega_2} \\ \epsilon\widetilde{\mathcal{K}}(\phi, \eta) + \epsilon^2 \mathcal{Q}(\phi, \eta) \end{pmatrix}. \tag{4.30}$$

The domain of $\mathcal{S}$ is taken to be

$$\mathcal{D}(\mathcal{S}) = \{(\phi, \eta) \in \mathcal{X} : \ \|(\phi, \eta - h)\|_{\mathcal{X}} < \bar{c}\}, \tag{4.31}$$

where $\bar{c}$ in (4.31) is the same as the $\bar{c}$ in (3.20). Consequently, $\mathcal{D}(\mathcal{S}) \subset \mathcal{D}(\mathcal{J})$.

Lemma 4.3 *It holds uniformly with respect to t that*

$$\mathcal{S}(0, h) = \big(O(|\gamma|\epsilon^4), O(|\gamma|\epsilon^4), O(|\gamma|\epsilon^4), O(|\gamma|\epsilon^4)\big).$$

Consequently, there exists $\widetilde{C} > 0$ such that $\|\mathcal{S}(0, h)\|_{\mathcal{Z}} \leq \widetilde{C}|\gamma|\epsilon^4$.

Proof. Calculations from (4.15) and (4.16) show that

$$\mathcal{K}_i(0, h) = \frac{1}{r_i}, \quad i = 1, 2, 0, \quad \text{and } \widetilde{\mathcal{K}}(0, h) = 2\sum_{i=0}^{2} \frac{d(\alpha_i \rho_i)}{d\eta}\Big|_{\eta=h}.$$

By (B.25) in Appendix B, $2\sum_{i=0}^{2} \frac{d(\alpha_i \rho_i)}{d\eta}|_{\eta=h} = 0$. Hence

$$\widetilde{\mathcal{K}}(0, h) = 0.$$

Consequently, by the virtue of the projection operator Π and the fact that $\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{r_0}$,

$$\mathcal{S}_s(0, h) = \Pi\epsilon \begin{pmatrix} \mathcal{K}_1(0, h) \\ \mathcal{K}_2(0, h) \\ \mathcal{K}_0(0, h) \\ \widetilde{\mathcal{K}}(0, h) \end{pmatrix} = \Pi\epsilon \begin{pmatrix} 1/r_1 \\ 1/r_2 \\ 1/r_0 \\ 0 \end{pmatrix} = 0. \tag{4.32}$$

Regarding $\mathcal{S}_l(0, h)$, let $\hat{\mathrm{r}}_i$ be the boundaries of the exact double bubble E, i.e.,

$$\hat{\mathbf{r}}_i(t) = \begin{cases} r_1 e^{\mathrm{i}(\pi - a_1 t)} + b_1 & \text{if} \quad i = 1, \\ r_i e^{\mathrm{i}a_i t} + b_i & \text{if} \quad i = 2, 0, \end{cases}$$

and $\mathbf{r}_i$ be the boundary of $T_{\epsilon,\xi,\theta}(E_i)$, i.e.,

$$\mathbf{r}_i(t) = \epsilon e^{\mathrm{i}\theta} \hat{\mathbf{r}}_i(t) + \xi.$$

One then deduces

$$\begin{aligned} \mathcal{I}_{ij}(0, h) &= \int_{T(E_i)} G(\mathbf{r}_j(t), y)\, dy \\ &= \int_{T(E_i)} \frac{1}{2\pi} \log \frac{1}{|\mathbf{r}_j(t) - y|}\, dy + \int_{T(E_i)} R(\mathbf{r}_j(t), y)\, dy \\ &= \epsilon^2 \int_{E_i} \frac{1}{2\pi} \log \frac{1}{\epsilon|\hat{\mathbf{r}}_j(t) - \hat{y}|}\, d\hat{y} + O(\epsilon^2) \\ &= \frac{\epsilon^2}{2\pi}\Big(\log \frac{1}{\epsilon}\Big)|E_i| + \epsilon^2 \int_{E_i} \frac{1}{2\pi} \log \frac{1}{|\hat{\mathbf{r}}_j(t) - \hat{y}|}\, d\hat{y} + O(\epsilon^2) \\ &= \frac{\epsilon^2}{2\pi}\Big(\log \frac{1}{\epsilon}\Big)|E_i| + O(\epsilon^2). \end{aligned}$$

Consequently, with the help of (4.13) of Lemma 4.2,

$$\begin{aligned}
\mathcal{Q}(0,h) &= \int_{-1}^{1}\Big[\gamma_{11}\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_1| + \gamma_{12}\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_2|\Big]\mathcal{E}_1(0,h)\,dt \\
&\quad + \int_{-1}^{1}\Big[\gamma_{12}\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_1| + \gamma_{22}\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_2|\Big]\mathcal{E}_2(0,h)\,dt \\
&\quad + \int_{-1}^{1}\Big[(\gamma_{11}-\gamma_{12})\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_1| + (\gamma_{12}-\gamma_{22})\frac{\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_2|\Big]\mathcal{E}_0(0,h)\,dt + O(|\gamma|\epsilon^2) \\
&= \frac{\gamma_{11}\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_1|\int_{-1}^{1}(\mathcal{E}_1(0,h)+\mathcal{E}_0(0,h))\,dt \\
&\quad + \frac{\gamma_{12}\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_2|\int_{-1}^{1}(\mathcal{E}_1(0,h)+\mathcal{E}_0(0,h))\,dt \\
&\quad + \frac{\gamma_{12}\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_1|\int_{-1}^{1}(\mathcal{E}_2(0,h)-\mathcal{E}_0(0,h))\,dt \\
&\quad + \frac{\gamma_{22}\epsilon^2}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)|E_2|\int_{-1}^{1}(\mathcal{E}_2(0,h)-\mathcal{E}_0(0,h))\,dt + O(|\gamma|\epsilon^2) \\
&= O(|\gamma|\epsilon^2).
\end{aligned}$$

Therefore

$$\mathcal{S}_l(0,h) = \frac{\epsilon^4}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\Pi\begin{pmatrix}\gamma_{11}|E_1|+\gamma_{12}|E_2| \\ \gamma_{12}|E_1|+\gamma_{22}|E_2| \\ (\gamma_{11}-\gamma_{12})|E_1|+(\gamma_{12}-\gamma_{22})|E_2| \\ 0\end{pmatrix} + O(|\gamma|\epsilon^4)$$

$$= \frac{\epsilon^4}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\left[\gamma_{11}|E_1|\Pi\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \gamma_{12}|E_2|\Pi\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \gamma_{12}|E_1|\Pi\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} + \gamma_{22}|E_2|\Pi\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right] + O(|\gamma|\epsilon^4)$$

$$= \frac{\epsilon^4}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)(\vec{0}+\vec{0}+\vec{0}+\vec{0}) + O(|\gamma|\epsilon^4) = O(|\gamma|\epsilon^4). \tag{4.33}$$

The lemma follows from (4.32) and (4.33). □

5 Second variation

The Fréchet derivative of the operator $\mathcal{S}$ at $(\phi,\eta)\in\mathcal{D}(\mathcal{S})$ is denoted $\mathcal{S}'(\phi,\eta)$. It is a linear operator from $\mathcal{X}$ to $\mathcal{Z}$. For every $(\psi,\zeta)\in\mathcal{X}$, it yields the second variation of $\mathcal{J}$:

$$\frac{d^2\mathcal{J}((\phi,\eta)+\varepsilon(\psi,\zeta))}{d\varepsilon^2}\Big|_{\varepsilon=0} = \langle\mathcal{S}'(\phi,\eta)(\psi,\zeta),(\psi,\zeta)\rangle. \tag{5.1}$$

Similar formulas hold if $\mathcal{J}$ is replaced by $\mathcal{J}_s$ and $\mathcal{S}$ replaced by $\mathcal{S}_s$, or $\mathcal{J}$ by $\mathcal{J}_l$ and $\mathcal{S}$ by $\mathcal{S}_l$.

In this section we show that the operator $\mathcal{S}'(0,h)$, the Fréchet derivative of $\mathcal{S}$ at the exact double bubble is positive definite and derives an upper bound for the inverse operator $(\mathcal{S}'(0,h))^{-1}$.

Define the ϵ independent part of $\mathcal{J}_s$ by $\mathcal{P}$ so that $\mathcal{J}_s=\epsilon\mathcal{P}$:

$$\mathcal{P}(\phi,\eta) = \sum_{i=0}^{2}\int_{-1}^{1}L_i(\phi_i',\phi_i,\eta)\,dt, \quad (\phi,\eta)\in\mathcal{Y}. \tag{5.2}$$

Calculations show that

$$\frac{\partial^2 L_i(0,0,\eta)}{\partial(\phi_i')^2} = \frac{1}{(\alpha_i\rho_i)^3}, \quad \frac{\partial^2 L_i(0,0,\eta)}{\partial\phi_i^2} = -\frac{1}{\alpha_i\rho_i^3}, \quad \frac{\partial^2 L_i(0,0,\eta)}{\partial\eta^2} = \frac{d^2(\alpha_i\rho_i)}{d\eta^2}, \tag{5.3}$$

$$\frac{\partial^2 L_i(0,0,\eta)}{\partial\phi_i'\partial\phi_i} = 0, \quad \frac{\partial^2 L_i(0,0,\eta)}{\partial\phi_i'\partial\eta} = 0, \quad \frac{\partial^2 L_i(0,0,\eta)}{\partial\phi_i\partial\eta} = \frac{d}{d\eta}\Big(\frac{1}{\rho_i}\Big). \tag{5.4}$$

The second variation of $\mathcal{P}$ at $(\phi,\eta) = (0,h)$ is

$$\frac{d^2\mathcal{P}(0+\varepsilon\psi, h+\varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} = \sum_{i=0}^{2}\int_{-1}^{1}\Big[\frac{1}{(a_ir_i)^3}(\psi_i'(t))^2 - \frac{1}{a_ir_i^3}\psi_i^2(t) + \frac{d^2(\alpha_i\rho_i)}{d\eta^2}\Big|_{\eta=h}\zeta^2 + 2\frac{d}{d\eta}\Big(\frac{1}{\rho_i}\Big)\Big|_{\eta=h}\psi_i(t)\zeta\Big]\,dt.$$

However the constraints (3.14) that the ψ_i's satisfy and the condition (3.4) on the ρ_i's imply that the integral of the last term vanishes. Hence

$$\frac{d^2\mathcal{P}(0+\varepsilon\psi, h+\varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} = \sum_{i=0}^{2}\int_{-1}^{1}\Big[\frac{1}{(a_ir_i)^3}(\psi_i'(t))^2 - \frac{1}{a_ir_i^3}\psi_i^2(t) + \frac{d^2(\alpha_i\rho_i)}{d\eta^2}\Big|_{\eta=h}\zeta^2\Big]\,dt. \tag{5.5}$$

This is a quadratic form on $\mathcal{Y}$. A simple lemma is needed at this point.

Lemma 5.1 *Let $q \in (0,\pi)$ and $\nu \in \mathbb{R}$. The inequality*

$$\int_{-1}^{1}((y'(t))^2 - q^2y^2(t))\,dt \geq \frac{\nu^2q^3}{2(\tan q - q)}$$

holds for all $y \in H_0^1(-1,1)$ that satisfy the constraint $\int_{-1}^{1} y(t)\,dt = \nu$.

The proof of this lemma is given in Appendix A.

Lemma 5.2 *There exists $d > 0$ such that*

$$\frac{d^2\mathcal{P}(0+\varepsilon\psi, h+\varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} \geq 2d\|(\psi,\zeta)\|_{\mathcal{Y}}^2 \tag{5.6}$$

for all $(\psi,\zeta) \in \mathcal{X}$. In other words for $(\psi,\zeta) \in \mathcal{X}$,

$$\langle \mathcal{S}_s'(0,h)(\psi,\zeta),(\psi,\zeta)\rangle \geq 2d\epsilon\|(\psi,\zeta)\|_{\mathcal{Y}}^2. \tag{5.7}$$

Proof. Let us set

$$\int_{-1}^{1}\psi_0\,dt = \nu, \quad \int_{-1}^{1}\psi_1\,dt = -\nu, \quad \int_{-1}^{1}\psi_2\,dt = \nu \tag{5.8}$$

because of the constraints (3.14). By Lemma 5.1, one deduces

$$\begin{aligned}
&\frac{d^2\mathcal{P}(0+\varepsilon\psi, h+\varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} - 2d\sum_{i=0}^{2}\|\psi_i\|_{H^1}^2 \\
&= \sum_{i=0}^{2}\int_{-1}^{1}\Big[\Big(\frac{1}{(a_ir_i)^3} - 2d\Big)(\psi_i'(t))^2 - \Big(\frac{1}{a_ir_i^3} + 2d\Big)\psi_i^2(t)\Big]\,dt + 2\zeta^2\sum_{i=0}^{2}\frac{d^2(\alpha_i\rho_i)}{d\eta^2}\Big|_{\eta=h} \\
&\geq \sum_{i=0}^{2}\frac{\Big(\frac{1}{(a_ir_i)^3} - 2d\Big)\nu^2q_i^3}{2(\tan q_i - q_i)} + 2\zeta^2\sum_{i=0}^{2}\frac{d^2(\alpha_i\rho_i)}{d\eta^2}\Big|_{\eta=h},
\end{aligned} \tag{5.9}$$

where

$$q_i = \sqrt{\frac{\frac{1}{a_i r_i^3} + 2d}{\frac{1}{(a_i r_i)^3} - 2d}}. \tag{5.10}$$

If $d \to 0$, then

$$\sum_{i=0}^{2} \frac{\left(\frac{1}{(a_i r_i)^3} - 2d\right) q_i^3}{2(\tan q_i - q_i)} \to \frac{1}{2} \sum_{i=0}^{2} \frac{1}{r_i^3 (\tan a_i - a_i)} = \frac{1}{2h^3} \sum_{i=0}^{2} \frac{\sin^3 a_i}{\tan a_i - a_i}. \tag{5.11}$$

By Lemma B.1 in Appendix B, (5.11) is positive. Hence for $d > 0$ sufficiently small,

$$\sum_{i=0}^{2} \frac{\left(\frac{1}{(a_i r_i)^3} - 2d\right) \nu^2 q_i^3}{2(\tan q_i - q_i)} \geq 0. \tag{5.12}$$

By (B.26) in Appendix B,

$$\sum_{i=0}^{2} \frac{d^2(\alpha_i \rho_i)}{d\eta^2}\Big|_{\eta=h} > 0. \tag{5.13}$$

Hence

$$2\zeta^2 \sum_{i=0}^{2} \frac{d^2(\alpha_i \rho_i)}{d\eta^2}\Big|_{\eta=h} \geq 2d\zeta^2 \tag{5.14}$$

if d is sufficiently small. The lemma now follows from (5.9), (5.12), and (5.14). $\square$

From the quadratic form (5.5), one finds the explicit formula for $\mathcal{S}_s'(0,h)$:

$$\mathcal{S}_s'(0,h)(\psi,\zeta) = \Pi\epsilon \begin{pmatrix} -\frac{1}{(a_1 r_1)^3}\psi_1'' - \frac{1}{a_1 r_1^3}\psi_1 \\ -\frac{1}{(a_2 r_2)^3}\psi_2'' - \frac{1}{a_2 r_2^3}\psi_2 \\ -\frac{1}{(a_0 r_0)^3}\psi_0'' - \frac{1}{a_0 r_0^3}\psi_0 \\ 2(\sum_{i=0}^{2} \frac{d^2(\alpha_i\rho_i)}{d\eta^2}|_{\eta=h})\zeta \end{pmatrix}. \tag{5.15}$$

Next study

$$\mathcal{S}_l'(0,h)(\psi,\zeta) = \Pi\epsilon^2 \begin{pmatrix} \gamma_{11}\mathcal{I}_{11}'(0,h)(\psi_1,\psi_0,\zeta) + \gamma_{12}\mathcal{I}_{21}'(0,h)(\psi_2,\psi_0,\zeta) \\ \gamma_{12}\mathcal{I}_{12}'(0,h)(\psi_1,\psi_0,\zeta) + \gamma_{22}\mathcal{I}_{22}'(0,h)(\psi_2,\psi_0,\zeta) \\ (\gamma_{11}-\gamma_{12})\mathcal{I}_{10}'(0,h)(\psi_1,\psi_0,\zeta) + (\gamma_{12}-\gamma_{22})\mathcal{I}_{20}'(0,h)(\psi_2,\psi_0,\zeta) \\ \mathcal{Q}'(0,h)(\psi,\zeta) \end{pmatrix}. \tag{5.16}$$

Lemma 5.3 *There exists $\check{C} > 0$ depending on D and m only such that*

$$\|\mathcal{S}_l'(0,h)(\psi,\zeta)\|_{\mathcal{Z}} \leq \check{C}|\gamma|\epsilon^4 \|(\psi,\zeta)\|_{\mathcal{Z}}$$

for all $(\psi,\zeta) \in \mathcal{X}$.

Proof. Recall that $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_0$ parametrize the boundaries of the perturbed double bubble Ω as in (3.10), and $(\phi,\eta) \in \mathcal{X}$ is the internal variable of Ω. The terms I_{Ω_1} and I_{Ω_2} in the first, second, and third components of (4.27) are the outcomes of the operators $\mathcal{I}_{ij}$ given in (4.28).

To compute the Fréchet derivatives of $\mathcal{I}_{ij}$, deform (ϕ,η) to $(\phi,\eta)+\varepsilon(\psi,\zeta)$ and denote the corresponding deformations of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_0$ by $\mathbf{r}_1^\varepsilon$, $\mathbf{r}_2^\varepsilon$, and $\mathbf{r}_0^\varepsilon$, respectively. Then for $i = 1,2$ and $j = 1,2,0$,

$$\mathcal{I}_{ij}'(\phi_i,\phi_0,\eta):\ (\psi_i,\psi_0,\zeta) \to \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_i^\varepsilon} G(\mathbf{r}_j(t),y)\,dy + \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_i} G(\mathbf{r}_j^\varepsilon,y)\,dy. \tag{5.17}$$

Apply Lemma 2.3 to the first term on the left side of (5.17) with $\Omega = T_{\epsilon,\xi,\theta}(E)$ whose boundaries are parametrized by

$$\mathbf{r}_1(t) = T_{\epsilon,\xi,\theta}(r_1 e^{\mathrm{i}(\pi - a_1 t)} + b_1), \quad \mathbf{r}_2(t) = T_{\epsilon,\xi,\theta}(r_2 e^{\mathrm{i}a_2 t} + b_2), \quad \mathbf{r}_0(t) = T_{\epsilon,\xi,\theta}(r_0 e^{\mathrm{i}a_0 t} + b_0), \tag{5.18}$$

to obtain

$$\begin{aligned} &\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_i^\varepsilon} G(\mathbf{r}_j(t), y)\, dy \qquad (5.19)\\ &= \begin{cases} -\displaystyle\int_{T(\partial E_1 \setminus \partial E_2)} G(\mathbf{r}_j(t), \mathbf{r}_1(\tau))\mathbf{N}_1 \cdot \mathbf{X}_1\, ds(\tau) - \int_{T(\partial E_1 \cap \partial E_2)} G(\mathbf{r}_j(t), \mathbf{r}_0(\tau))\mathbf{N}_0 \cdot \mathbf{X}_0\, ds(\tau) & \text{if } i = 1,\\ -\displaystyle\int_{T(\partial E_2 \setminus \partial E_1)} G(\mathbf{r}_j(t), \mathbf{r}_2(\tau))\mathbf{N}_2 \cdot \mathbf{X}_2\, ds(\tau) + \int_{T(\partial E_1 \cap \partial E_2)} G(\mathbf{r}_j(t), \mathbf{r}_0(\tau))\mathbf{N}_0 \cdot \mathbf{X}_0\, ds(\tau) & \text{if } i = 2. \end{cases} \end{aligned}$$

With the help of (4.8), one finds that

$$\begin{aligned} &-\int_{T(\partial E_1 \setminus \partial E_2)} G(\mathbf{r}_j(t), \mathbf{r}_1(\tau))\mathbf{N}_1 \cdot \mathbf{X}_1\, ds(\tau)\\ &= \epsilon^2 \int_{-1}^{1} G(\mathbf{r}_j(t), \mathbf{r}_1(\tau))(\psi_1 + \mathcal{E}_1(0,h)\zeta)\, d\tau\\ &= \int_{-1}^{1} \frac{\epsilon^2}{2\pi}\Big(\log \frac{1}{|\mathbf{r}_j(t) - \mathbf{r}_1(\tau)|}\Big)(\psi_1 + \mathcal{E}_1(0,h)\zeta)\, d\tau + \epsilon^2 \int_{-1}^{1} R(\mathbf{r}_j(t), \mathbf{r}_1(\tau))(\psi_1 + \mathcal{E}_1(0,h)\zeta)\, d\tau\\ &= \frac{\epsilon^2}{2\pi}\Big(\log \frac{1}{\epsilon}\Big) \int_{-1}^{1} (\psi_1 + \mathcal{E}_1(0,h)\zeta)\, d\tau + \frac{\epsilon^2}{2\pi} \int_{-1}^{1} \Big(\log \frac{1}{|r_j e^{\mathrm{i}a_j t} + b_j - r_1 e^{\mathrm{i}(\pi - a_1 \tau)} - b_1|}\Big)(\psi_1 + \mathcal{E}_1(0,h)\zeta)\, d\tau\\ &\quad + \epsilon^2 \int_{-1}^{1} R(\mathbf{r}_j(t), \mathbf{r}_1(\tau))(\psi_1 + \mathcal{E}_1(0,h)\zeta)\, d\tau\\ &= \frac{\epsilon^2}{2\pi}\Big(\log \frac{1}{\epsilon}\Big) \int_{-1}^{1} (\psi_1 + \mathcal{E}_1(0,h)\zeta)\, d\tau + O(\epsilon^2)(\|\psi_1\|_{L^2} + |\zeta|). \end{aligned}$$

The above estimate holds uniformly with respect to t. Also the term $r_j e^{\mathrm{i}a_j t}$ above is valid if $j = 0, 2$; if $j = 1$, it should be replaced by $r_1 e^{\mathrm{i}(\pi - a_1 t)}$. Similar estimates hold for the other three terms in (5.19). By the constraints (3.14) on ψ_i and (4.13) of Lemma 4.2 one deduces that

$$\begin{aligned} &\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_i^\varepsilon} G(\mathbf{r}_j(t), y)\, dy\\ &= \begin{cases} \dfrac{\epsilon^2}{2\pi}\Big(\log \dfrac{1}{\epsilon}\Big) \displaystyle\int_{-1}^{1} (\psi_1 + \mathcal{E}_1(0,h)\zeta)\, d\tau + \dfrac{\epsilon^2}{2\pi}\Big(\log \dfrac{1}{\epsilon}\Big) \int_{-1}^{1} (\psi_0 + \mathcal{E}_0(0,h)\zeta)\, d\tau + O(\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}},\\ \dfrac{\epsilon^2}{2\pi}\Big(\log \dfrac{1}{\epsilon}\Big) \displaystyle\int_{-1}^{1} (\psi_2 + \mathcal{E}_2(0,h)\zeta)\, d\tau - \dfrac{\epsilon^2}{2\pi}\Big(\log \dfrac{1}{\epsilon}\Big) \int_{-1}^{1} (\psi_0 + \mathcal{E}_0(0,h)\zeta)\, d\tau + O(\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}} \end{cases}\\ &= O(\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}} \qquad (5.20) \end{aligned}$$

holds uniformly with respect to t.

The second part on the right side of (5.17), for $(\phi, \eta) = (0, h)$, is written as

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \int_{T(E_i)} G(\mathbf{r}_j^\varepsilon(t), y)\, dy = \int_{T(E_i)} \nabla G(\mathbf{r}_j(t), y) \cdot \mathbf{X}_j(t)\, dy, \tag{5.21}$$

where ∇G stands for the gradient of G with respect to its first argument. Clearly

$$\int_{T(E_i)} |\nabla G(\mathbf{r}_j(t), y)|\, dy = O(\epsilon) \tag{5.22}$$

holds uniformly with respect to t. Calculations from (4.3) and (4.5) show that

$$\mathbf{X}_j(t) = \begin{cases} \epsilon e^{\mathrm{i}\theta}\Big[\dfrac{1}{a_1 r_1}(\psi_1 + a_1 r_1 \rho_1' \zeta)e^{\mathrm{i}(\pi - a_1 t)} + r_1 \alpha_1' \zeta t e^{\mathrm{i}(\pi - a_1 t)}(-\mathrm{i}) + \beta_1' \zeta\Big] & \text{if} \quad j = 1, \\ \epsilon e^{\mathrm{i}\theta}\Big[\dfrac{1}{a_j r_j}(\psi_j + a_j r_j \rho_j' \zeta)e^{\mathrm{i}a_j t} + r_j \alpha_j' \zeta t e^{\mathrm{i}a_j t}\mathrm{i} + \beta_j' \zeta\Big] & \text{if} \quad j = 2, 0, \end{cases} \tag{5.23}$$

where ρ_j', α_j', and β_j' refer to the derivatives of ρ_j, α_j, and β_j with respect to η at η equal to h, respectively. Then (5.22) and (5.23) imply

$$\Big\| \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \int_{T(E_i)} G(\mathbf{r}_j^{\varepsilon}(t), y)\, dy \Big\|_{L^2} = O(\epsilon^2)(\|\psi_j\|_{L^2} + |\zeta|). \tag{5.24}$$

By (5.20) and (5.24) we find that

$$\|\mathcal{I}_{ij}'(0,h)(\psi_i, \psi_0, \zeta)\|_{L^2} = O(\epsilon^2)\|(\psi, \zeta)\|_{\mathcal{Z}}. \tag{5.25}$$

This allows us to handle the first three components of $\mathcal{S}_l'$ in (5.16).

Finally consider $\mathcal{Q}'$ in the last component of $\mathcal{S}_l'$. Note that

$$\begin{aligned} \mathcal{Q}'(0,h)(\psi,\zeta) \quad = \quad & \int_{-1}^{1} (\gamma_{11}\mathcal{I}_{11}'(0,h)(\psi_1,\psi_0,\zeta) + \gamma_{12}\mathcal{I}_{21}'(0,h)(\psi_2,\psi_0,\zeta))\mathcal{E}_1(0,h)\, dt \\ & + \int_{-1}^{1} (\gamma_{12}\mathcal{I}_{12}'(0,h)(\psi_1,\psi_0,\zeta) + \gamma_{22}\mathcal{I}_{22}'(0,h)(\psi_2,\psi_0,\zeta))\mathcal{E}_2(0,h)\, dt \\ & + \int_{-1}^{1} ((\gamma_{11} - \gamma_{12})\mathcal{I}_{10}'(0,h)(\psi_1,\psi_0,\zeta) + (\gamma_{12} - \gamma_{22})\mathcal{I}_{20}'(0,h)(\psi_2,\psi_0,\zeta))\mathcal{E}_0(0,h)\, dt \\ & + \int_{-1}^{1} (\gamma_{11}\mathcal{I}_{11}(0,h) + \gamma_{12}\mathcal{I}_{21}(0,h))\mathcal{E}_1'(0,h)(\psi_1,\zeta)\, dt \\ & + \int_{-1}^{1} (\gamma_{12}\mathcal{I}_{12}(0,h) + \gamma_{22}\mathcal{I}_{22}(0,h))\mathcal{E}_2'(0,h)(\psi_2,\zeta)\, dt \\ & + \int_{-1}^{1} ((\gamma_{11} - \gamma_{12})\mathcal{I}_{10}(0,h) + (\gamma_{12} - \gamma_{22})\mathcal{I}_{20}(0,h))\mathcal{E}_0'(0,h)(\psi_0,\zeta)\, dt. \end{aligned} \tag{5.26}$$

Denote the six terms on the right side of (5.26) by I, II, III, IV, V, and VI, respectively. Then the estimate (5.25) implies that

$$I,\ II,\ III = O(|\gamma|\epsilon^2)\, \|(\psi,\zeta)\|_{\mathcal{Z}}. \tag{5.27}$$

Regarding IV, V, and VI, note that

$$\begin{aligned} \mathcal{I}_{ij}(0,h) \quad &= \quad \int_{T(E_i)} G(\mathbf{r}_j(t), y)\, dy \\ &= \quad \int_{T(E_i)} \frac{1}{2\pi} \log \frac{1}{|\mathbf{r}_j(t) - y|}\, dy + \int_{T(E_i)} R(\mathbf{r}_j(t), y)\, dy \\ &= \quad \frac{|E_i|}{2\pi}\Big(\log \frac{1}{\epsilon}\Big)\epsilon^2 + \epsilon^2 A_{ij}(t), \end{aligned}$$

where

$$A_{ij}(t) = \begin{cases} \displaystyle\int_{E_i} \Big(\frac{1}{2\pi} \log \frac{1}{|r_1 e^{\mathrm{i}(\pi - a_1 t)} - \hat{y}|} + R(\mathbf{r}_1(t), T(\hat{y}))\Big)\, d\hat{y} & \text{if} \quad j = 1, \\ \displaystyle\int_{E_i} \Big(\frac{1}{2\pi} \log \frac{1}{|r_j e^{\mathrm{i}a_j t} - \hat{y}|} + R(\mathbf{r}_j(t), T(\hat{y}))\Big)\, d\hat{y} & \text{if} \quad j = 2, 0. \end{cases} \tag{5.28}$$

Then

$$\int_{-1}^{1} \mathcal{I}_{ij}(0,h)\mathcal{E}_j'(0,h)(\psi_j,\zeta)\,dt = \frac{|E_i|}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\epsilon^2\int_{-1}^{1}\mathcal{E}_j'(0,h)(\psi_j,\zeta)\,dt + \epsilon^2\int_{-1}^{1}A_{ij}(t)\mathcal{E}_j'(0,h)(\psi_j,\zeta)\,dt. \quad (5.29)$$

Calculations from (3.13) and (4.9) show that

$$\mathcal{E}_j'(0,h)(\psi_j,\zeta) = e_j'(t) \quad (5.30)$$

where $e_j'(t)$ stands for the derivative of $e_j(t)$ with respect to t and

$$e_j(t) = \begin{cases} \Big(-\dfrac{\alpha_1'(h)t}{a_1} - \dfrac{\beta_1'(h)}{a_1 r_1}\sin a_1 t\Big)\psi_1 + \Big(\dfrac{d}{d\eta}\Big|_{\eta=h}(\alpha_1\rho_1\rho_1' t - \rho_1\beta_1'\sin\alpha_1 t)\Big)\zeta & \text{if} \quad j=1 \\ \Big(-\dfrac{\alpha_j'(h)t}{a_j} + \dfrac{\beta_j'(h)}{a_j r_j}\sin a_j t\Big)\psi_j + \Big(\dfrac{d}{d\eta}\Big|_{\eta=h}(\alpha_j\rho_j\rho_j' t + \rho_j\beta_j'\sin\alpha_j t)\Big)\zeta & \text{if} \quad j=2,0 \end{cases}. \quad (5.31)$$

One then estimates the second term on the right side of (5.29) via integration by parts:

$$\epsilon^2\int_{-1}^{1}A_{ij}(t)\mathcal{E}_j'(0,h)(\psi_j,\zeta)\,dt = \epsilon^2 A_{ij}(t)e_j(t)\Big|_{-1}^{1} - \epsilon^2\int_{-1}^{1}A_{ij}'(t)e_j(t)\,dt. \quad (5.32)$$

Then

$$\epsilon^2 A_{ij}(t)e_j(t)\Big|_{-1}^{1} = \epsilon^2 A_{ij}(t)\Big[\frac{d}{d\eta}\Big|_{\eta=h}(\alpha_j\rho_j\rho_j' t + (-1)^j\rho_j\beta_j'\sin\alpha_j t)\Big]\zeta\Big|_{-1}^{1} = O(\epsilon^2)|\zeta|, \quad (5.33)$$

$$\Big|\epsilon^2\int_{-1}^{1}A_{ij}'(t)e_j(t)\,dt\Big| \le \epsilon^2\|A_{ij}'\|_{L^2}\|e_j\|_{L^2} = O(\epsilon^2)(\|\psi_j\|_{L^2} + |\zeta|) \quad (5.34)$$

since $A_{ij}'(t)$ is bounded with respect to t. By (5.29), (5.32), (5.33), and (5.34) one concludes that

$$\int_{-1}^{1}\mathcal{I}_{ij}(0,h)\mathcal{E}_j'(0,h)(\psi_j,\zeta)\,dt = \frac{|E_i|}{2\pi}\Big(\log\frac{1}{\epsilon}\Big)\epsilon^2\int_{-1}^{1}\mathcal{E}_j'(0,h)(\psi_j,\zeta)\,dt + O(\epsilon^2)(\|\psi_j\|_{L^2} + |\zeta|). \quad (5.35)$$

By (4.13) of Lemma 4.2,

$$\int_{-1}^{1}\mathcal{E}_1'(0,h)(\psi_1,\zeta)\,dt + \int_{-1}^{1}\mathcal{E}_0'(0,h)(\psi_0,\zeta)\,dt = \int_{-1}^{1}\mathcal{E}_2'(0,h)(\psi_2,\zeta)\,dt - \int_{-1}^{1}\mathcal{E}_0'(0,h)(\psi_0,\zeta)\,dt = 0. \quad (5.36)$$

Following (5.35) and (5.36) one arrives at

$$IV + V + VI = O(|\gamma|\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}}. \quad (5.37)$$

By (5.27) and (5.37), (5.26) becomes

$$\mathcal{Q}'(0,h)(\psi,\zeta) = O(|\gamma|\epsilon^2)\|(\psi,\zeta)\|_{\mathcal{Z}}. \quad (5.38)$$

By (5.25) and (5.38) we deduce that there exists $\check{C} > 0$ such that

$$\|\mathcal{S}_l'(0,h)(\psi,\zeta)\|_{\mathcal{Z}} \le \check{C}|\gamma|\epsilon^4\|(\psi,\zeta)\|_{\mathcal{Z}} \quad (5.39)$$

for all $(\psi,\zeta) \in \mathcal{X}$. □

Combining Lemmas 5.2 and 5.3 we obtain the following lemma.

Lemma 5.4 *There exist $d > 0$ and $\sigma > 0$ such that when $|\gamma|\epsilon^3 < \sigma$,*

$$\langle \mathcal{S}'(0,h)(\psi,\zeta),(\psi,\zeta)\rangle \geq d\epsilon\|(\psi,\zeta)\|_{\mathcal{Y}}^2$$

for all $(\psi,\zeta) \in \mathcal{X}$.

Proof. Let d be the positive number given in Lemma 5.2 and $\sigma = \frac{d}{\check{C}}$ where $\check{C}$ comes from Lemma 5.3. Then Lemma 5.3 shows that for $|\gamma|\epsilon^3 < \sigma$,

$$\|\mathcal{S}_l'(0,h)(\psi,\zeta)\|_{\mathcal{Z}} \leq \check{C}|\gamma|\epsilon^4\|(\psi,\zeta)\|_{\mathcal{Z}} \leq \check{C}\sigma\epsilon\|(\psi,\zeta)\|_{\mathcal{Z}} = d\epsilon\|(\psi,\zeta)\|_{\mathcal{Z}} \tag{5.40}$$

for all $(\psi,\zeta) \in \mathcal{X}$. By Lemma 5.2 and (5.40)

$$\begin{aligned}\langle \mathcal{S}'(0,h)(\psi,\zeta),(\psi,\zeta)\rangle &= \langle \mathcal{S}_s'(0,h)(\psi,\zeta),(\psi,\zeta)\rangle + \langle \mathcal{S}_l'(0,h)(\psi,\zeta),(\psi,\zeta)\rangle \\ &\geq 2d\epsilon\|(\psi,\zeta)\|_{\mathcal{Y}}^2 - d\epsilon\|(\psi,\zeta)\|_{\mathcal{Z}}^2 \;\geq\; d\epsilon\|(\psi,\zeta)\|_{\mathcal{Y}}^2\end{aligned}$$

for all $(\psi,\zeta) \in \mathcal{X}$. □

A consequence of the positivity of $\mathcal{S}'(0,h)$ is its invertibility.

Lemma 5.5 *Let σ be the number given in Lemma 5.4.*

1. *There exists $\tilde{d} > 0$ such that if $|\gamma|\epsilon^3 < \sigma$, $\|\mathcal{S}'(0,h)(\psi,\zeta)\|_{\mathcal{Z}} \geq \tilde{d}\epsilon\|(\psi,\zeta)\|_{\mathcal{X}}$ holds for all $(\psi,\zeta) \in \mathcal{X}$.*
2. *The linear map $\mathcal{S}'(0,h)$ is one to one and onto from $\mathcal{X}$ to $\mathcal{Z}$; moreover, $\|(\mathcal{S}'(0,h))^{-1}\| \leq \frac{1}{d\epsilon}$ where $\|(\mathcal{S}'(0,h))^{-1}\|$ is the operator norm of $(\mathcal{S}'(0,h))^{-1}$.*

Proof. By Lemma 5.4 it is easy to see that if $|\gamma|\epsilon^3 < \sigma$, then for all $(\psi,\zeta) \in \mathcal{X}$

$$\|(\psi,\zeta)\|_{\mathcal{Z}} \leq \frac{1}{d\epsilon}\|\mathcal{S}'(0,h)(\psi,\zeta)\|_{\mathcal{Z}}. \tag{5.41}$$

The first part of Lemma 5.5 asserts that the $\mathcal{Z}$-norm of (ψ,ζ) on the left side of (5.41) can be strengthened to the stronger $\mathcal{X}$-norm, if d is replaced by a possibly smaller $\tilde{d}$.

If part 1 is false, then there exist γ_n, ϵ_n, and $(\psi_n,\zeta_n) \in \mathcal{X}$ such that $|\gamma_n|\epsilon_n^3 < \sigma$, $\|(\psi_n,\zeta_n)\|_{\mathcal{X}} = 1$, and with $\epsilon = \epsilon_n$ and $\gamma = \gamma_n$ in $\mathcal{S}'$,

$$\|\epsilon_n^{-1}\mathcal{S}'(0,0)(\psi_n,\zeta_n)\|_{\mathcal{Z}} \to 0 \;\text{ as } n \to \infty. \tag{5.42}$$

By (5.41),

$$\|(\psi_n,\zeta_n)\|_{\mathcal{Z}} \to 0. \tag{5.43}$$

Moreover, due to the compactness of the embedding $H^2(-1,1) \to C^1[-1,1]$ and $\|(\psi_n,\zeta_n)\|_{\mathcal{X}} = 1$, $\|\psi_{n,i}\|_{C^1} \to 0$ and in particular

$$\psi_{n,i}'(\pm 1) \to 0,\; i = 1,2,0,\; \text{as } n \to \infty. \tag{5.44}$$

Since $\mathcal{S}'(0,h) = \mathcal{S}_s'(0,h) + \mathcal{S}_l'(0,h)$, and (5.40) and (5.43) imply that

$$\|\epsilon_n^{-1}\mathcal{S}_l'(0,h)(\psi_n,\zeta_n)\|_{\mathcal{Z}} \to 0, \tag{5.45}$$

one derives from (5.42) and (5.45) that

$$\|\epsilon_n^{-1}\mathcal{S}_s'(0,h)(\psi_n,\zeta_n)\|_{\mathcal{Z}} \to 0. \tag{5.46}$$

By (5.15) write

$$\epsilon_n^{-1}\mathcal{S}_s'(0,h)(\psi_n,\zeta_n) = \Pi\begin{pmatrix} -\frac{1}{(a_1r_1)^3}\psi_{n,1}'' \\ -\frac{1}{(a_2r_2)^3}\psi_{n,2}'' \\ -\frac{1}{(a_0r_0)^3}\psi_{n,0}'' \\ 0 \end{pmatrix} + \Pi\begin{pmatrix} -\frac{1}{a_1r_1^3}\psi_{n,1} \\ -\frac{1}{a_2r_2^3}\psi_{n,2} \\ -\frac{1}{a_0r_0^3}\psi_{n,0} \\ 2(\sum_{i=0}^2 \frac{d^2(\alpha_i\rho_i)}{d\eta^2}|_{\eta=h})\zeta_n \end{pmatrix}. \tag{5.47}$$

By (5.43) one finds that

$$\left\|\Pi\begin{pmatrix} -\frac{1}{a_1r_1^3}\psi_{n,1} \\ -\frac{1}{a_2r_2^3}\psi_{n,2} \\ -\frac{1}{a_0r_0^3}\psi_{n,2} \\ 2(\sum_{i=0}^2 \frac{d^2(\alpha_i\rho_i)}{d\eta^2}|_{\eta=h})\zeta_n \end{pmatrix}\right\|_{\mathcal{Z}} \to 0. \tag{5.48}$$

Then (5.46), (5.47), and (5.48) show that

$$\left\|\Pi\begin{pmatrix} -\frac{1}{(a_1r_1)^3}\psi_{n,1}'' \\ -\frac{1}{(a_2r_2)^3}\psi_{n,2}'' \\ -\frac{1}{(a_0r_0)^3}\psi_{n,0}'' \\ 0 \end{pmatrix}\right\|_{\mathcal{Z}} \to 0. \tag{5.49}$$

By the definition of Π, (4.24),

$$\Pi\begin{pmatrix} -\frac{1}{(a_1r_1)^3}\psi_{n,1}'' \\ -\frac{1}{(a_2r_2)^3}\psi_{n,2}'' \\ -\frac{1}{(a_0r_0)^3}\psi_{n,0}'' \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{(a_1r_1)^3}\psi_{n,1}'' \\ -\frac{1}{(a_2r_2)^3}\psi_{n,2}'' \\ -\frac{1}{(a_0r_0)^3}\psi_{n,0}'' \\ 0 \end{pmatrix} + \begin{pmatrix} \left(\frac{1}{3(a_1r_1)^3}\psi_{n,1}' + \frac{1}{6(a_2r_2)^3}\psi_{n,2}' + \frac{1}{6(a_0r_0)^3}\psi_{n,0}'\right)\Big|_{-1}^{1} \\ \left(\frac{1}{6(a_1r_1)^3}\psi_{n,1}' + \frac{1}{3(a_2r_2)^3}\psi_{n,2}' - \frac{1}{6(a_0r_0)^3}\psi_{n,0}'\right)\Big|_{-1}^{1} \\ \left(\frac{1}{6(a_1r_1)^3}\psi_{n,1}' - \frac{1}{6(a_2r_2)^3}\psi_{n,2}' + \frac{1}{3(a_0r_0)^3}\psi_{n,0}'\right)\Big|_{-1}^{1} \\ 0 \end{pmatrix}. \tag{5.50}$$

Moreover, (5.44) implies that

$$\begin{pmatrix} \left(\frac{1}{3(a_1r_1)^3}\psi_{n,1}' + \frac{1}{6(a_2r_2)^3}\psi_{n,2}' + \frac{1}{6(a_0r_0)^3}\psi_{n,0}'\right)\Big|_{-1}^{1} \\ \left(\frac{1}{6(a_1r_1)^3}\psi_{n,1}' + \frac{1}{3(a_2r_2)^3}\psi_{n,2}' - \frac{1}{6(a_0r_0)^3}\psi_{n,0}'\right)\Big|_{-1}^{1} \\ \left(\frac{1}{6(a_1r_1)^3}\psi_{n,1}' - \frac{1}{6(a_2r_2)^3}\psi_{n,2}' + \frac{1}{3(a_0r_0)^3}\psi_{n,0}'\right)\Big|_{-1}^{1} \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4. \tag{5.51}$$

Therefore, by (5.49), (5.50), and (5.51),

$$\|\psi_{n,i}''\|_{L^2} \to 0, \quad i = 1,2,0, \quad \text{as } n \to \infty. \tag{5.52}$$

From (5.43) and (5.52) we deduce that $\|(\psi_n,\zeta_n)\|_{\mathcal{X}} \to 0$, a contradiction to our assumption at the beginning that $\|(\psi_n,\zeta_n)\|_{\mathcal{X}} = 1$.

For part 2, it suffices to show that $\mathcal{S}'(0,h)$ is onto. First, note that by the standard theory of second order linear differential equations, $\mathcal{S}'(0,h)$ is an unbounded self-adjoint operator on $\mathcal{Z}$ with the domain $\mathcal{X} \subset \mathcal{Z}$. Second, if $(\tilde{\psi},\tilde{\zeta}) \in \mathcal{Z}$ is perpendicular to the range of $\mathcal{S}'(0,h)$, i.e. $\langle \mathcal{S}'(0,h)(\psi,\zeta),(\tilde{\psi},\tilde{\zeta})\rangle = 0$ for all $(\psi,\zeta) \in \mathcal{X}$, then the self-adjointness of $\mathcal{S}'(0,h)$ implies that $(\tilde{\psi},\tilde{\zeta}) \in \mathcal{X}$ and $\mathcal{S}'(0,h)(\tilde{\psi},\tilde{\zeta}) = 0$. By (5.41), $(\tilde{\psi},\tilde{\zeta})$ is zero. Hence, the range of $\mathcal{S}'(0,h)$ is dense in $\mathcal{Z}$. Finally (5.41) implies that the range of $\mathcal{S}'(0,h)$ is a closed subset of $\mathcal{Z}$. Therefore $\mathcal{S}'(0,h)$ is onto. $\square$

Finally in this section we state two properties regarding $\mathcal{S}''$, the second Fréchet derivative of $\mathcal{S}$ or the third variation of $\mathcal{J}$.

Lemma 5.6 *There exists $\widehat{C} > 0$ such that for all $(\phi, \eta) \in \mathcal{D}(\mathcal{S})$,*

$$\|\mathcal{S}''(\phi,\eta)((\tilde{\psi},\tilde{\zeta}),(\psi,\zeta))\|_{\mathcal{Z}} \leq \widehat{C}(\epsilon + |\gamma|\epsilon^4)\|(\tilde{\psi},\tilde{\zeta})\|_{\mathcal{X}}\|(\psi,\zeta)\|_{\mathcal{X}}$$

holds for all (ψ, ζ) and $(\tilde{\psi}, \tilde{\zeta}) \in \mathcal{X}$.

The proof, which is skipped, is straightforward estimation, similar to the proofs of [30, Lemma 3.2] and [29, Lemma 6.1].

Lemma 5.7 *There exists $\widehat{C} > 0$ such that for all $(\phi, \eta) \in \mathcal{D}(\mathcal{S})$,*

$$|\langle \mathcal{S}''(\phi,\eta)((\tilde{\psi},\tilde{\zeta}),(\psi,\zeta)),(\psi,\zeta)\rangle| \leq \widehat{C}(\epsilon + |\gamma|\epsilon^4)\|(\tilde{\psi},\tilde{\zeta})\|_{\mathcal{X}}\|(\psi,\zeta)\|_{\mathcal{Y}}^2$$

holds for (ψ, ζ) and $(\tilde{\psi}, \tilde{\zeta}) \in \mathcal{X}$

See [30, Lemma 4.1] or [29, Lemma 7.2] for the proofs of similar formulas.

6 Minimization in a restricted class

For each $(\xi, \theta) \in \overline{D_{\bar{\delta}}} \times \mathbb{S}^1$ that specifies the transformation $T_{\epsilon,\xi,\theta}$, we find a locally $\mathcal{J}$ minimizing perturbed double bubble in the restricted class. One starts by solving

$$\mathcal{S}(\phi, \eta) = 0. \tag{6.1}$$

Lemma 6.1 *There exists $\sigma > 0$ such that (6.1) admits a solution $(\phi^*, \eta^*) \in \mathcal{D}(\mathcal{S}) \subset \mathcal{X}$ satisfying $\|(\phi^*, \eta^* - h)\|_{\mathcal{X}} \leq \frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}$, provided $|\gamma|\epsilon^3 < \sigma$.*

Proof. For $(\phi, \eta) \in \mathcal{D}(\mathcal{S})$ write

$$\mathcal{S}(\phi,\eta) = \mathcal{S}(0,h) + \mathcal{S}'(0,h)(\phi,\eta-h) + \mathcal{R}(\phi,\eta), \tag{6.2}$$

where $\mathcal{R}(\phi, \eta)$ is a higher order term defined by (6.2). Define an operator $\mathcal{T}$ from $\mathcal{D}(\mathcal{S}) \subset \mathcal{X}$ into $\mathcal{X}$ by

$$\mathcal{T}(\phi,\eta) = (0,h) - (\mathcal{S}'(0,h))^{-1}(\mathcal{S}(0,h) + \mathcal{R}(\phi,\eta)), \tag{6.3}$$

and rewrite the equation $\mathcal{S}(\phi, \eta) = 0$ as a fixed point problem $\mathcal{T}(\phi, \eta) = (\phi, \eta)$.

Let $c \in (0, \bar{c})$, where $\bar{c}$ is given in (4.31), and define a closed ball

$$\mathcal{W} = \{(\phi,\eta) \in \mathcal{X} : \ \|(\phi,\eta-h)\|_{\mathcal{X}} \leq c\} \subset \mathcal{D}(\mathcal{S}).$$

For $(\phi, \eta) \in \mathcal{W}$,

$$\|\mathcal{R}(\phi,\eta)\|_{\mathcal{Z}} \leq \frac{1}{2}\sup_{\tau\in(0,1)}\|\mathcal{S}''((1-\tau)(0,h)+\tau(\phi,\eta))((\phi,\eta-h),(\phi,\eta-h))\|_{\mathcal{Z}} \leq \frac{\widehat{C}(\epsilon+|\gamma|\epsilon^4)}{2}\|(\phi,\eta-h)\|_{\mathcal{X}}^2 \tag{6.4}$$

by Lemma 5.6. Then by Lemmas 4.3 and 5.5,

$$\begin{aligned}
\|\mathcal{T}(\phi,\eta) - (0,h)\|_{\mathcal{X}} &\leq \|(\mathcal{S}'(0,h))^{-1}\|(\|\mathcal{S}(0,h)\|_{\mathcal{Z}} + \|\mathcal{R}(\phi,\eta)\|_{\mathcal{Z}}) \\
&\leq \frac{1}{\epsilon\tilde{d}}\Big(\widetilde{C}|\gamma|\epsilon^4 + \frac{\widehat{C}(\epsilon+|\gamma|\epsilon^4)}{2}c^2\Big) \\
&\leq \frac{\widetilde{C}\sigma}{\tilde{d}} + \frac{\widehat{C}+\widehat{C}\sigma}{2\tilde{d}}c^2.
\end{aligned} \tag{6.5}$$

Let $(\tilde{\phi},\tilde{\eta})\in\mathcal{W}$. Consider

$$\begin{aligned}
\|\mathcal{T}(\phi,\eta)-\mathcal{T}(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}} &\le \|(\mathcal{S}'(0,h))^{-1}\|\ \|\mathcal{R}(\phi,\eta)-\mathcal{R}(\tilde{\phi},\tilde{\eta})\|_{\mathcal{Z}}\\
&\le \frac{1}{\epsilon\tilde{d}}\|\mathcal{S}(\phi,\eta)-\mathcal{S}(\tilde{\phi},\tilde{\eta})-\mathcal{S}'(0,h)((\phi,\eta)-(\tilde{\phi},\tilde{\eta}))\|_{\mathcal{Z}}\\
&\le \frac{1}{\epsilon\tilde{d}}\|\mathcal{S}(\phi,\eta)-\mathcal{S}(\tilde{\phi},\tilde{\eta})-\mathcal{S}'(\tilde{\phi},\tilde{\eta})((\phi,\eta)-(\tilde{\phi},\tilde{\eta}))\|_{\mathcal{Z}}\\
&\quad+\frac{1}{\epsilon\tilde{d}}\|(\mathcal{S}'(\tilde{\phi},\tilde{\eta})-\mathcal{S}'(0,h))((\phi,\eta)-(\tilde{\phi},\tilde{\eta}))\|_{\mathcal{Z}}\\
&\le \frac{1}{2\epsilon\tilde{d}}\sup_{\tau\in(0,1)}\|\mathcal{S}''((1-\tau)(\tilde{\phi},\tilde{\eta})+\tau(\phi,\eta))\|\ \|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}}^2\\
&\quad+\frac{1}{\epsilon\tilde{d}}\sup_{\tau\in(0,1)}\|\mathcal{S}''((1-\tau)(0,h)+\tau(\tilde{\phi},\tilde{\eta}))\|\ \|(\tilde{\phi},\tilde{\eta}-h)\|_{\mathcal{X}}\|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}}\\
&\le \frac{\widehat{C}(\epsilon+|\gamma|\epsilon^4)}{\epsilon\tilde{d}}\Big(c+c\Big)\|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}}\\
&\le \frac{2\widehat{C}(1+\sigma)c}{\tilde{d}}\|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}}. \qquad (6.6)
\end{aligned}$$

Take

$$c=\min\Big\{\frac{\tilde{d}}{6\widehat{C}},\ \frac{\bar{c}}{2}\Big\}. \qquad (6.7)$$

Let σ be small enough so that Lemma 5.5 holds, and moreover

$$\sigma\le\min\Big\{1,\ \frac{\tilde{d}c}{2\widetilde{C}}\Big\}. \qquad (6.8)$$

It follows from (6.5) and (6.6) that

$$\|\mathcal{T}(\phi,\eta)-(0,h)\|_{\mathcal{X}}\le c\ \text{ and }\ \|\mathcal{T}(\phi,\eta)-\mathcal{T}(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}}\le\frac{2}{3}\|(\phi,\eta)-(\tilde{\phi},\tilde{\eta})\|_{\mathcal{X}} \qquad (6.9)$$

for all $(\phi,\eta),\ (\tilde{\phi},\tilde{\eta})\in\mathcal{W}$. The contraction mapping principle says that $\mathcal{T}$ has a fixed point in $\mathcal{W}$. This fixed point is denoted by (ϕ^*,η^*). It solves (6.1).

To prove the estimate of (ϕ^*,η^*), revisit the equation $(\phi,\eta)=\mathcal{T}(\phi,\eta)$, satisfied by (ϕ^*,η^*), and derive from (6.3) and (6.4) that

$$\begin{aligned}
\|(\phi^*,\eta^*-h)\|_{\mathcal{X}} &\le \|(\mathcal{S}'(0,h))^{-1}\|(\|\mathcal{S}(0,h)\|_{\mathcal{Z}}+\|\mathcal{R}(\phi^*,\eta^*)\|_{\mathcal{Z}})\\
&\le \frac{1}{\epsilon\tilde{d}}\Big(\widetilde{C}|\gamma|\epsilon^4+\frac{\widehat{C}(\epsilon+|\gamma|\epsilon^4)}{2}\|(\phi^*,\eta^*-h)\|_{\mathcal{X}}^2\Big).
\end{aligned}$$

Rewrite the above as

$$\Big(1-\frac{\widehat{C}(1+|\gamma|\epsilon^3)}{2\tilde{d}}\|(\phi^*,\eta^*-h)\|_{\mathcal{X}}\Big)\|(\phi^*,\eta^*-h)\|_{\mathcal{X}}\le\frac{\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}. \qquad (6.10)$$

In (6.10) estimate

$$\frac{\widehat{C}(1+|\gamma|\epsilon^3)}{2\tilde{d}}\|(\phi^*,\eta^*-h)\|_{\mathcal{X}}\le\frac{\widehat{C}(1+|\gamma|\epsilon^3)}{2\tilde{d}}c\le\frac{\widehat{C}c(1+\sigma)}{2\tilde{d}}\le\frac{1}{6} \qquad (6.11)$$

by (6.7) and (6.8). The estimate of (ϕ^*,η^*) follows from (6.10). □

The first part of the next lemma shows that the perturbed double bubble (ϕ^*,η^*) is locally energy minimizing, hence stable, within the restricted class of perturbed double bubbles. The second part gives a measurement on the non-degeneracy of (ϕ^*,η^*) within the restricted class.

Lemma 6.2 *1. There exist $\hat{d} > 0$ and $\sigma > 0$ such that if $|\gamma|\epsilon^3 < \sigma$, then the solution (ϕ^*, η^*) found in Lemma 6.1 satisfies $\langle \mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta), (\psi, \zeta)\rangle \geq \hat{d}\epsilon\|(\psi, \zeta)\|^2_{\mathcal{Y}}$ for all $(\psi, \zeta) \in \mathcal{X}$.*

2. There exist $\check{d} > 0$ and $\sigma > 0$ such that if $|\gamma|\epsilon^3 < \sigma$, the solution (ϕ^*, η^*) satisfies $\|\mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta)\|_{\mathcal{Z}} \geq \check{d}\epsilon\|(\psi, \zeta)\|_{\mathcal{X}}$ for all $(\psi, \zeta) \in \mathcal{X}$.

Proof. There exists $\tilde{\tau} \in (0, 1)$ such that

$$\langle \mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta), (\psi, \zeta)\rangle = \langle \mathcal{S}'(0, h)(\psi, \zeta), (\psi, \zeta)\rangle + \langle \mathcal{S}''((1 - \tilde{\tau})(0, h) + \tilde{\tau}(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\psi, \zeta)), (\psi, \zeta)\rangle.$$

By Lemma 5.7,

$$|\langle \mathcal{S}''((1 - \tilde{\tau})(0, h) + \tilde{\tau}(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\psi, \zeta)), (\psi, \zeta)\rangle| \leq \widehat{C}(\epsilon + |\gamma|\epsilon^4)\|(\phi^*, \eta^* - h)\|_{\mathcal{X}}\|(\psi, \zeta)\|^2_{\mathcal{Y}}. \quad (6.12)$$

Consequently by Lemmas 5.4 and 6.1

$$\begin{aligned}
\langle \mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta), (\psi, \zeta)\rangle &\geq d\epsilon\|(\psi, \zeta)\|^2_{\mathcal{Y}} - \widehat{C}(\epsilon + |\gamma|\epsilon^4)\frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}\|(\psi, \zeta)\|^2_{\mathcal{Y}} \\
&\geq \epsilon\Big(d - \frac{2\widehat{C}\widetilde{C}(\sigma + \sigma^2)}{\tilde{d}}\Big)\|(\psi, \zeta)\|^2_{\mathcal{Y}} \;\geq\; \frac{d\epsilon}{2}\|(\psi, \zeta)\|^2_{\mathcal{Y}}
\end{aligned}$$

if σ is sufficiently small. The first part follows if $\hat{d} = \frac{d}{2}$.

By Lemmas 5.5, 5.6, and 6.1,

$$\begin{aligned}
\|\mathcal{S}'(\phi^*, \eta^*)(\psi, \zeta)\|_{\mathcal{Z}} &\geq \|\mathcal{S}'(0, h)(\psi, \zeta)\|_{\mathcal{Z}} - \sup_{\tau \in (0,1)} \|\mathcal{S}''((1 - \tau)(0, h) + \tau(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\psi, \zeta))\|_{\mathcal{Z}} \\
&\geq \tilde{d}\epsilon\|(\psi, \zeta)\|_{\mathcal{X}} - \widehat{C}(\epsilon + |\gamma|\epsilon^4)\|(\phi^*, \eta^* - h)\|_{\mathcal{X}}\|(\psi, \zeta)\|_{\mathcal{X}} \\
&\geq \Big(\tilde{d}\epsilon - \widehat{C}(\epsilon + |\gamma|\epsilon^4)\frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}\Big)\|(\psi, \zeta)\|_{\mathcal{X}} \\
&\geq \epsilon\Big(\tilde{d} - \frac{2\widehat{C}\widetilde{C}(\sigma + \sigma^2)}{\tilde{d}}\Big)\|(\psi, \zeta)\|_{\mathcal{X}} \;\geq\; \frac{\tilde{d}\epsilon}{2}\|(\psi, \zeta)\|_{\mathcal{X}}
\end{aligned}$$

if σ is sufficiently small. Part 2 follows if $\check{d} = \frac{\tilde{d}}{2}$. □

One interprets the equation $\mathcal{S}(\phi^*, \eta^*) = 0$ and proves the following.

Lemma 6.3 *The perturbed double bubble described by (ϕ^*, η^*) satisfies (1.2)-(1.4). Moreover at the triple points,*

$$\sum_{i=0}^{2} \mathbf{T}_i \cdot \mathbf{X}^S\Big|_{-1}^{1} = 0, \quad (6.13)$$

where the $\mathbf{T}_i$'s are unit tangent vectors of the boundaries and $\mathbf{X}^S$ is given in Lemma 4.1.

Proof. By the virtue of the projection operator Π, the first three components of $\mathcal{S}$ in (4.30) imply that there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\begin{aligned}
\epsilon\mathcal{K}_1(\phi_1^*, \eta^*) + \epsilon^2(\gamma_{11}I_{\Omega_1^*} + \gamma_{12}I_{\Omega_2^*}) &= \lambda_1, \\
\epsilon\mathcal{K}_2(\phi_2^*, \eta^*) + \epsilon^2(\gamma_{12}I_{\Omega_1^*} + \gamma_{22}I_{\Omega_2^*}) &= \lambda_2, \\
\epsilon\mathcal{K}_0(\phi_0^*, \eta^*) + \epsilon^2(\gamma_{11} - \gamma_{12})I_{\Omega_1^*} + \epsilon^2(\gamma_{12} - \gamma_{22})I_{\Omega_2^*} &= \lambda_1 - \lambda_2.
\end{aligned} \quad (6.14)$$

Here $\Omega^* = (\Omega_1^*, \Omega_2^*)$ is the perturbed double bubble represented by (ϕ^*, η^*). Hence Ω^* satisfies the first three equations (1.2)-(1.4) for stationary points of $\mathcal{J}$. The constants λ_1 and λ_2 here are equal to λ_1 and λ_2 in (1.2)-(1.4) multiplied by ϵ^2, respectively.

From the fourth component of $\mathcal{S}$ in (4.30) one sees that

$$\epsilon\widetilde{\mathcal{K}}(\phi^*,\eta^*)+\epsilon^2\mathcal{Q}(\phi^*,\eta^*)=0.$$

By the expression of $\widetilde{\mathcal{K}}$ in (4.18) and the definition (4.20) of $\mathcal{Q}$, the last equation asserts

$$\sum_{i=0}^{2}\mathbf{T}_i\cdot\mathbf{X}^S\Big|_{-1}^{1}+\int_{-1}^{1}\big(\epsilon\mathcal{K}_1(\phi_1^*,\eta^*)+\epsilon^2(\gamma_{11}I_{\Omega_1^*}+\gamma_{12}I_{\Omega_2^*})\big)\mathcal{E}_1(\phi_1^*,\eta^*)\,dt$$

$$+\int_{-1}^{1}\big(\epsilon\mathcal{K}_2(\phi_2^*,\eta^*)+\epsilon^2(\gamma_{12}I_{\Omega_2^*}+\gamma_{22}I_{\Omega_2^*})\big)\mathcal{E}_2(\phi_2^*,\eta^*)\,dt$$

$$+\int_{-1}^{1}\big(\epsilon\mathcal{K}_0(\phi_0^*,\eta^*)+\epsilon^2(\gamma_{11}-\gamma_{12})I_{\Omega_1^*}+\epsilon^2(\gamma_{12}-\gamma_{22})I_{\Omega_2^*})\big)\mathcal{E}_0(\phi_0^*,\eta^*)\,dt=0.$$

By (6.14) the last equation is simplified to

$$\sum_{i=0}^{2}\mathbf{T}_i\cdot\mathbf{X}^S\Big|_{-1}^{1}+\int_{-1}^{1}\lambda_1\mathcal{E}_1(\phi_1^*,\eta^*)\,dt+\int_{-1}^{1}\lambda_2\mathcal{E}_2(\phi_2^*,\eta^*)\,dt+\int_{-1}^{1}(\lambda_1-\lambda_2)\mathcal{E}_0(\phi_0^*,\eta^*)\,dt=0.$$

Formula (4.13) of Lemma 4.2 further simplifies the above to (6.13) completing the proof. □

Equation (6.13) does not imply the fourth equation (1.5) for stationary points of $\mathcal{J}$. For most values of $\xi\in D_{\overline{\delta}}$ and $\theta\in\mathbb{S}^1$ that define the transformation $T_{\epsilon,\xi,\theta}$ in our setting, the perturbed double bubble given by (ϕ^*,η^*) does not satisfy (1.5), so it is not a stationary point of $\mathcal{J}$. In the next section we will find suitable ξ and θ in the transformation $T_{\epsilon,\xi,\theta}$. They will yield two more equations which together with (6.13) will imply (1.5).

7 Minimization beyond restricted classes

In the last section we found a particular perturbed double bubble (ϕ^*,η^*) in each restricted lass. Since a restricted class is specified by $(\xi,\theta)\in\overline{D_{\overline{\delta}}}\times\mathbb{S}^1$, the perturbed double bubble (ϕ^*,η^*) depends on (ξ,θ). In this section we emphasize this dependence and often denote the quadruple (ϕ^*,η^*) by $(\phi^*(\cdot,\xi,\theta),\eta^*(\xi,\theta))$.

We shall minimize $\mathcal{J}(\phi^*(\cdot,\xi,\theta),\eta^*(\xi,\theta))$ with respect to $(\xi,\theta)\in\overline{D_{\overline{\delta}}}\times\mathbb{S}^1$ to obtain a minimum (ξ^*,θ^*). With the particular ξ^* and θ^*, the corresponding perturbed double bubble $(\phi^*(\cdot,\xi^*,\theta^*),\eta^*(\xi^*,\theta^*))$ will yield the final solution.

The first lemma gives an estimate on the difference between the energy of $(\phi^*(\cdot,\xi,\theta),\eta^*(\xi,\theta))$ and the energy of the exact double bubble $T_{\epsilon,\xi,\theta}(E)$. Note that in the restricted class specified by (ξ,θ), the exact double bubble $T_{\epsilon,\xi,\theta}(E)$ is represented by the quadruple $(0,h)$.

Lemma 7.1 *If σ is small, then*

$$|\mathcal{J}(\phi^*(\cdot,\xi,\theta),\eta^*(\xi,\theta))-\mathcal{J}(T_{\epsilon,\xi,\theta}(E))|\le|\gamma|\epsilon^4\Big(\frac{\widetilde{C}^2}{\tilde{d}}|\gamma|\epsilon^3+\frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\epsilon^3)^2+\frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\epsilon^3)^3\Big)$$

holds uniformly for all $(\xi,\theta)\in\overline{D_{\overline{\delta}}}\times\mathbb{S}^1$.

Proof. Expanding $\mathcal{J}(\phi^*,\eta^*)$ yields

$$\begin{aligned}\mathcal{J}(\phi^*,\eta^*) &= \mathcal{J}(0,h)+\langle\mathcal{S}(0,h),(\phi^*,\eta^*-h)\rangle+\frac{1}{2}\langle\mathcal{S}'(0,h)(\phi^*,\eta^*-h),(\phi^*,\eta^*-h)\rangle\\ &\quad+\frac{1}{6}\langle\mathcal{S}''((1-\tilde{\tau})(0,h)+\tilde{\tau}(\phi^*,\eta^*))((\phi^*,\eta^*-h),(\phi^*,\eta^*-h)),(\phi^*,\eta^*-h)\rangle \qquad (7.1)\end{aligned}$$

for some $\tilde{\tau} \in (0,1)$. Also expanding $\mathcal{S}(\phi^*, \eta^*)$ gives

$$\begin{aligned} & \|\mathcal{S}(\phi^*, \eta^*) - \mathcal{S}(0,h) - \mathcal{S}'(0,h)(\phi^*, \eta^* - h)\|_{\mathcal{Z}} \\ \leq & \sup_{\tau \in (0,1)} \frac{1}{2} \|\mathcal{S}''((1-\tau)(0,h) + \tau(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\phi^*, \eta^* - h))\|_{\mathcal{Z}}. \end{aligned} \tag{7.2}$$

Since $\mathcal{S}(\phi^*, \eta^*) = 0$, (7.2) shows that

$$\|\mathcal{S}(0,h) + \mathcal{S}'(0,h)(\phi^*, \eta^* - h)\|_{\mathcal{Z}} \leq \sup_{\tau \in (0,1)} \frac{1}{2} \|\mathcal{S}''((1-\tau)(0,h) + \tau(\phi^*, \eta^* - h))((\phi^*, \eta^*), (\phi^*, \eta^* - h))\|_{\mathcal{Z}},$$

which implies that

$$\begin{aligned} & |\langle \mathcal{S}(0,h), (\phi^*, \eta^* - h)\rangle + \langle \mathcal{S}'(0,h)(\phi^*, \eta^* - h), (\phi^*, \eta^* - h)\rangle| \\ \leq & \Big(\frac{1}{2} \sup_{\tau \in (0,1)} \|\mathcal{S}''((1-\tau)(0,h) + \tau(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\phi^*, \eta^* - h))\|_{\mathcal{Z}}\Big) \|(\phi^*, \eta^* - h)\|_{\mathcal{X}}. \end{aligned} \tag{7.3}$$

By (7.3), (7.1) yields that

$$\begin{aligned} & \Big|\mathcal{J}(\phi^*, \eta^*) - \mathcal{J}(0,h) - \frac{1}{2}\langle \mathcal{S}(0,h), (\phi^*, \eta^* - h)\rangle\Big| \\ \leq & \Big(\frac{5}{12} \sup_{\tau \in (0,1)} \|\mathcal{S}''((1-\tau)(0,h) + \tau(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\phi^*, \eta^* - h))\|_{\mathcal{Z}}\Big) \|(\phi^*, \eta^* - h)\|_{\mathcal{X}}. \end{aligned} \tag{7.4}$$

Lemmas 4.3, 5.6, and 6.1 show that

$$\begin{aligned} & |\mathcal{J}(\phi^*, \eta^*) - \mathcal{J}(0,h)| \\ \leq & \frac{1}{2}|\langle \mathcal{S}(0,h), (\phi^*, \eta^* - h)\rangle| + \\ & \Big(\frac{5}{12} \sup_{\tau \in (0,1)} \|\mathcal{S}''((1-\tau)(0,h) + \tau(\phi^*, \eta^*))((\phi^*, \eta^* - h), (\phi^*, \eta^* - h))\|_{\mathcal{Z}}\Big) \|(\phi^*, \eta^* - h)\|_{\mathcal{X}} \\ \leq & \frac{1}{2}(\widetilde{C}|\gamma|\epsilon^4)\frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}} + \frac{5}{12}\widehat{C}(\epsilon + |\gamma|\epsilon^4)\Big(\frac{2\widetilde{C}|\gamma|\epsilon^3}{\tilde{d}}\Big)^3 \\ = & |\gamma|\epsilon^4\Big(\frac{\widetilde{C}^2}{\tilde{d}}|\gamma|\epsilon^3 + \frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\epsilon^3)^2 + \frac{10\widehat{C}\widetilde{C}^3}{3\tilde{d}^3}(|\gamma|\epsilon^3)^3\Big) \end{aligned} \tag{7.5}$$

which proves the lemma. □

Now let ξ vary in $\overline{D_{\bar{\delta}}}$, θ vary in $\mathbb{S}^1$, and set

$$J(\xi, \theta) = \mathcal{J}(\phi^*(\cdot, \xi, \theta), \eta^*(\xi, \theta)) \quad \text{and} \quad \bar{J}(\xi, \theta) = \mathcal{J}(T_{\epsilon,\xi,\theta}(E)). \tag{7.6}$$

Both J and $\bar{J}$ are treated as functions of $(\xi, \theta) \in \overline{D_{\bar{\delta}}} \times \mathbb{S}^1$.

Lemma 7.2 *When δ and σ are sufficiently small, the function J defined on $\overline{D_{\bar{\delta}}} \times \mathbb{S}^1$ attains a minimum in $D_{\bar{\delta}} \times \mathbb{S}^1$, the interior of $\overline{D_{\bar{\delta}}} \times \mathbb{S}^1$. Every minimum of J on $\overline{D_{\bar{\delta}}} \times \mathbb{S}^1$ must be in $D_{\bar{\delta}} \times \mathbb{S}^1$.*

Proof. Let $(\xi, \theta) \in \partial D_{\bar{\delta}} \times \mathbb{S}^1$ and $(\tilde{\xi}, \tilde{\theta}) \in D_{\bar{\delta}} \times \mathbb{S}^1$, with $\tilde{\xi}$ being a minimum of $R(z,z)$ in D, i.e. $R(\tilde{\xi}, \tilde{\xi}) = \min_{z \in D} R(z,z)$. Here $\tilde{\theta}$ may be arbitrary. Recall that by (2.13) every minimum of $R(z,z)$ in D must be in $D_{\bar{\delta}}$. By Lemma 7.1,

$$J(\xi, \theta) - J(\tilde{\xi}, \tilde{\theta}) \geq \bar{J}(\xi, \theta) - \bar{J}(\tilde{\xi}, \tilde{\theta}) - 2|\gamma|\epsilon^4\Big(\frac{\widetilde{C}^2\sigma}{\tilde{d}} + \frac{10\widehat{C}\widetilde{C}^3\sigma^2}{3\tilde{d}^3} + \frac{10\widehat{C}\widetilde{C}^3\sigma^3}{3\tilde{d}^3}\Big). \tag{7.7}$$

Lemma 2.1 shows that

$$\begin{aligned}
&\bar{J}(\xi,\theta)-\bar{J}(\tilde{\xi},\tilde{\theta})\\
&\quad\geq\ \frac{(\sum_{i,j=1}^{2}\gamma_{ij}m_im_j)\epsilon^4}{2}(R(\xi,\xi)-R(\tilde{\xi},\tilde{\xi}))-4\Big(\sum_{i,j=1}^{2}\gamma_{ij}m_im_j\Big)\epsilon^5\max\{r_1,r_2\}\max_{x,y\in\overline{D_{\overline{\delta}}}}|\nabla R(x,y)|. \quad (7.8)
\end{aligned}$$

The condition (1.8) implies that

$$\sum_{i,j=1}^{2}\gamma_{ij}m_im_j\geq\frac{c|\gamma|\,|m|^2}{4},\quad |m|^2=m_1^2+m_2^2=m^2+(1-m)^2, \quad (7.9)$$

holds for some c that depends on ι only. Then (7.7), (7.8), and (7.9) show that

$$\begin{aligned}
&J(\xi,\theta)-J(\tilde{\xi},\tilde{\theta})\\
&\quad\geq\ \frac{c|\gamma|\epsilon^4|m|^2}{8}(R(\xi,\xi)-R(\tilde{\xi},\tilde{\xi}))-\\
&\qquad\Big[2|\gamma|\epsilon^4\Big(\frac{\widetilde{C}^2\sigma}{\tilde{d}}+\frac{10\widehat{C}\widetilde{C}^3\sigma^2}{3\tilde{d}^3}+\frac{10\widehat{C}\widetilde{C}^3\sigma^3}{3\tilde{d}^3}\Big)+4\Big(\sum_{i,j=1}^{2}|\gamma_{ij}|m_im_j\Big)\epsilon^5\max\{r_1,r_2\}\max_{x,y\in\overline{D_{\overline{\delta}}}}|\nabla R(x,y)|\Big]\\
&\quad\geq\ |\gamma|\epsilon^4\Big\{\frac{c|m|^2}{8}(R(\xi,\xi)-R(\tilde{\xi},\tilde{\xi}))-\\
&\qquad\Big(\frac{2\widetilde{C}^2\sigma}{\tilde{d}}+\frac{20\widehat{C}\widetilde{C}^3\sigma^2}{3\tilde{d}^3}+\frac{20\widehat{C}\widetilde{C}^3\sigma^3}{3\tilde{d}^3}+8\delta|m|^2\max\{r_1,r_2\}\max_{x,y\in\overline{D_{\overline{\delta}}}}|\nabla R(x,y)|\Big)\Big\}. \quad (7.10)
\end{aligned}$$

To reach the last line, note that $|\gamma_{ij}|\leq|\gamma|$ and $\sum_{i,j=1}^{2}m_im_j\leq 2|m|^2$. Because of (2.13), if σ and δ are sufficiently small, then (7.10) asserts

$$J(\xi,\theta)-J(\tilde{\xi},\tilde{\theta})>0 \quad (7.11)$$

for all $(\xi,\theta)\in\partial D_{\overline{\delta}}\times\mathbb{S}^1$ and $(\tilde{\xi},\tilde{\theta})\in D_{\overline{\delta}}\times\mathbb{S}^1$, with $\tilde{\xi}$ being a minimum of $R(z,z)$. Therefore any minimum of J on $\overline{D_{\overline{\delta}}}\times\mathbb{S}^1$ must be in $D_{\overline{\delta}}\times\mathbb{S}^1$, the interior of $\overline{D_{\overline{\delta}}}\times\mathbb{S}^1$. □

Note that this is the first time after (2.14) that δ is required to be small. It is also the first time that the condition (1.8) is used. Only from this moment on, δ and σ become dependent on ι.

The dependence of $(\phi^*,\eta^*)=(\phi^*(t,\xi,\theta),\eta^*(\xi,\theta))$ on $\xi=(\xi^1,\xi^2)$, and θ is investigated in the next lemma.

Lemma 7.3 *When σ is sufficiently small, $\|\frac{\partial(\phi^*,\eta^*)}{\partial\xi^l}\|_{\mathcal{X}}=O(|\gamma|\epsilon^3)$, $l=1,2$, and $\|\frac{\partial(\phi^*,\eta^*)}{\partial\theta}\|_{\mathcal{X}}=O(|\gamma|\epsilon^4)$ uniformly with respect to all $(\xi,\theta)\in D_{\overline{\delta}}\times S^1$.*

Proof. Equation (6.1) is now written as

$$\mathcal{S}(\phi,\eta,\xi,\theta)=0, \quad (7.12)$$

with the operator $\mathcal{S}$ acting as

$$\mathcal{S}:\ (\phi,\eta)\times(\xi,\theta)\to\mathcal{S}(\phi,\eta,\xi,\theta) \quad (7.13)$$

from $\mathcal{D}(\mathcal{S})\times D_{\overline{\delta}}\times\mathbb{S}^1$ to $\mathcal{Z}$. Estimate $\frac{D\mathcal{S}(\phi,\alpha,\xi,\theta)}{D\xi^l}$ and $\frac{D\mathcal{S}(\phi,\alpha,\xi,\theta)}{D\theta}$, the Fréchet derivatives of $\mathcal{S}$ with respect to ξ^l and θ, respectively. Let F be the perturbed double bubble so that $T_{\epsilon,\xi,\theta}(F)=\Omega$, where Ω is represented by (ϕ,η), and $\hat{\mathbf{r}}_j(t)$ correspond to $\mathbf{r}_j$, the boundaries of Ω, via $\mathbf{r}_j=T_{\epsilon,\xi,\theta}(\hat{\mathbf{r}}_j)$. Here F and $\hat{\mathbf{r}}_j$ are independent

of ξ and θ. The operator $\mathcal{S}$ acts on ξ and θ via the transformation $T_{\epsilon,\xi,\theta}$, and only the parts involving I_{Ω_i} in $\mathcal{S}$ depend on ξ and θ as follows:

$$\begin{aligned} I_{\Omega_i} &= \int_{\Omega_i} G(\mathbf{r}_j(t), y)\, dy = \int_{\Omega_i} \frac{1}{2\pi} \log \frac{1}{|\mathbf{r}_j(t) - y|} dy + \int_{\Omega_i} R(\mathbf{r}_j(t), y)\, dy \\ &= \int_{F_i} \frac{\epsilon^2}{2\pi} \log \frac{1}{\epsilon|\hat{\mathbf{r}}_j(t) - \hat{y}|}\, d\hat{y} + \epsilon^2 \int_{F_i} R(\epsilon e^{\mathrm{i}\theta}\hat{\mathbf{r}}_j(t) + \xi, \epsilon e^{\mathrm{i}\theta}\hat{y} + \xi)\, d\hat{y}. \end{aligned}$$

Then clearly

$$\frac{\partial I_{\Omega_i}}{\partial \xi^l} = O(\epsilon^2) \ \text{ and } \ \frac{\partial I_{\Omega_i}}{\partial \theta} = O(\epsilon^3) \tag{7.14}$$

hold uniformly with respect to t, ξ, and θ. Consequently

$$\Big\| \frac{D\mathcal{S}(\phi, \eta, \xi, \theta)}{D\xi^l} \Big\| = O(|\gamma|\epsilon^4) \ \text{ and } \ \Big\| \frac{D\mathcal{S}(\phi, \eta, \xi, \theta)}{D\theta} \Big\| = O(|\gamma|\epsilon^5). \tag{7.15}$$

Here the Fréchet derivatives are operators from $\mathbb{R}$ to $\mathcal{Z}$ and the above are estimates on the norms of these operators. On the other hand, Lemma 6.2(2) shows that at $(\phi^*(\cdot, \xi, \theta), \eta^*(\xi, \theta))$, the solution found in Lemma 6.1,

$$\Big\| \Big(\frac{D\mathcal{S}(\phi^*, \eta^*, \xi, \theta)}{D(\phi, \eta)} \Big)^{-1} \Big\| \le \frac{1}{\check{d}\epsilon} \tag{7.16}$$

if σ is small. Note that $\frac{D\mathcal{S}(\phi^*, \eta^*, \xi, \theta)}{D(\phi, \eta)}$ here is the same as $\mathcal{S}'(\phi^*, \eta^*)$ in Lemma 6.2. The implicit function theorem asserts that when σ is small enough,

$$\Big\| \frac{D(\phi^*, \eta^*)}{D\xi^l} \Big\| = O(|\gamma|\epsilon^3) \ \text{ and } \ \Big\| \frac{D(\phi^*, \eta^*)}{D\theta} \Big\| = O(|\gamma|\epsilon^4). \tag{7.17}$$

Since

$$\Big\| \frac{D(\phi^*, \eta^*)}{D\xi^l} \Big\| = \Big\| \frac{\partial(\phi^*, \eta^*)}{\partial \xi^l} \Big\|_{\mathcal{X}} \ \text{ and } \ \Big\| \frac{D(\phi^*, \eta^*)}{D\theta} \Big\| = \Big\| \frac{\partial(\phi^*, \eta^*)}{\partial \theta} \Big\|_{\mathcal{X}} \tag{7.18}$$

the lemma follows. □

Finally we complete the proof of the main theorem.

Proof of Theorem 1.1. Let the three curves of $(\phi^*(\cdot, \xi, \theta), \eta^*(\xi, \theta))$ found in Lemma 6.1 be parametrized by $\mathbf{r}_i^*(t, \xi, \theta)$. Without loss of generality we assume that $J(\xi, \theta)$ given in (7.6) is minimized at $(\vec{0}, 0)$, i.e. $\xi^* = \vec{0}$ and $\theta^* = 0$. For $(\xi, \theta) \in D_{\bar{\delta}} \times \mathbb{S}^1$ one views $\mathbf{r}_i^*(t, \xi, \theta)$ as a three parameter family of deformations of $\mathbf{r}_i^*(t, \vec{0}, 0)$. If $(\xi, \theta) = (\varepsilon, 0, 0)$, then it is approximately a horizontal deformation whose infinitesimal element is

$$\mathbf{X}_i^H(t) = \frac{\partial \mathbf{r}_i^*(t, \xi, \theta)}{\partial \xi^1}\Big|_{(\xi,\theta)=(\vec{0},0)}.$$

Since

$$\mathbf{r}_i^*(t, \xi, \theta) = \epsilon e^{\mathrm{i}\theta}(u_i^* e^{\mathrm{i}\alpha_i(\eta^*)t} + \beta_i(\eta^*)) + \xi$$

and $2\phi_i^* = \alpha_i(\eta^*)(u_i^*)^2 - \alpha_i(\eta^*)\rho_i^2(\eta^*)$, Lemma 7.3 implies that

$$\mathbf{X}_i^H(t) = \frac{\partial \mathbf{r}_i^*(t, \xi, \theta)}{\partial \xi^1}\Big|_{(\xi,\theta)=(\vec{0},0)} = (1, 0) + O(|\gamma|\epsilon^4) \tag{7.19}$$

uniformly with respect to t. If $(\xi, \theta) = (0, \varepsilon, 0)$, then it is nearly a vertical deformation whose infinitesimal element is

$$\mathbf{X}_i^V(t) = \frac{\partial \mathbf{r}_i^*(t, \xi, \theta)}{\partial \xi^2}\Big|_{(\xi,\theta)=(\vec{0},0)} = (0, 1) + O(|\gamma|\epsilon^4); \tag{7.20}$$

and if $(\xi,\theta)=(0,0,\varepsilon)$, then it is almost a rotational deformation whose infinitesimal element is

$$\mathbf{X}_i^R(t)=\frac{\partial \mathbf{r}_i^*(t,\xi,\theta)}{\partial\theta}\Big|_{(\xi,\theta)=(\vec{0},0)}=\mathrm{i}\ \mathbf{r}_i^*(t,\vec{0},0)+O(|\gamma|\epsilon^5) \tag{7.21}$$

uniformly with respect to t. Note that these three deformations are no longer in the restricted class.

By Lemma 7.3, since $(\vec{0},0)$ is an interior minimum of J,

$$\frac{\partial J(\xi,\theta)}{\partial\xi^1}\Big|_{(\xi,\theta)=(\vec{0},0)}=\frac{\partial J(\xi,\theta)}{\partial\xi^2}\Big|_{(\xi,\theta)=(\vec{0},0)}=\frac{\partial J(\xi,\theta)}{\partial\theta}\Big|_{(\xi,\theta)=(\vec{0},0)}=0. \tag{7.22}$$

On the other hand Lemma 2.4, which holds for both restricted deformations and non-restricted deformations, shows that $\frac{\partial J(\xi,\theta)}{\partial\xi^1}\Big|_{(\xi,\theta)=(\vec{0},0)}$, $\frac{\partial J(\xi,\theta)}{\partial\xi^2}\Big|_{(\xi,\theta)=(\vec{0},0)}$, and $\frac{\partial J(\xi,\theta)}{\partial\theta}\Big|_{(\xi,\theta)=(\vec{0},0)}$ are equal to

$$(\sum_{i=0}^{2}\mathbf{T}_i)\cdot\mathbf{X}\Big|_{-1}^{1}-\int_{\partial\Omega_1\setminus\partial\Omega_2}(\kappa_1+\gamma_{11}I_{\Omega_1}+\gamma_{12}I_{\Omega_2})\mathbf{N}_1\cdot\mathbf{X}\,ds-\int_{\partial\Omega_2\setminus\partial\Omega_1}(\kappa_2+\gamma_{12}I_{\Omega_1}+\gamma_{22}I_{\Omega_2})\mathbf{N}_2\cdot\mathbf{X}\,ds$$
$$-\int_{\partial\Omega_1\cap\partial\Omega_2}(\kappa_0+(\gamma_{11}-\gamma_{12})I_{\Omega_1}+(\gamma_{12}-\gamma_{22})I_{\Omega_2})\mathbf{N}_0\cdot\mathbf{X}\,ds \tag{7.23}$$

with $\mathbf{X}$ being $\mathbf{X}^H$, $\mathbf{X}^V$, and $\mathbf{X}^R$, respectively. In (7.23) $\mathbf{T}_i$ and $\mathbf{N}_i$ are the tangent and normal vectors of the curves $\mathbf{r}_i^*(t,\vec{0},0)$. But these curves satisfy the first three equations (1.2)-(1.4). Hence, the three integral terms in (7.23) are simplified to

$$-\int_{\partial\Omega_1\setminus\partial\Omega_2}\lambda_1\mathbf{N}_1\cdot\mathbf{X}_1\,ds-\int_{\partial\Omega_2\setminus\partial\Omega_1}\lambda_2\mathbf{N}_2\cdot\mathbf{X}_2\,ds-\int_{\partial\Omega_1\cap\partial\Omega_2}(\lambda_1-\lambda_2)\mathbf{N}_0\cdot\mathbf{X}_0\,ds.$$

By (2.29) and (2.30) of Lemma 2.4, the above is equal to

$$\lambda_1\frac{d|\Omega_1^\varepsilon|}{d\varepsilon}\Big|_{\varepsilon=0}+\lambda_2\frac{d|\Omega_2^\varepsilon|}{d\varepsilon}\Big|_{\varepsilon=0}=0$$

by the constraints (1.7). Since the three intergrals in (7.23) vanish, (7.22) and (7.23) imply

$$(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)\cdot\mathbf{X}\Big|_{-1}^{1}=0 \tag{7.24}$$

for $\mathbf{X}$ equal to $\mathbf{X}^H$, $\mathbf{X}^V$, $\mathbf{X}^R$.

By Lemma 6.3, (7.24) also holds for $\mathbf{X}=\mathbf{X}^S$, where $\mathbf{X}^S$ is the vector given in Lemma 4.1. Under the assumption $(\xi,\theta)=(\vec{0},0)$,

$$\mathbf{X}^S(\pm1)=\pm\epsilon\,\mathrm{i}. \tag{7.25}$$

Unlike $\mathbf{X}^H$, $\mathbf{X}^V$, and $\mathbf{X}^R$, this $\mathbf{X}^S$ is the infinitesimal element of a restricted deformation. The equations (7.24) form a four by four linear homogeneous system for the two components of the vector $(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)(1)$ and the two components of the vector $(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)(-1)$. The coefficients of the matrix are the components of $\mathbf{X}^H(\pm1)$, $\mathbf{X}^V(\pm1)$, $\mathbf{X}^R(\pm1)$, and $\mathbf{X}^S(\pm1)$ given in (7.19), (7.20), (7.21), and (7.25). In the case of $\mathbf{X}^R(\pm1)$,

$$\mathbf{X}^R(\pm1)=\mathrm{i}\ \epsilon(0,\pm\eta^*(\vec{0},0))+O(|\gamma|\epsilon^5)=(\mp\epsilon\eta^*(\vec{0},0),0)+O(|\gamma|\epsilon^5).$$

The system (7.24), including (6.13), can be written as

$$\left(\begin{bmatrix}1&0&-1&0\\0&1&0&-1\\-\eta^*(\vec{0},0)&&-\eta^*(\vec{0},0)&0\\0&1&0&1\end{bmatrix}+O(|\gamma|\epsilon^4)\right)\begin{bmatrix}(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)(1)\\(\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_0)(-1)\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}. \tag{7.26}$$

Since the matrix on the left side is non-singular when δ and σ are small,

$$(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0)(1) = (\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_0)(-1) = \vec{0}. \tag{7.27}$$

In (1.5) the ν_i's are the unit inward tangential vectors at the triple points, so $\nu_i = -\mathbf{T}_i$ at the upper triple point corresponding to $t = 1$ and $\nu_i = \mathbf{T}_i$ at the lower triple corresponding to $t = -1$. Hence (7.27) implies (1.5).

According to Lemma 6.1 the solution $(\phi^*(\cdot, \vec{0}, 0), \eta^*(\vec{0}, 0))$ is found in the space $\mathcal{X}$, so the functions $\phi_i^*(\cdot, \vec{0}, 0)$ are in $H^2(-1, 1)$. The standard boot-strapping argument applied to the second order integro-differential equations (1.2)-(1.4) shows that the $\phi_i^*(\cdot, 0, 0)$'s are all C^∞. Hence the two bubbles of the solution are enclosed by continuous curves that are C^∞ except at the triple points.

A systematic study of stability of solutions to (1.2)-(1.5) is beyond the scope of this paper. Our assertion that the solution $(\phi^*(\cdot, \vec{0}, 0), \eta^*(\vec{0}, 0))$ is stable is interpreted by its local minimization property. Recall that the solution $(\phi^*(\cdot, \vec{0}, 0), \eta^*(\vec{0}, 0))$ is found in two steps. First for each $(\xi, \theta) \in \overline{D_\delta} \times \mathbb{S}^1$, a fixed point $(\phi^*(\cdot, \xi, \theta), \eta^*(\xi, \theta))$ is constructed in a restricted class of perturbed double bubbles. This fixed point is shown to be locally minimizing $\mathcal{J}$ in the restricted class in Lemma 6.2(1). In the second step $\mathcal{J}$ is minimized among the $(\phi^*(\cdot, \xi, \theta), \eta^*(\xi, \theta))$'s where (ξ, θ) ranges over $\overline{D_\delta} \times \mathbb{S}^1$, and $(\phi^*(\cdot, \vec{0}, 0), \eta^*(\vec{0}, 0))$ emerges as a minimum. As a minimum of locally minimizing perturbed double bubbles from restricted classes, $(\phi^*(\cdot, \vec{0}, 0), \eta^*(\vec{0}, 0))$ is a local minimizer of $\mathcal{J}$ with respect to both restricted deformations and non-restricted deformations; hence, we claim that $(\phi^*(\cdot, \vec{0}, 0), \eta^*(\vec{0}, 0))$ is stable.

The amount of deviation of our solution from an exact double bubble is given by $\|(\phi^*(\vec{0}, 0), \eta^*(\vec{0}, 0)) - (0, h)\|_{\mathcal{X}}$ and this quantity is of the order $|\gamma|\epsilon^3$ by Lemma 6.1. Therefore, the smaller $|\gamma|\epsilon^3$ is, the closer the solution is to an exact double bubble. $\square$

The proof of Theorem 1.1 above actually shows that any critical point of J gives rise to a stationary point of $\mathcal{J}$. We used a global minimum of J found in Lemma 7.2 to deduce the theorem. However one can just as well find other critical points of J to obtain other stationary points of $\mathcal{J}$. A local minimum of J would lead to a stationary point that is in some sense stable. A saddle point of J gives an unstable stationary point. Indeed one can use a mini-max type argument to show that J always admits a saddle point. Intuitively this saddle point of $J = J(\xi, \theta)$ for $(\xi, \theta) \in \overline{D_\delta} \times \mathbb{S}^1$ is a minimum with respect to ξ and a maximum with respect to θ.

If we denote the minimum of J on $\overline{D_\delta} \times \mathbb{S}^1$ found in Lemma 7.2 by (ξ^*, θ^*), one can see from the proof of that lemma that as $\epsilon \to 0$ and $|\gamma|\epsilon^3 \to 0$, $(\xi^*, \theta^*) \to (\tilde{\xi}, \tilde{\theta})$ possibly along a subsequence so that $R(\tilde{\xi}, \tilde{\xi}) = \min_{z \in D} R(z, z)$. Therefore the perturbed double-bubble solution sits close to a minimum of the function $z \to R(z, z)$ when ϵ and $|\gamma|\epsilon^3$ are small.

However the proof of Lemma 7.2 does not tell us what $\tilde{\theta}$ is, so we do not know the direction of the double bubble. One possible way to resolve this problem is to find a better approximate solution. In this paper we used the exact double bubble $E = (E_1, E_2)$ to build an approximate solution. For a better approximation, we could use a solution of a *profile* problem, a problem defined on the entire plane $\mathbb{R}^2$. The idea is that if we enlarge our double-bubble solution by a factor ϵ^{-1} and let $\epsilon \to 0$, it should converge to a solution of the profile problem.

The free energy of the profile problem is

$$\mathcal{H}(P_1, P_2) = \frac{1}{2} \sum_{i=1}^{3} \mathcal{P}(P_i) + \frac{1}{2} \sum_{i,j=1}^{2} \Gamma_{ij} \int_{P_i} \mathcal{N}(\chi_{P_j})\, dx \tag{7.28}$$

for Lebesgue measurable sets P_1, P_2 in $\mathbb{R}^2$ such that $|P_1| = m$, $|P_2| = 1 - m$, and $|P_1 \cap P_2| = 0$. Here $m \in (0, 1)$ is the same number as the one introduced before (1.7), $P_3 = \mathbb{R}^2 \backslash (P_1 \cup P_2)$, $\mathcal{P}(P_i)$ is the perimeter of P_i in $\mathbb{R}^2$, the Γ_{ij}'s form a positive definite matrix Γ, and $\mathcal{N}$ is the Newtonian potential operator given by

$$\mathcal{N}(f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Big(\log \frac{1}{|x - y|} \Big) f(y)\, dy. \tag{7.29}$$

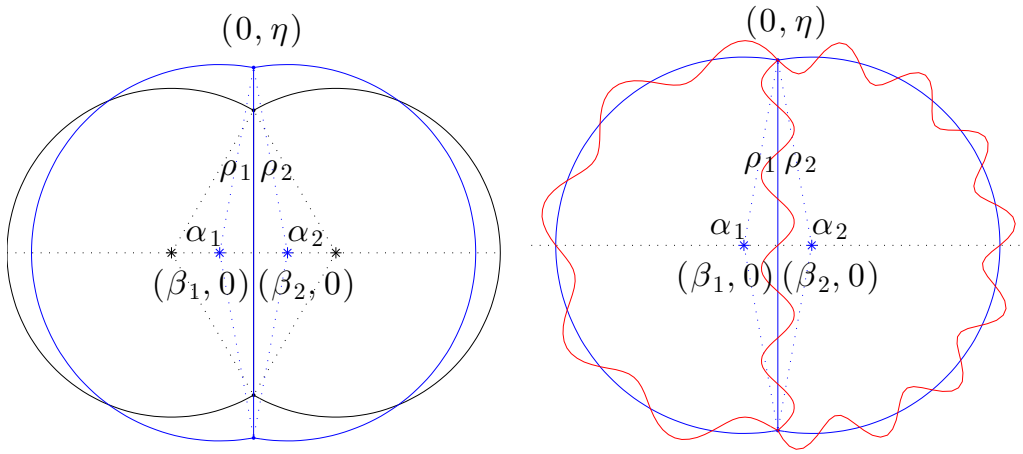

Figure 6: Left plot: the first step of perturbation. Right plot: the second step of perturbation without the exact double bubble showing.

A stationary point of the functional $\mathcal{H}$ is a solution of

$$\begin{aligned}
\kappa_1 + \Gamma_{11}\mathcal{N}(\chi_{P_1}) + \Gamma_{12}\mathcal{N}(\chi_{P_2}) &= \lambda_1 \ \text{ on } \partial P_1 \backslash \partial P_2, && (7.30)\\
\kappa_2 + \Gamma_{12}\mathcal{N}(\chi_{P_1}) + \Gamma_{22}\mathcal{N}(\chi_{P_2}) &= \lambda_2 \ \text{ on } \partial P_2 \backslash \partial P_1, && (7.31)\\
\kappa_0 + (\Gamma_{11} - \Gamma_{12})\mathcal{N}(\chi_{P_1}) + (\Gamma_{12} - \Gamma_{22})\mathcal{N}(\chi_{P_2}) &= \lambda_1 - \lambda_2 \ \text{ on } \partial P_1 \cap \partial P_2, && (7.32)\\
\nu_1 + \nu_2 + \nu_0 &= \vec{0} \ \text{ at } \partial P_1 \cap \partial P_2 \cap \partial P_3. && (7.33)
\end{aligned}$$

Here ν_1, ν_2, and ν_0 are the inward unit tangent vectors of $\partial P_1 \backslash \partial P_2$, $\partial P_2 \backslash \partial P_1$, and $\partial P_1 \cap \partial P_2$, respectively.

If one can find a solution $\widetilde{E} = (\widetilde{E}_1, \widetilde{E}_2)$ to (7.30)-(7.33) that resembles a double bubble, then the image $T_{\epsilon,\xi,\theta}(\widetilde{E})$ can be used as an approximate solution to (1.2)-(1.5) with $\gamma_{ij} = \epsilon^{-3}\Gamma_{ij}$. This new approximate solution based on $\widetilde{E}$ rather than E may yield a more refined estimate than Lemma 2.1. It may also improve Lemma 7.1 and reveal the limiting direction $\tilde{\theta}$.

8 The symmetric case

In this section we prove Theorem 1.1 in the symmetric case where $m = \frac{1}{2}$. In this case the middle arc of the exact double bubble E becomes a straight line. Consequently, $a_1 = a_2 = \frac{2\pi}{3}$, $a_0 = 0$, $r_1 = r_2$, $r_0 = \infty$, and $b_1 = -b_2$ as in Figure 3. The proof proceeds along the same lines, so we will only present the differences, all of which are related to the middle line.

In the first step of perturbation within the restricted class, the two triple points $(0, \pm h)$ again move vertically in opposite directions to $(0, \pm\eta)$. The centers $(b_1, 0)$ and $(b_2, 0)$ move to $(\beta_1, 0)$ and $(\beta_2, 0)$ with $\beta_1 = -\beta_2$, and the radii $r_1 = r_2$ become $\rho_1 = \rho_2$; see the left plot of Figure 6. The constraints on the areas of the two parts now read

$$\rho_i^2(\alpha_i - \cos\alpha_i \sin\alpha_i) = \frac{1}{2}, \ i = 1, 2. \tag{8.1}$$

The μ_i's in this case are constants: $\mu_1 = \mu_2 = \frac{1}{2}$ and $\mu_0 = 0$.

In the second step of perturbation we again introduce functions $u_i(t)$, $i = 1, 2, 0$, for $t \in (-1, 1)$ to form curves $\hat{\mathbf{r}}_i(t)$; see the right plot of Figure 6. The left and right curves are given by the same formula

$$\hat{\mathbf{r}}_1(t) = u_1(t)e^{\mathrm{i}(\pi - \alpha_1 t)} + \beta_1, \quad \hat{\mathbf{r}}_2(t) = u_2(t)e^{\mathrm{i}\alpha_2 t} + \beta_2. \tag{8.2}$$

However, this time the middle curve is parametrized differently:

$$\hat{\mathbf{r}}_0(t) = (u_0(t), \eta t), \quad t \in (-1, 1), \tag{8.3}$$

with the boundary condition

$$u_0(\pm 1) = 0. \tag{8.4}$$

While the internal variables ϕ_1 and ϕ_2 are defined in the same way as in (3.13) with the same boundary condition $\phi_i(\pm 1) = 0$ $(i = 1, 2)$, ϕ_0 is given by

$$\phi_0(t) = \eta u_0(t) \tag{8.5}$$

with the boundary condition $\phi_0(\pm 1) = 0$. The area constraints on ϕ_1, ϕ_2, and ϕ_0 remain (3.14). Moreover

$$-\mathbf{N}_0 \cdot \mathbf{X}_0 ds = \epsilon^2(\psi_0 + \mathcal{E}_0(\phi_0, \eta)\zeta)\, dt, \quad \mathcal{E}_0(\phi_0, \eta) = -\frac{\phi_0 + \phi_0' t}{\eta}. \tag{8.6}$$

The length of the middle curve is

$$\int_{-1}^{1} \sqrt{\frac{(\phi_0'(t))^2}{\eta^2} + \eta^2}\, dt. \tag{8.7}$$

Once Lemma 5.2 is established, the rest of the proof is similar to the one in the asymmetric case.

To this end, let

$$\int_{-1}^{1} \psi_0\, dt = \nu, \quad \int_{-1}^{1} \psi_1\, dt = -\nu, \quad \int_{-1}^{1} \psi_2\, dt = \nu \tag{8.8}$$

and by Lemma 5.1 one derives

$$\begin{aligned}
&\frac{d^2\mathcal{P}(0 + \varepsilon\psi, h + \varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} - 2d\sum_{i=0}^{2} \|\phi_i\|_{H^1}^2 \\
&= \sum_{i=1}^{2} \int_{-1}^{1} \Big[\frac{1}{(a_i r_i)^3}(\psi_i'(t))^2 - \frac{1}{a_i r_i^3}\psi_i^2(t)\Big]\, dt + \int_{-1}^{1} \frac{1}{h^3}(\psi_0'(t))^2\, dt + 2\zeta^2 \frac{d^2}{d\eta^2}\Big|_{\eta=h}(\alpha_1\rho_1 + \alpha_2\rho_2 + \eta) \\
&\quad -2d\sum_{i=0}^{2} \|\phi_i\|_{H^1}^2 \\
&= \sum_{i=1}^{2} \int_{-1}^{1} \Big[\Big(\frac{1}{(a_i r_i)^3} - 2d\Big)(\psi_i'(t))^2 - \Big(\frac{1}{a_i r_i^3} + 2d\Big)\psi_i^2(t)\Big]\, dt + \int_{-1}^{1} \Big[\Big(\frac{1}{h^3} - 2d\Big)(\psi_0'(t))^2 - 2d\psi_0^2(t)\Big]\, dt \\
&\quad +2\zeta^2 \frac{d^2}{d\eta^2}\Big|_{\eta=h}(\alpha_1\rho_1 + \alpha_2\rho_2 + \eta) \\
&\geq \sum_{i=1}^{2} \frac{\big(\frac{1}{(a_i r_i)^3} - 2d\big)\nu^2 q_i^3}{2(\tan q_i - q_i)} + \frac{\big(\frac{1}{h^3} - 2d\big)\nu^2 q_0^3}{2(\tan q_0 - q_0)} + 2\zeta^2 \frac{d^2}{d\eta^2}\Big|_{\eta=h}(\alpha_1\rho_1 + \alpha_2\rho_2 + \eta),
\end{aligned} \tag{8.9}$$

where

$$q_i = \sqrt{\frac{\frac{1}{a_i r_i^3} + 2d}{\frac{1}{(a_i r_i)^3} - 2d}}, \quad i = 1, 2; \quad q_0 = \sqrt{\frac{2d}{\frac{1}{h^3} - 2d}}. \tag{8.10}$$

As $d \to 0$,

$$\begin{aligned}
&\sum_{i=1}^{2} \frac{\big(\frac{1}{(a_i r_i)^3} - 2d\big)q_i^3}{2(\tan q_i - q_i)} + \frac{\big(\frac{1}{h^3} - 2d\big)q_0^3}{2(\tan q_0 - q_0)} \\
&\quad \to \frac{1}{2}\sum_{i=1}^{2} \frac{1}{r_i^3(\tan a_i - a_i)} + \frac{3}{2h^3} = \frac{\sin^3 \frac{2\pi}{3}}{(\tan \frac{2\pi}{3} - \frac{2\pi}{3})h^3} + \frac{3}{2h^3} = \frac{1.3303...}{h^3} > 0.
\end{aligned} \tag{8.11}$$

Hence for $d > 0$ sufficiently small,

$$\sum_{i=1}^{2} \frac{\left(\frac{1}{(a_i r_i)^3} - 2d\right)\nu^2 q_i^3}{2(\tan q_i - q_i)} + \frac{\left(\frac{1}{h^3} - 2d\right)\nu^2 q_0^3}{2(\tan q_0 - q_0)} \geq 0. \tag{8.12}$$

Regarding the remaining term in (8.9), let

$$P = 2\sum_{i=1}^{2} \alpha_i \rho_i + 2\eta = 2\sum_{i=1}^{2} \frac{\eta \alpha_i}{\sin \alpha_i} + 2\eta \tag{8.13}$$

since

$$\eta = \rho_i \sin \alpha_i. \tag{8.14}$$

Implicit differentiation from (8.1) and (8.14) shows that

$$\frac{d\alpha_i}{d\eta} = -\frac{(\alpha_i - \cos\alpha_i \sin\alpha_i)\sin\alpha_i}{\eta(\sin\alpha_i - \alpha_i\cos\alpha_i)}. \tag{8.15}$$

It follows that

$$\frac{dP}{d\eta} = 2\cos\alpha_1 + 2\cos\alpha_2 + 2. \tag{8.16}$$

Note that at the exact double bubble where α_i is $\frac{2\pi}{3}$,

$$\frac{dP}{d\eta}\Big|_{\eta=h} = 0. \tag{8.17}$$

Moreover

$$\frac{d^2P}{d\eta^2} = \frac{2}{\eta}\sum_{i=1}^{2} \frac{(\alpha_i - \cos\alpha_i\sin\alpha_i)\sin^2\alpha_i}{\sin\alpha_i - \alpha_i\cos\alpha_i}. \tag{8.18}$$

At the exact double bubble

$$\frac{d^2P}{d\eta^2}\Big|_{\eta=h} = \frac{2}{h}\sum_{i=1}^{2} \frac{(\frac{2\pi}{3} - \cos\frac{2\pi}{3}\sin\frac{2\pi}{3})\sin^2\frac{2\pi}{3}}{\sin\frac{2\pi}{3} - \frac{2\pi}{3}\cos\frac{2\pi}{3}} = \frac{3.9631...}{h} > 0. \tag{8.19}$$

The proof of the counterpart of (8.19) in the asymmetric case is more complex; see Appendix B. By (8.19)

$$2\zeta^2 \frac{d^2}{d\eta^2}\Big|_{\eta=h}(\alpha_1\rho_1 + \alpha_2\rho_2 + \eta) \geq 2d\zeta^2 \tag{8.20}$$

if d is sufficiently small.

It follows from (8.9), (8.12), and (8.20) that

$$\frac{d^2\mathcal{P}(0 + \varepsilon\psi, h + \varepsilon\zeta)}{d\varepsilon^2}\Big|_{\varepsilon=0} \geq 2d\|(\psi,\zeta)\|_{\mathcal{Y}}^2. \tag{8.21}$$

Hence Lemma 5.2 holds in the symmetric case.

Appendix A

We prove Lemma 5.1. Let $\mathcal{F}$ be the functional

$$\mathcal{F}(y) = \int_{-1}^{1} ((y'(t))^2 - q^2 y^2(t))\, dt \tag{A.1}$$

for $y \in H_0^1(-1,1)$ and $\int_{-1}^{1} y(t)\, dt = \nu$, where $q \in (0,\pi)$.

Step 1. $\mathcal{F}$ is bounded below.

Let $e_1 = (\frac{\pi}{2})^2$, $e_2 = \pi^2$, and $e_3 = (\frac{3\pi}{2})^2$ be the first three eigenvalues of the problem

$$-f'' = ef, \quad f \in H_0^1(-1,1),$$

and $f_1(t) = \cos\frac{\pi t}{2}$ and $f_2(t) = \sin \pi t$ be eigenfunctions corresponding to λ_1 and λ_2. Note that

$$\int_{-1}^{1} f_1^2(t)\, dt = \int_{-1}^{1} f_2^2(t)\, dt = 1, \quad \text{and} \quad \int_{-1}^{1} f_1(t)\, dt = \frac{4}{\pi}, \quad \int_{-1}^{1} f_2(t)\, dt = 0.$$

For every $y \in H_0^1(-1,1)$, decompose $y = c_1 f_1 + c_2 f_2 + z$, where $z \in H_0^1(-1,1)$ is perpendicular to f_1 and f_2: $\int_{-1}^{1} f_1(t) z(t)\, dt = \int_{-1}^{1} f_2(t) z(t)\, dt = 0$. By the variational characterization of the eigenvalues,

$$\mathcal{F}(y) = c_1^2(e_1 - q^2) + c_2^2(e_2 - q^2) + \mathcal{F}(z) \geq c_1^2(e_1 - q^2) + c_2^2(e_2 - q^2) + (e_3 - q^2)\int_{-1}^{1} z^2(t)\, dt. \tag{A.2}$$

Note

$$\nu = \int_{-1}^{1} y(t)\, dt = \frac{4c_1}{\pi} + \int_{-1}^{1} z(t)\, dt.$$

Then

$$\Big(\nu - \frac{4c_1}{\pi}\Big)^2 = (\int_{-1}^{1} z(t)\, dt)^2 \leq 2\int_{-1}^{1} z^2(t)\, dt$$

and

$$\begin{aligned} \mathcal{F}(y) &\geq c_1^2(e_1 - q^2) + c_2^2(e_2 - q^2) + \frac{1}{2}(e_3 - q^2)\Big(\nu - \frac{4c_1}{\pi}\Big)^2 \\ &= \Big(e_1 - q^2 + (e_3 - q^2)\frac{8}{\pi^2}\Big)c_1^2 - (e_3 - q^2)\Big(\frac{4\nu}{\pi}\Big)c_1 + (e_3 - q^2)\frac{\nu^2}{2} + (e_2 - q^2)c_2^2. \end{aligned}$$

Since

$$e_1 - \pi^2 + (e_3 - \pi^2)\frac{8}{\pi^2} = -\frac{3\pi^2}{4} + 10 > 0 \ \text{ and } e_2 - q^2 > \pi^2 - \pi^2 = 0,$$

$\mathcal{F}(y)$ is bounded below for all $y \in H_0^1(-1,1)$ with $\int_{-1}^{1} y(t)\, dt = \nu$.

Step 2. A minimizing sequence is bounded in $H_0^1(-1,1)$.

Let y_n be a minimizing sequence. Decompose, as above, $y_n = c_1^n f_1 + c_2^n f_2 + z_n$. Then

$$\mathcal{F}(y_n) \geq \Big(e_1 - q^2 + (e_3 - q^2)\frac{8}{\pi^2}\Big)(c_1^n)^2 - (e_3 - q^2)\Big(\frac{4\nu}{\pi}\Big)c_1^n + (e_3 - q^2)\frac{\nu^2}{2} + (e_2 - q^2)(c_2^n)^2.$$

Since $\mathcal{F}(y_n)$ is bounded below and above (for y_n is minimizing), $|c_1^n|$ and $|c_2^n|$ are bounded with respect to n. By (A.2), $\int_{-1}^{1} z_n^2(t)\, dt$ is also bounded. Consequently $\int_{-1}^{1} y_n^2(t)\, dt$ is bounded. From (A.1) we deduce that $\int_{-1}^{1} (y_n'(t))^2\, dt$ is bounded. Hence y_n is bounded in $H_0^1(-1,1)$.

Step 3. A minimizer w exists.

From the minimizing sequence y_n, there is a subsequence again denoted by y_n that converges weakly in $H_0^1(-1,1)$ and strongly in $L^2(-1,1)$ to a limit $w \in H_0^1(-1,1)$ with $\int_{-1}^1 w(t) = \nu$. By the weak lower semi-continuity of the H^1 norm,

$$\mathcal{F}(w) \leq \liminf_{n\to\infty} \mathcal{F}(y_n).$$

Hence w is a minimizer.

Step 4. $\mathcal{F}(w) = \frac{\nu^2 q^3}{2(\tan q - q)}$.

As a minimizer, w satisfies the equation $-w'' - q^2 w = \lambda$, $w(\pm 1) = 0$, for some $\lambda \in \mathbb{R}$. Solving the equation, we find $w(t) = C\cos(qt) - \frac{\lambda}{q^2}$, $\lambda = Cq^2 \cos q$. Hence $w(t) = C(\cos(qt) - \cos q)$ and $\nu = \int_{-1}^1 w(t)\,dt = C\Big(\frac{2\sin q}{q} - 2\cos q\Big)$. It follows that $C = \frac{\nu}{\frac{2\sin q}{q} - 2\cos q}$ and

$$w(t) = \frac{\nu(\cos(qt) - \cos q)}{\frac{2\sin q}{q} - 2\cos q}.$$

If we multiply the equation for w by w and integrate, then

$$\mathcal{F}(w) = \lambda \int_{-1}^1 w(t)\,dt = \lambda\nu = C\nu q^2 \cos q = \frac{\nu^2 q^3}{2(\tan q - q)}.$$

This proves Lemma 5.1.

Appendix B

We start in a somewhat different way to carry out the first step of perturbing an exact double bubble, and will later return to the perturbation setting described in Section 3. Note that we continue to assume that $r_1 < r_2$, i.e. $0 < m < \frac{1}{2}$.

From the exact double bubble E, move the triple points $(0, \pm h)$ vertically by the same distance in opposing directions to $(0, \pm\eta)$. Connect the new triple points by three arcs with the radii ρ_i, the angles α_i, and the centers $(\beta_i, 0)$ for $i = 1, 2, 0$. However at this point we do not impose the condition $\frac{1}{\rho_1} - \frac{1}{\rho_2} = \frac{1}{\rho_0}$.

Define

$$\mu_i = \rho_i^2(\alpha_i - \cos\alpha_i \sin\alpha_i), \quad i = 1, 2, 0, \tag{B.1}$$

as before. Since $\rho_i \sin\alpha_i = \eta$, one can rewrite μ_i as

$$\mu_i = \frac{\eta^2(\alpha_i - \cos\alpha_i \sin\alpha_i)}{\sin^2\alpha_i}. \tag{B.2}$$

The μ_i's must still satisfy the area constraints

$$\mu_1 + \mu_0 = m, \quad \mu_2 - \mu_0 = 1 - m. \tag{B.3}$$

If α_i is treated as a function of μ_i and η, implicit differentiation shows that

$$\frac{\partial\alpha_i}{\partial\mu_i} = \frac{\sin^3\alpha_i}{2\eta^2(\sin\alpha_i - \alpha_i\cos\alpha_i)}, \tag{B.4}$$

$$\frac{\partial\alpha_i}{\partial\eta} = -\frac{(\alpha_i - \cos\alpha_i\sin\alpha_i)\sin\alpha_i}{\eta(\sin\alpha_i - \alpha_i\cos\alpha_i)}. \tag{B.5}$$

The total length of the three arcs is

$$P = 2\sum_{i=0}^{2} \alpha_i \rho_i = 2\sum_{i=0}^{2} \frac{\eta \alpha_i}{\sin \alpha_i}. \tag{B.6}$$

Since α_i depends on μ_i and η, and the μ_i's are subject to the constraints (B.3), we take μ_0 and η as the independent variables and treat α_i and P as functions of μ_0 and η.

Compute $\frac{\partial P}{\partial \mu_0}$. Since

$$\frac{\partial P}{\partial \mu_0} = 2\eta \sum_{i=0}^{2} \frac{\partial}{\partial \alpha_i}\Big(\frac{\alpha_i}{\sin \alpha_i}\Big) \frac{\partial \alpha_i}{\partial \mu_i} \frac{\partial \mu_i}{\partial \mu_0} \tag{B.7}$$

and

$$\frac{\partial}{\partial \alpha_i}\Big(\frac{\alpha_i}{\sin \alpha_i}\Big) = \frac{\sin \alpha_i - \alpha_i \cos \alpha_i}{\sin^2 \alpha_i}, \tag{B.8}$$

one deduces by (B.4) and (B.3) that

$$\frac{\partial P}{\partial \mu_0} = \sum_{i=0}^{2} \frac{(-1)^i \sin \alpha_i}{\eta}. \tag{B.9}$$

Note that the right side of (B.9) is $-\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_0}$.

Next compute $\frac{\partial P}{\partial \eta}$. Note that

$$\frac{\partial P}{\partial \eta} = 2\sum_{i=0}^{2} \Big(\frac{\alpha_i}{\sin \alpha_i} + \eta \frac{\partial}{\partial \alpha_i}\Big(\frac{\alpha_i}{\sin \alpha_i}\Big) \frac{\partial \alpha_i}{\partial \eta}\Big).$$

By (B.5) and (B.8) one finds

$$\frac{\partial P}{\partial \eta} = 2\sum_{i=0}^{2} \cos \alpha_i. \tag{B.10}$$

Note that at a critical point where $\frac{\partial P}{\partial \mu_0} = \frac{\partial P}{\partial \eta} = 0$,

$$-\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_0 = 0, \tag{B.11}$$

$$\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_0 = 0, \tag{B.12}$$

which imply that $\alpha_1 = \frac{2\pi}{3} - \alpha_0$ and $\alpha_2 = \frac{2\pi}{3} + \alpha_0$, i.e. an exact double bubble.

Now proceed to calculate the second derivatives of P. First

$$\frac{\partial^2 P}{\partial \mu_0^2} = \frac{\partial}{\partial \mu_0}\Big(\sum_{i=0}^{1} \frac{(-1)^i \sin \alpha_i}{\eta}\Big) = \sum_{i=0}^{2} \frac{(-1)^i \cos \alpha_i}{\eta} \frac{\partial \alpha_i}{\partial \mu_i} \frac{\partial \mu_i}{\partial \mu_0} = \sum_{i=0}^{2} \frac{\cos \alpha_i}{\eta} \frac{\partial \alpha_i}{\partial \mu_i}.$$

By (B.4)

$$\frac{\partial^2 P}{\partial \mu_0^2} = \frac{1}{2\eta^3} \sum_{i=0}^{2} \frac{\cos \alpha_i \sin^3 \alpha_i}{\sin \alpha_i - \alpha_i \cos \alpha_i}. \tag{B.13}$$

Next

$$\frac{\partial^2 P}{\partial \mu_0 \partial \eta} = \frac{\partial}{\partial \eta}\Big(\sum_{i=0}^{2} \frac{(-1)^i \sin \alpha_i}{\eta}\Big) = \sum_{i=0}^{2} (-1)^i \Big(-\frac{\sin \alpha_i}{\eta^2} + \frac{\cos \alpha_i}{\eta} \frac{\partial \alpha_i}{\partial \eta}\Big).$$

Using (B.5) one finds

$$\frac{\partial^2 P}{\partial \mu_0 \partial \eta} = -\frac{1}{\eta^2} \sum_{i=0}^{2} \frac{(-1)^i \sin^4 \alpha_i}{\sin \alpha_i - \alpha_i \cos \alpha_i}. \tag{B.14}$$

Finally

$$\frac{\partial^2 P}{\partial \eta^2} = \frac{\partial}{\partial \eta}\Big(2 \sum_{i=0}^{2} \cos \alpha_i\Big) = -2 \sum_{i=0}^{2} \sin \alpha_i \frac{\partial \alpha_i}{\partial \eta}.$$

By (B.5) one derives

$$\frac{\partial^2 P}{\partial \eta^2} = \frac{2}{\eta} \sum_{i=0}^{2} \frac{(\alpha_i - \cos \alpha_i \sin \alpha_i) \sin^2 \alpha_i}{\sin \alpha_i - \alpha_i \cos \alpha_i}. \tag{B.15}$$

In summary, the Hessian matrix of P is

$$D^2 P = \begin{bmatrix} \frac{1}{2\eta^3} \sum_{i=0}^{2} \frac{\cos \alpha_i \sin^3 \alpha_i}{\sin \alpha_i - \alpha_i \cos \alpha_i} & -\frac{1}{\eta^2} \sum_{i=0}^{2} \frac{(-1)^i \sin^4 \alpha_i}{\sin \alpha_i - \alpha_i \cos \alpha_i} \\ -\frac{1}{\eta^2} \sum_{i=0}^{2} \frac{(-1)^i \sin^4 \alpha_i}{\sin \alpha_i - \alpha_i \cos \alpha_i} & \frac{2}{\eta} \sum_{i=0}^{2} \frac{(\alpha_i - \cos \alpha_i \sin \alpha_i) \sin^2 \alpha_i}{\sin \alpha_i - \alpha_i \cos \alpha_i} \end{bmatrix}. \tag{B.16}$$

This matrix is evaluated at the exact double bubble where $\alpha_i = a_i$ and $\eta = h$. The a_i's satisfy $a_1 = \frac{2\pi}{3} - a_0$ and $a_2 = \frac{2\pi}{3} + a_0$.

Regarding the (1,1) entry, one has the following.

Lemma B.1 *For $a_0 \in (0, \frac{\pi}{3})$, $\sum_{i=0}^{2} \frac{\sin^3 a_i}{\tan a_i - a_i} > 0$. Hence the (1,1) entry of $D^2 P$ at the exact double bubble is positive.*

Proof. Set $f(a) = \frac{\sin^3 a}{\tan a - a}$. First note that $f'(a) < 0$ if $a \in (0, \frac{\pi}{2})$, since

$$\begin{aligned} f'(a) &= \frac{1}{(\tan a - a)^2} [3 \sin^2 a \cos a (\tan a - a) - \sin^3 a (\sec^2 a - 1)] \\ &= \frac{\sin^2 a \cos a}{(\tan a - a)^2} [3(\tan a - a) - \tan^3 a]. \end{aligned}$$

Let $\tilde{f}(a) = 3(\tan a - a) - \tan^3 a$. Then

$$\tilde{f}'(a) = 3(\sec^2 a - 1) - 3 \tan^2 a \sec^2 a = 3 \tan^2 a (1 - \sec^2 a) = -3 \tan^4 a < 0,$$

and consequently

$$\tilde{f}(a) < \tilde{f}(0) = 0.$$

Therefore

$$f'(a) < 0. \tag{B.17}$$

To prove the lemma consider three cases.

Case (1). $0 < a_0 < \frac{\pi}{6}$. Then $a_1 \in (\frac{\pi}{2}, \frac{2\pi}{3})$ and $a_2 \in (\frac{2\pi}{3}, \frac{5\pi}{6})$. By (B.17)

$$\begin{aligned} |f(a_1)| &= \frac{\sin^3 a_1}{|\tan a_1| + a_1} < \frac{1}{\sqrt{3} + \frac{\pi}{2}}, \\ |f(a_2)| &= \frac{\sin^3 a_2}{|\tan a_2| + a_2} < \frac{\left(\frac{\sqrt{3}}{2}\right)^3}{\frac{1}{\sqrt{3}} + \frac{2\pi}{3}}, \end{aligned}$$

$$f(a_0) \;>\; f(\frac{\pi}{6}) \;=\; \frac{\left(\frac{1}{2}\right)^3}{\frac{1}{\sqrt{3}} - \frac{\pi}{6}}.$$

Therefore

$$f(a_0) + f(a_1) + f(a_2) \;>\; \frac{\left(\frac{1}{2}\right)^3}{\frac{1}{\sqrt{3}} - \frac{\pi}{6}} - \frac{1}{\sqrt{3} + \frac{\pi}{2}} - \frac{\left(\frac{\sqrt{3}}{2}\right)^3}{\frac{1}{\sqrt{3}} + \frac{2\pi}{3}} \;=\; 1.7796... \;>\; 0.$$

Case (2). $\frac{\pi}{6} < a_0 < \frac{\pi}{3}$. Then $a_1 \in (\frac{\pi}{3}, \frac{\pi}{2})$ and $a_2 \in (\frac{5\pi}{6}, \pi)$. By (B.17)

$$\begin{aligned} f(a_1) \;&>\; 0, \\ |f(a_2)| \;&<\; \frac{\left(\frac{1}{2}\right)^3}{0 + \frac{5\pi}{6}}, \\ f(a_0) \;&>\; f(\frac{\pi}{3}) \;=\; \frac{\left(\frac{\sqrt{3}}{2}\right)^3}{\sqrt{3} - \frac{\pi}{3}}. \end{aligned}$$

Therefore

$$f(a_0) + f(a_1) + f(a_2) \;>\; \frac{\left(\frac{\sqrt{3}}{2}\right)^3}{\sqrt{3} - \frac{\pi}{3}} - \frac{\left(\frac{1}{2}\right)^3}{0 + \frac{5\pi}{6}} \;=\; 0.9006... \;>\; 0.$$

Case (3). First take $a_0 \in (0, \frac{\pi}{6})$ and then pass the limit $a_0 \to \frac{\pi}{6}$ in the first case to conclude that the lemma holds for $a_0 = \frac{\pi}{6}$. $\square$

Next show that the matrix has the positive determinant at the exact double bubble. Define

$$t_{11} = \sum_{i=0}^{2} \frac{\cos a_i \sin^3 a_i}{\sin a_i - a_i \cos a_i}, \quad t_{12} = \sum_{i=0}^{2} \frac{(-1)^i \sin^4 a_i}{\sin a_i - a_i \cos a_i}, \quad t_{22} = \sum_{i=0}^{2} \frac{(a_i - \cos a_i \sin a_i) \sin^2 a_i}{\sin a_i - a_i \cos a_i}. \tag{B.18}$$

Lemma B.2 *$t_{11} > |t_{12}|$ and $t_{22} > |t_{12}|$. Hence the determinant of D^2P at the exact double bubble is positive.*

Proof. To prove $t_{11} > |t_{12}|$ consider three cases: (1) $a_0 \in \left(0, \frac{\pi}{6}\right)$, (2) $a_0 \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$, and (3) $a_0 = \frac{\pi}{6}$. For the first two cases write

$$t_{11} = \sum_{i=0}^{2} \frac{\sin^3 a_i}{\tan a_i - a_i}, \quad t_{12} = \sum_{i=0}^{2} \frac{(-1)^i \tan a_i \sin^3 a_i}{\tan a_i - a_i}, \quad t_{22} = \sum_{i=0}^{2} \frac{(a_i - \cos a_i \sin a_i) \tan a_i \sin a_i}{\tan a_i - a_i}. \tag{B.19}$$

Then

$$t_{11} \pm t_{12} = \frac{(1 \pm \tan a_0) \sin^3 a_0}{\tan a_0 - a_0} + \frac{(1 \mp \tan a_1) \sin^3 a_1}{\tan a_1 - a_1} + \frac{(1 \pm \tan a_2) \sin^3 a_2}{\tan a_2 - a_2}.$$

For case (1), note from the proof of Lemma B.1 that $f(a) = \frac{\sin^3 a}{\tan a - a}$ is decreasing on $a \in \left(0, \frac{\pi}{2}\right)$ and $f\left(\frac{\pi}{6}\right) = 2.3255...$ Then

$$\frac{(1 - \tan a_0) \sin^3 a_0}{\tan a_0 - a_0} > \left(1 - \tan \frac{\pi}{6}\right) f\left(\frac{\pi}{6}\right) = 0.9829....$$

In case (1), $a_1 \in (\frac{\pi}{2}, \frac{2\pi}{3})$ and $\tan a_1 < 0$. Then

$$\left| \frac{\sin^3 a_1}{\tan a_1 - a_1} \right| < \frac{1}{-\tan \frac{2\pi}{3} + \frac{\pi}{2}} = 0.3026....$$

Also since $\frac{\tan a_1}{\tan a_1 - a_1}$ is positive and decreasing,

$$\frac{\tan a_1 \sin^3 a_1}{\tan a_1 - a_1} > \frac{\tan \frac{2\pi}{3}}{\tan \frac{2\pi}{3} - \frac{2\pi}{3}} \sin^3 \frac{2\pi}{3} = 0.2940....$$

In this case $a_2 \in \left(\frac{2\pi}{3}, \frac{5\pi}{6}\right)$. Then

$$\left|\frac{(1-\tan a_2)\sin^3 a_2}{\tan a_2 - a_2}\right| < \frac{(1-\tan\frac{2\pi}{3})\sin^3\frac{2\pi}{3}}{-\tan\frac{5\pi}{6}+\frac{2\pi}{3}} = 0.8856....$$

Hence

$$t_{11} - t_{12} > 0.9829 - 0.3026 + 0.2940 - 0.8856 = 0.08876 > 0.$$

For $t_{11} + t_{12}$, note

$$\begin{aligned}
\frac{(1+\tan a_0)\sin^3 a_0}{\tan a_0 - a_0} &> (1+\tan 0)\frac{\sin^3\frac{\pi}{6}}{\tan\frac{\pi}{6}-\frac{\pi}{6}} = 2.3255...,\\
\left|\frac{\sin^3 a_1}{\tan a_1 - a_1}\right| &< \frac{\sin\frac{\pi}{2}}{-\tan\frac{2\pi}{3}+\frac{\pi}{2}} = 0.3028...,\\
\left|-\frac{\tan a_1 \sin^3 a_1}{\tan a_1 - a_1}\right| &< 1\sin^3\frac{\pi}{2} = 1,\\
\left|\frac{(1+\tan a_2)\sin^3 a_2}{\tan a_2 - a_2}\right| &< \frac{|1+\tan\frac{2\pi}{3}|\sin^3\frac{2\pi}{3}}{-\tan\frac{5\pi}{6}+\frac{2\pi}{3}} = 0.1780....
\end{aligned}$$

Then

$$t_{11} + t_{12} > 2.3255... - 0.3028... - 1 - 0.1780... = 0.8447... > 0.$$

In case (2), $a_0 \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$, $a_1 \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, and $a_2 \in \left(\frac{5\pi}{6}, \pi\right)$. Then

$$\begin{aligned}
t_{11} - t_{12} &= \frac{(1-\tan a_0)\sin^3 a_0}{\tan a_0 - a_0} + \frac{(1+\tan a_1)\sin^3 a_1}{\tan a_1 - a_1} + \frac{(1-\tan a_2)\sin^3 a_2}{\tan a_2 - a_2}\\
&> -\left|1-\tan\frac{\pi}{3}\right|\frac{\sin^3\frac{\pi}{6}}{\tan\frac{\pi}{6}-\frac{\pi}{6}} + (1+\tan\frac{\pi}{3})\frac{\sin^3\frac{\pi}{3}}{\tan\frac{\pi}{3}-\frac{\pi}{3}} - \frac{(1-\tan\frac{5\pi}{6})\sin^3\frac{5\pi}{6}}{-\tan\frac{5\pi}{6}+\frac{5\pi}{6}}\\
&= -1.7024... + 2.5911... - 0.0617... \;=\; 0.8270... \;>\; 0.
\end{aligned}$$

And since $\tan a_2 - a_2$ is increasing,

$$0 < \frac{\tan a_1 - 1}{\tan a_1 - a_1} = 1 + \frac{a_1 - 1}{\tan a_1 - a_1} < 1 + \frac{\frac{\pi}{2}-1}{\tan\frac{\pi}{3}-\frac{\pi}{3}}.$$

Then

$$\begin{aligned}
t_{11} + t_{12} &= \frac{(1+\tan a_0)\sin^3 a_0}{\tan a_0 - a_0} + \frac{(1-\tan a_1)\sin^3 a_1}{\tan a_1 - a_1} + \frac{(1+\tan a_2)\sin^3 a_2}{\tan a_2 - a_2}\\
&> \left(1+\tan\frac{\pi}{6}\right)\frac{\sin^3\frac{\pi}{3}}{\tan\frac{\pi}{3}-\frac{\pi}{3}} - \left(1+\frac{\frac{\pi}{2}-1}{\tan\frac{\pi}{3}-\frac{\pi}{3}}\right)\sin^3\frac{\pi}{3} - \frac{(1+\tan\frac{5\pi}{6})\sin^3\frac{5\pi}{6}}{-\tan\pi+\frac{5\pi}{6}}\\
&= 1.4960... - 1.1909... - 0.0202... = 0.2849... \;>\; 0.
\end{aligned}$$

For case (3) one can first take $a_0 \in (0, \frac{\pi}{6})$ and then pass the limit $a_0 \to \frac{\pi}{6}$ in (B.18) to deduce that $t_{11} \pm t_{12} > 0$ in case (3).

Next show that $t_{22} \pm t_{12} > 0$. First rewrite t_{22} as

$$\begin{aligned} t_{22} &= -\sum_{i=0}^{2} \cos a_i \sin^2 a_i + \sum_{i=0}^{2} \frac{a_i \sin^4 a_i}{\sin a_i - a_i \cos a_i} \\ &= \sum_{i=0}^{2} \Big(\frac{\cos 3a_i}{4} - \frac{\cos a_i}{4} \Big) + \sum_{i=0}^{2} \frac{a_i \sin^4 a_i}{\sin a_i - a_i \cos a_i}. \end{aligned}$$

By (B.12) and the relations $a_1 = \frac{2\pi}{3} - a_0$ and $a_2 = \frac{2\pi}{3} + a_0$, one has

$$t_{22} = \frac{3\cos 3a_0}{4} + \sum_{i=0}^{2} \frac{a_i \sin^4 a_i}{\sin a_i - a_i \cos a_i}. \tag{B.20}$$

Again consider three cases: (1) $a_0 \in \big(0, \frac{\pi}{6}\big)$, (2) $a_0 \in \big(\frac{\pi}{6}, \frac{\pi}{3}\big)$, and (3) $a_0 = \frac{\pi}{6}$. In cases (1) and (2) write

$$t_{22} \pm t_{12} = \frac{3\cos 3a_0}{4} + \frac{(a_0 \pm 1)\tan a_0 \sin^3 a_0}{\tan a_0 - a_0} + \frac{(a_1 \mp 1)\tan a_1 \sin^3 a_1}{\tan a_1 - a_1} + \frac{(a_2 \pm 1)\tan a_2 \sin^3 a_2}{\tan a_2 - a_2}. \tag{B.21}$$

In case (1), where $a_0 \in (0, \frac{\pi}{6})$, $a_1 \in (\frac{\pi}{2}, \frac{2\pi}{3})$ and $a_2 \in (\frac{2\pi}{3}, \frac{5\pi}{6})$, one has

$$\frac{(a_1 + 1)\tan a_1 \sin^3 a_1}{\tan a_1 - a_1} > \Big(\frac{\pi}{2} + 1\Big)\Big(\frac{\tan \frac{2\pi}{3}}{\tan \frac{2\pi}{3} - \frac{2\pi}{3}}\Big) \sin^3 \frac{2\pi}{3} = 0.7558...$$

since $\frac{\tan a_1}{\tan a_1 - a_1}$ is positive and decreasing and

$$\frac{(a_2 - 1)\tan a_2 \sin^3 a_2}{\tan a_2 - a_2} > \Big(\frac{2\pi}{3} - 1\Big)\Big(\frac{\tan \frac{5\pi}{6}}{\tan \frac{5\pi}{6} - \frac{5\pi}{6}}\Big) \sin^3 \frac{5\pi}{6} = 0.0247....$$

For the first two terms that involve a_0 consider two subcases $a_0 < \frac{1}{2}$ and $a_0 \geq \frac{1}{2}$. When $a_0 < \frac{1}{2}$,

$$\frac{3\cos 3a_0}{4} > \frac{3\cos \frac{3}{2}}{4} = 0.0531...,$$

$$\Big|\frac{(a_0 - 1)\tan a_0 \sin^3 a_0}{\tan a_0 - a_0}\Big| < \Big(-\Big(\frac{1}{2} - 1\Big) \tan \frac{1}{2}\Big) f(0) = 0.8195...$$

where f is given in Lemma B.1 and $f(0) = \lim_{a \to 0} f(a) = 3$. Here one also used the fact that $|(a_0 - 1)\tan a_0| = -(a_0 - 1)\tan a_0$ is increasing. If $a_0 \geq \frac{1}{2}$, then

$$\frac{3\cos 3a_0}{4} > \frac{3\cos \frac{\pi}{2}}{4} = 0$$

and

$$\Big|\frac{(a_0 - 1)\tan a_0 \sin^3 a_0}{\tan a_0 - a_0}\Big| < \Big(-\Big(\frac{\pi}{6} - 1\Big) \tan \frac{\pi}{6}\Big) f\Big(\frac{1}{2}\Big) = 0.6546...,$$

again by $-(a_0 - 1)\tan a_0$ being increasing. Hence

$$t_{22} - t_{12} > \begin{cases} 0.0531... - 0.8195... + 0.7558... + 0.0247... \;=\; 0.0141... \;>\; 0 & \text{if} \quad a_0 < \frac{1}{2}, \\ 0 - 0.6546... + 0.7558... + 0.0247... \;=\; 0.1259... \;>\; 0 & \text{if} \quad a_0 \geq \frac{1}{2}. \end{cases}$$

Also

$$\begin{aligned} t_{22} + t_{12} &= \frac{3\cos 3a_0}{4} + \frac{(a_0 + 1)\tan a_0 \sin^3 a_0}{\tan a_0 - a_0} + \frac{(a_1 - 1)\tan a_1 \sin^3 a_1}{\tan a_1 - a_1} + \frac{(a_2 + 1)\tan a_2 \sin^3 a_2}{\tan a_2 - a_2} \\ &> 0 + 0 + 0 + \Big(\frac{2\pi}{3} + 1\Big) \frac{\tan \frac{5\pi}{6} \sin^3 \frac{5\pi}{6}}{\tan \frac{2\pi}{3} - \frac{5\pi}{6}} \;=\; 0.0513... \;>\; 0. \end{aligned}$$

In case (2) where $a_0 \in (\frac{\pi}{6}, \frac{\pi}{3})$, $a_1 \in (\frac{\pi}{3}, \frac{\pi}{2})$, and $a_2 \in (\frac{5\pi}{6}, \pi)$, one has

$$\begin{aligned} t_{22} - t_{12} &= \frac{3\cos 3a_0}{4} + \frac{(a_0-1)\tan a_0 \sin^3 a_0}{\tan a_0 - a_0} + \frac{(a_1+1)\tan a_1 \sin^3 a_1}{\tan a_1 - a_1} + \frac{(a_2-1)\tan a_2 \sin^3 a_2}{\tan a_2 - a_2} \\ &> \frac{3\cos\pi}{4} + \Big(\frac{\pi}{6} - 1\Big)\tan\frac{\pi}{6}\Big(\frac{\sin^3\frac{\pi}{6}}{\tan\frac{\pi}{6} - \frac{\pi}{6}}\Big) + \Big(\frac{\pi}{3}+1\Big)1 + 0 \\ &= -0.75 - 0.6396... + 2.0472... \;=\; 0.6576... \;>\; 0. \end{aligned}$$

Here for the second term one used the fact that $(a_0-1)\tan(a_0)$ is increasing for a_0 from $\frac{\pi}{6}$ where $(a_0-1)\tan a_0$ is -0.275 to $\frac{\pi}{3}$ where $(a_0-1)\tan a_0$ is 0.0817; for the third term note that $\frac{\tan a_1 \sin^3 a_1}{\tan a_1 - a_1}$ is decreasing and approaches 1 as $a_1 \to \frac{\pi}{2}$. Also

$$\begin{aligned} t_{22} + t_{12} &= \frac{3\cos 3a_0}{4} + \frac{(a_0+1)\tan a_0 \sin^3 a_0}{\tan a_0 - a_0} + \frac{(a_1-1)\tan a_1 \sin^3 a_1}{\tan a_1 - a_1} + \frac{(a_2+1)\tan a_2 \sin^3 a_2}{\tan a_2 - a_2} \\ &> \frac{3\cos\pi}{4} + \Big(\frac{\pi}{6} + 1\Big)\tan\frac{\pi}{6}\Big(\frac{\sin^3\frac{\pi}{3}}{\tan\frac{\pi}{3} - \frac{\pi}{3}}\Big) + 0 + 0 \\ &= -0.75 + 0.8343... \;=\; 0.0843... \;>\; 0. \end{aligned}$$

For case (3) where $a_0 = \frac{\pi}{6}$, again first take $a_0 \in (0, \frac{\pi}{6})$ and then pass the limit $a_0 \to \frac{\pi}{6}$ to deduce that $t_{22} \pm t_{12} > 0$. $\square$

Combining Lemmas B.1 and B.2 one sees that D^2P is positive definite at the exact double bubble.

Last we connect the setting here with the setting in the rest of the paper regarding P versus μ_0 and η. After P is treated as a function of μ_0 and η here, one sets up the equation

$$\frac{\partial P(\mu_0, \eta)}{\partial \mu_0} = 0, \tag{B.22}$$

and uses it to define μ_0 as a function of η implicitly. This can be done near the exact double bubble because

$$\frac{\partial^2 P}{\partial \mu_0^2}\Big|_{\mu_0 = r_0^2(a_0 - \cos a_0 \sin a_0),\ \eta = h} \neq 0 \tag{B.23}$$

by Lemma B.1. As seen after (B.9), (B.22) is just the condition

$$\frac{1}{\rho_1} - \frac{1}{\rho_2} = \frac{1}{\rho_0}, \tag{B.24}$$

precisely the one requirement, (3.4), in the setting of restrictedly perturbed double bubbles in Section 3 that is not implemented in this appendix before (B.22).

Once $\mu_0 = \mu_0(\eta)$ becomes a dependent variable, $P = P(\mu_0(\eta), \eta)$ is a function of η only, and

$$\begin{aligned} \frac{dP}{d\eta} &= \frac{\partial P}{\partial \mu_0}\frac{d\mu_0}{d\eta} + \frac{\partial P}{\partial \eta} \;=\; \frac{\partial P}{\partial \eta}, \\ \frac{d^2P}{d\eta^2} &= \frac{\partial^2 P}{\partial \mu_0 \partial \eta}\frac{d\mu_0}{d\eta} + \frac{\partial^2 P}{\partial \eta^2} \;=\; \frac{\partial^2 P}{\partial \mu_0 \partial \eta}\Big(-\frac{\frac{\partial^2 P}{\partial \mu_0 \partial \eta}}{\frac{\partial^2 P}{\partial \mu_0^2}}\Big) + \frac{\partial^2 P}{\partial \eta^2} \\ &= \frac{\frac{\partial^2 P}{\partial \mu_0^2}\frac{\partial^2 P}{\partial \eta^2} - (\frac{\partial^2 P}{\partial \mu_0 \partial \eta})^2}{\frac{\partial^2 P}{\partial \mu_0^2}}. \end{aligned}$$

Consequently by (B.10),

$$\frac{dP}{d\eta}\Big|_{\eta=h} = \frac{\partial P}{\partial \eta}\Big|_{\mu_0 = r_0^2(a_0 - \cos a_0 \sin a_0),\ \eta = h} = 2\sum_{i=0}^{2} \cos a_i = 0, \tag{B.25}$$

and by Lemmas B.1 and B.2,

$$\frac{d^2P}{d\eta^2}\Big|_{\eta=h} > 0. \tag{B.26}$$

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