Vassiliev-Gusarov Skein Modules of 3-Manifolds and Criteria for Periodicity of Knots

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Abstract. We show how to use invariants of knots discovered by Vassiliev [Va1] and Gusarov [Gu2] to analyze periodicity of knots. We analyze also Vassiliev-Gusarov invariants as an example of co-skein modules of 3-manifolds.

0 Introduction

Vassiliev-Gusarov skein modules (also called finite degree or finite type skein modules) are nice computable (at least for $S^3$) examples of general skein modules of 3-manifolds, see [P3], [Tu1] or [HP3]. Given a 3-manifold $M$ and a commutative ring with 1, we can associate with $M$ various “skein modules” based on knots and links which $M$ supports. Skein modules have their origin in the Alexander observation ([Al], 1928) that his polynomials of three links $L_+, L_-$ and $L_0$ in $S^3$ are linearly related (here $L_+$, $L_-$ and $L_0$ denote three links which are identical except in a small ball where they look as in Fig. 0.1). Conway rediscovered the Alexander observation and normalized the Alexander polynomial so that it satisfies the skein relation

$$\Delta_{L_+}(z) - \Delta_{L_-}(z) = z \Delta_{L_0}(z)$$

([Co1], 1969). In the late seventies Conway advocated the idea of considering the free $\mathbb{Z}[z]$-module over oriented links in an oriented 3-manifold and dividing it by the submodule generated by his skein relation [Co2] (cited in [Gi]) and [Co3] (cited in [Kal]). However there is no published account of the content of Conway's talks except when $S^3$ or its submanifolds are analyzed.

![Figure 0.1](image-url)
I introduced skein modules when thinking of colored generalizations of Jones type polynomials. I just had read a preliminary version of the Hoste and Kidwell paper [HK] and I realized that they really worked with knot theory in the solid torus where links were considered modulo the skein relation. I was familiar with Conway’s fundamental paper [Co1] and with [Gi],[Ka1], [LM] and [Li] but not with Conway’s talks [Co2], [Co3].

Skein modules were also introduced independently by Turaev [Tu1].

In the first part, we make a general overview of skein modules of 3-manifolds discussing in more details a possibility of torsion in skein modules.

In the second part, we describe Vassiliev-Gusarov invariants of degree $n$ of links in 3-manifolds (not necessary orientable) as elements of co-skein modules dual to certain skein modules (Vassiliev-Gusarov skein modules of degree $n$). We describe the V-G skein module as a completion and introduce a Hopf algebra structure on it. We analyze also a concept of an $m$-trivial knot.

In the third part, we describe how V-G invariants can be used for periodicity criteria for links and how some other criteria ([Tr1] and [P2]) can be deduced from it.

In the fourth part, we describe how strong are V-G invariants and propose various modifications of them.

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1 Skein Modules of 3-Manifolds

The definition of skein modules follows the standard (after Poincaré [Po]) procedure in modern mathematics of considering formal linear combinations of a large class of objects modulo some natural relations. In short skein modules are quotients of modules over isotopy classes of links in a 3-manifold by properly chosen local (skein) relations. In the choice of relations we are guided by polynomial invariants of links in $S^3$. In the second part we will explain how Vassiliev-Gusarov invariants can be described in this setting, but before we will give some older examples of skein modules (see [HP3] for a survey of skein modules till the summer of 1990).

Example 1.1 (see [P3], Example 2(b)) Let $M$ be a 3-manifold, $R$ a commutative ring with unit, $\mathcal{L}$ the set of all oriented links in $M$ up to ambient isotopy and $RL$ the free $R$-module generated by $\mathcal{L}$. Let $\mathcal{M}$ be the submodule of $RL$ generated by skein expressions $L_+ - L_-$ where $L_+$ and $L_-$ are two links in $M$ which differ only in a small ball where they look as in Fig.0.1. Notice that $\mathcal{M}$ need not to be oriented (or orientable) in order to define $\mathcal{M}$. The skein module $S(M;R,L_+ - L_-)$ is defined as $RL/\mathcal{M}$.

This skein module has a natural multiplication $L_1 \cdot L_2 = L_1 \cup L_2$ (i.e a disjoint sum - it does not depend on a relative position of $L_1$ with respect to $L_2$). The simplest description of our skein module uses the above multiplication. $S(M;R,L_+ - L_-)$ with the multiplication is a ring without a unit so it is convenient to adjoin it. Let $\mathcal{L}^{alg}$ denote the extension of $\mathcal{L}$ by the empty knot, $\emptyset$.

*According to Dieudonné [D]: “Before Poincaré the only similar construction of that type was the formation of “divisors” on an algebraic curve by Dedekind and Weber”.
and $S^{alg}(M; R, L_+ - L_-)$ denote the corresponding extension of $S(M; R, L_+ - L_-)$. Then $S^{alg}(M; R, L_+ - L_-)$ is an $R$-algebra with $0$ as its unit, and it is the $R$-algebra isomorphic to the symmetric tensor algebra over $R\hat{\pi}$, denoted by $S(R\hat{\pi})$, where $R\hat{\pi}$ is the free $R$-module generated by $\hat{\pi}$, the set of conjugacy classes of the fundamental group $\pi = \pi_1(M)$.

We devoted so much time to this easy example because it happen to be the Vassiliev-Gusarov skein module of degree 0 (see part 2).

**Example 1.2 ([P3, Tu1])** Let $M$ be an oriented 3-manifold, $R = \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$, $\mathcal{L}$ the set of all oriented links in $M$ up to ambient isotopy of $M$ and $\mathcal{M}_3$ the submodule of $R\mathcal{L}$ generated by the skein expressions $v^{-1}L_+ - vL_- - zL_0$. Then the third skein module of $M$ is defined to be:

$$S_3(M) = S_3(M; \mathbb{Z}[v^{\pm 1}, z^{\pm 1}], v^{-1}L_+ - vL_- - zL_0) = R\mathcal{L}/\mathcal{M}_3.$$

The third skein module is the straightforward generalization of the skein (Homfly) polynomial of links in $S^3$ and the existence of the skein polynomial has been generalized as follows:

**Theorem 1.3 ([P5])** Let $M = F \times I$ where $F$ is an oriented surface. Then

(a) $S_3(M)$ is a free $R$-module.

(b) We define a multiplication, $\cdot$, in the module by writing $L_1 \cdot L_2$ for the link obtained by placing $L_1$ over $L_2$ in $F \times I$. If we extend $S_3(M)$ by an empty knot, $\emptyset$, and add the relation $v^{-1}\emptyset - v\emptyset - T_1$ where $T_1$ is the unknot, we get the $R$ algebra $S^{alg}_3(M)$ which is $R$-module isomorphic to the symmetric tensor algebra $S(R\hat{\pi}^0)$, where $\hat{\pi}^0 = \hat{\pi} - \{1\}$.

Furthermore we can probably prove the following fact, but because some details should be still worked out and having in mind [JQ] we propose it as:

**Conjecture 1.4** If $M = M_1 \# M_2$ and $\pi_1(M) \neq 0$ then $S_3(M)$ has a torsion.

Conjecture 1.4 would contradict conjectures in [P5],[HP3]. We will discuss it with more details in the forthcoming paper [P4].

Theorem 1.3 and Conjecture 1.4 suggest the following conjecture:

**Conjecture 1.5** Let $M$ be a compact 3-manifold and $\check{M}$ the manifold obtained from $M$ by capping off all 2-sphere boundary components of $M$. Then $S_3(M)$ is a free module if and only if $\check{M}$ is an irreducible 3-manifold.

We should remark here that it was shown in [P3] that $S_3(S^1 \times S^2)$ has a torsion. Namely let the knot $K = S^1 \times \ast$ then $K \neq 0$ in $S_3(S^1 \times S^2)$ and $(v^{-1} - v)^2 - z^2)K = 0$ in $S_3(S^1 \times S^2)$. 
Example 1.6 If we modify slightly the definition of \( S_3(M) \) by considering a ring in which \( z \) is not invertible, say \( R' = \mathbb{Z}[z^{\pm 1}, z] \) in place of \( R = \mathbb{Z}[z^{\pm 1}, z^{\pm 1}] \), we obtain the skein module \( S'_3(M) = R'\mathcal{L}/(M_3 \cap R'\mathcal{L}) \). Let \( I \) be the ideal in \( R' \) generated by \( v^2 - 1 \) and \( z \), then one can analyze quotients of \( S' \) by powers of the ideal \( I \):

\[
S'_3(M)/(I^{k+1}S'_3(M)) = S'_3(M) \otimes R'/I^{k+1},
\]

which are still very useful but easier to compute - they are quotients of the tensor algebra over \( (R'/I^{k+1})\mathfrak{h}(M) \) (see [PP] and [Tr4]). In fact these quotients are exactly “Vassiliev-Gusarov parts” of our skein module. Notice that \( S'_3(M) \otimes (R'/I) \) is \( Z'-\text{isomorphic to } S(M; Z', L_+, L_-) \) from Example 1.1, where \( Z' = R'/I = \mathbb{Z}[v]/(v^2 - 1) \). We will discuss it in more general setting in the part 2.

2 Vassiliev-Gusarov Skein Modules

We will describe in this part the W-G skein modules. For \( M = S^3 \) our description is based mostly on [Gu3], [Lin2], [BN3] and [St1]. For some generalizations of V-G skein modules we refer to part 4.

Let \( M \) be any 3-manifold (orientable or not) and \( \mathcal{K} \) the set of (ambient) isotopy classes of oriented knots in \( M \). Let \( R \) be a commutative ring with 1 (it will be usually \( \mathbb{Z} \) or \( Q \)) and \( RK \) be the free \( R \) module over \( \mathcal{K} \). The \( m' \)th V-G skein module of \( M \) is obtained by dividing \( RK \) by the submodule, \( C_m+1 \), generated by V-G skein expressions. We will give two equivalent descriptions of generators of \( C_m \): first related to Gusarov work [Gu2, Gu3] and second (in Remark 2.1) related to Vassiliev approach [Val].

\( C_m \) is the submodule of \( RK \) defined by the following expressions: Consider \( 2^m \) knots in \( M \) which are the same except in one oriented ball (orientation agrees with that of \( M \) if \( M \) is oriented) in which they are \( 2m \)- braids of the form

\[
\sigma_1^e \sigma_3^{-1} ... \sigma_{2m-1}^{-1},
\]

where \( e_i = +1 \) or \( -1 \), compare Fig.2.1.

![Figure 2.1](image)

The V-G expression is the alternating sum of described \( 2^m \) knots (if two knots differ by an odd number of crossings then they have different sign in the sum). The \( m' \)th Vassiliev-Gusarov skein module \( W_m(M, R) \) is defined by \( W_m(M, R) = RK/C_{m+1} \).

We have:

\[
... \subset C_m \subset ... \subset C_1 \subset C_0 = RK
\]

and therefore we have also the sequence of epimorphisms \( \{1\} \leftarrow W_0 \leftarrow W_1 \leftarrow W_2 \leftarrow ... \leftarrow W_m \leftarrow ... \). We define the V-G skein module \( W_{\infty}(M, R) \) as the inverse limit \( W_{\infty}(M, R) = \lim_{\leftarrow} W_m(M, R) \). Equivalently, the V-G skein module is the completion \( \hat{RK} \) of \( RK \) with respect to the topology...
yielded by the sequence of descending submodules $C_i$. The kernel of the natural $R$-homomorphism $\theta : R\mathcal{K} \to R\mathcal{K}$ is equal to $\bigcap_{i=0}^{\infty} C_i$. We will discuss it with more details later.

If instead of knots, we repeat the above construction for links (starting from $\mathcal{L}$ and $C_m^{\mathcal{L}} \subset R\mathcal{L}$ we will get the stratified (by the number of components) V-G skein modules for links which we denote by $W_m^{\mathcal{L}}(M, R)$ and $W^{\mathcal{L}}(M, R)$.

**Remark 2.1** To be closer to the original Vassiliev definition (and to [BL] and [St1]) we can equivalently define the $m$th V-G skein module, $W_m(M, R)$, as follows: Let $K^{\mathcal{S}}$ denote the set of singular oriented knots in $M$ where we allow only immersions of $S^1$ with, possibly, double points, up to ambient isotopy; additionally for any double point we choose orientation of a small ball around it (if $M$ is oriented the chosen orientation of the ball agrees with that of $M$). In $R\mathcal{K}^{\mathcal{S}}$ we consider resolving singularity relations $\approx$: $K_{cr} = K_+ - K_-$; see Fig. 2.2.

Figure 2.2

$R\mathcal{K}^{\mathcal{S}} / \approx$ is obviously $R$-isomorphic to $R\mathcal{K}$. Let $C_m^{\mathcal{S}}$ be a submodule of $R\mathcal{K}^{\mathcal{S}} / \approx$ generated by immersed knots with $m$ double points. Of course $C_m^{\mathcal{S}}$ corresponds to $C_m$ under the isomorphism of $R\mathcal{K}^{\mathcal{S}} / \approx$ and $R\mathcal{K}$. Therefore $W_m(M, R) = (R\mathcal{K}^{\mathcal{S}} / \approx)/C_m^{\mathcal{S}}$.

A V-G invariant of degree $m$ of knots is defined as an element of the dual space $V^{m}(M, R) = W_m^{*}(M, R) = \text{Hom}(W_m(M, R), R)$ (sometimes it is defined as an element of $\text{Hom}_Z(W_m(M, Z), A)$, where $A$ is an abelian group) \(^1\). The space of invariants of finite degree is defined as the direct limit $V^{\infty}(M, R) = \text{lim}_\to V^{m}(M, R)$. V-G invariants of links are defined analogously. The space of V-G invariants of links of degree $m$ is denoted by $V^{\mathcal{L}}_m(M)$ and $V^{\mathcal{L}}_\infty(M) = \text{lim}_\to V^{\mathcal{L}}_m(M)$.

One of the reasons that V-G invariants are of great importance is that a group $W_m(S^3, Z)$ is finitely generated (more generally if $\pi_1(M)$ is finite then $W_m(M, R)$ is finitely generated) and there is an algorithm to find these groups explicitly. Furthermore for a given V-G invariant of knots in $S^3$ the value of the invariant can be computed in polynomial time with respect to the number of crossings of a knot diagram. The advantage of the Vassiliev approach to $C_m$ is that the above facts are almost immediately seen.

Now we concentrate for a moment on a topological module $R\mathcal{K}$ with topology yielded by the filtration $\{C_i\}_{i=0}^{\infty}$ of $R\mathcal{K}$ (to be in agreement with standard terminology we should have written $C_{-1}$ instead of $C_i$). $R\mathcal{K}$ has a standard structure

\(^1\) Sometimes one restricts the notion of invariants of degree $m$ to elements of $V^m(M, R) - V^{m-1}(M, R)$.
of a co-algebra with knots being group like elements. That is the comultiplication
\( \Delta : RK \rightarrow RK \otimes RK \) is given by \( \Delta(K) = K \otimes K \) and the counit \( \varepsilon : RK \rightarrow R \) is
given by \( \varepsilon(K) = 1 \).

**Lemma 2.2**

(a) \( RK \) is a filtered co-algebra, i.e. \( \Delta(C_n) \subseteq \Sigma_{i=0}^n C_i \otimes C_{n-i} \).

(b) \( RK \) is a topological co-algebra.

**Proof:** (a) Let \( K = K^{p_1, p_2, \ldots, p_n} \) be a singular knot with \( n \) double points \( p_1, p_2, \ldots, p_n \)
and orientation chosen around each double point. Then \( \Delta(K) \) has the following
form:

\[
\Delta(K) = \Sigma_{e_1, e_2, \ldots, e_n} K^{p_1, p_2, \ldots, p_n} \otimes K^{p_1, p_2, \ldots, p_n}_{e_1, e_2, \ldots, e_n-1} \in \Sigma_{i=0}^n C_i \otimes C_{n-i}
\]

where \( e_i \) is 1 or 0 and a sub-index 0, 1 or \( -1 \) under \( e_i \) indicates whether we deal
with a double point, a positive crossing or a negative crossing.

(b) follows easily from (a). \( \square \)

We will limit ourself for a while to the case of \( M = S^3 \). Knots in \( S^3 \) form
a commutative semigroup under the connected sum addition, with unknotted as the
neutral element. \( K \) is a unique decomposition semigroup so it can be embedded
in the group \( K^\pm \) by adding elements of the form \( g^{-1} \) to \( K \). Consider now the
semigroup ring \( RK \) (respectively group ring \( RK^\pm \)). \( RK \) has a natural structure of a
commutative and co-commutative bialgebra (respectively \( RK^\pm \) is a Hopf algebra).
Namely all knots are group like elements that is the comultiplication \( \Delta : RK \rightarrow
RK \otimes RK \) is defined by \( \Delta(K) = K \otimes K \), counit \( \varepsilon : RK \rightarrow R \) is defined by \( \varepsilon(K) = 1 \).
Antipodism, in the case of \( RK^\pm \), \( S : RK^\pm \rightarrow RK^\pm \) is given by \( S(K) = K^{-1} \).

**Lemma 2.3** \( RK \) is a topological bi-algebra.

**Proof:** It follows, essentially, from the fact that if \( P_1 \in C_{m_1} \), then \( P_1 \# P_2 \in C_{m_1+m_2} \)
and \( \Delta P_1 \in \Sigma_{i=0}^n C_i \otimes C_{m_1-i} \). \( \square \)

The bi-algebra structure on \( RK \) extends/descends to its completion \( \hat{RK} = W_{\infty}(S^3, R) \) and because for any knot \( K \), \( (K-1) \in C_1 \) therefore \( K \) is invertible
in the completion \( (K^{-1} = 1 - (K-1)^2 - (K-1)^3 + \ldots) \) and therefore \( W_{\infty}(S^3, R) \) is a Hopf algebra \( (S(K) \) is the inverse of \( K )) \).

The associated graded \( R \)-Hopf algebra is defined by \( W_{\infty} = \bigoplus_{i=0}^\infty W_{(i)} \)
where \( W_{(i)} = W_i/W_{i+1} \). It is a fundamental open question whether we loose some
information about knots when going from \( W_{\infty} \) to \( W_{(\infty)} \) and whether modules \( W_i \)
and \( W_{(i)} \) are torsion free (it is the case for \( i \leq 7 \) [Gu3, BN3, St1]).

If we consider \( R \) with discrete topology then the set of continuous elements
of \( \text{Hom}(W_{\infty}, R) \) is exactly the space of Vassiliev-Gusarov invariants of finite degree
\( W^\infty \). This space inherits the Hopf algebra structure from \( W_{\infty} \). This Hopf algebra
of V-G invariants was studied in [Gu3, Lin2, Lin3] where it was observed that
because it is commutative and co-commutative it is the symmetric tensor algebra
over primitive elements (i.e. functions additive under the connected sum).

The natural generalization of the semigroup \( K, \# \) is the semigroup \( (M, K'), \# \)
of oriented 3-manifolds with embedded knots and with multiplication given by the
connected sum of pairs. The $R$-free module generated by $(M, K)$, denoted by $R(M, K)$, is the $R$-(semigroup) algebra (in fact bialgebra). It has the filtration by ideals $C_m(M, K)$ generated by $m$-singular knots. As before, we can take the completion, $\hat{R}(M, K)$, with respect to the topology given by this filtration and treat it as the $R\mathcal{K}(S^3)$ algebra. We leave the more detailed investigation of $R(M, K)$ to a future paper.

From the fact that knots are invertible in the completion $W_\infty(S^3, R)$ it follows that:

**Corollary 2.4** For any $m$, knots are invertible in the $V$-$G$ skein module $W_m$.

In fact Gussarov proved a stronger result from which it follows that knots form a group in any $W_m$.

**Theorem 2.5** For any $m$ and any knot $K_1$ in $K$ there is a knot $K_2$ such that $K_1 \# K_2$ is equal to the unknot in $W_m(S^3, R)$.

To prove Theorem 2.5, Gussarov is using the concept of an $m$-trivial knot. This concept was introduced by Ohyama [Oh] and Yamamoto [Ya]:

**Definition 2.6** (compare [Oh, Gu3, Ya]) Let $K$ be a knot in a 3-manifold $M$. Consider $2^{m+1}$ knots which are the same except in $m + 1$ small 3-balls, where they differ, possibly, by the mirror image (i.e. orientation reversing involution of a 3-ball which keeps the part of the knot lying on the boundary of the ball pointwise fixed). If $K$ is one of these knots and all other are trivial knots then we say that $K$ is an $m$-trivial knot (see Fig 2.4).

The importance, from our point of view, of $m$-trivial knots lies in the following observation:

**Lemma 2.7** (Gussarov [Gu3]) If $K$ is an $m$-trivial knot and $T_1$ denotes the unknot then $(K - T_1) \in C_{m+1}(M, R)$ or equivalently $K = T_1$ in $W_m(M, R)$.

**Proof:** (Sketch) The main observation, from which the result follows quickly, is that if $c$ and $c'$ are in $C_m$ and they are identical except in a small ball where they look as in Fig. 2.3, then $(c - c') \in C_{m+1}$.

![Figure 2.3](image-url)
Yamamoto [Ya] proved that any knot in $S^3$ is 1-trivial. It was also proved independently by Gusarov [Gu].

Ohyama [Oh] proved an existence, for any $m$, of an $m$-trivial knot. Ohyama mentions also a construction of $m$-trivial knots by Taniyama.\(^5\) Namely Taniyama [Ta] showed that if $K$ is $m$-trivial then its untwisted double is $(m+1)$-trivial (Fig.2.4 illustrates the fact that the untwisted double of a trefoil knot is a 2-trivial knot).

![Figure 2.4](image)

On a more general level, examples of Taniyama can be summarized as follows (compare also [Og, OO]):

**Theorem 2.8** Let $Q$ be an $(s-1)$-trivial knot in a solid torus, $S^1 \times D^2$, such that $Q$ is the unknot in $S^3$ (for the standard embedding of $S^1 \times D^2$ in $S^3$ which determines a longitude of $\theta(S^1 \times D^2)$). Let $K$ be an $(m-1)$-trivial knot in a 3-manifold $M$ (in particular $K$ is homotopic to 0 in $M$ so has a preferred framing in $M$). Let $\text{sat}_Q K$ be the satellite of $K$ with pattern $Q$. Then $\text{sat}_Q K$ is $(s + m - 1)$-trivial in $M$.

\(^1\)The 1-triviality of any knot can be proved quickly using ideas of Stanford [St2]: Let $K$ be a knot and $\gamma_K$ a braid representing $K$. We can use Markov moves to get a new braid $\gamma'_K$ representing $K$ such that $\gamma'_K = \alpha_1 \alpha_2$ and $\alpha_1$ is a pure braid with the sum of exponents equal to 0 and the closure of $\alpha_2$ is the unknot. $\alpha_1$ is in the commutator subgroup of the braid group so it is a product of commutants. Now we can easily find two families of crossings in $\alpha_1$ so that all changes along them (we have three possibilities) produce the unit braid. For example if $\alpha_1 = \beta_1 \beta_2 \beta_3^{-1} \beta_2^{-1}$, then we choose, as the first family, crossings of $\beta_3$ whose change trivialize $\beta_3$ together with corresponding crossings of $\beta_3^{-1}$; the second group is made from analogous crossings for $\beta_2$ and $\beta_2^{-1}$. If we use the same two groups of crossings for the closure of $\gamma'_K = \alpha_1 \alpha_2$ we find that it is a 1-trivial knot.

\(^5\)Stanford [St2] found infinitely many simple (non-satellite) $m$-trivial knots using the lower central series of pure braid groups.
Proof: (Outline) To find $(m + s)$ 3-balls from Definition 2.6 we should properly use images in $M$ of $s$ 3-balls in the solid torus and $m$ 3-balls in $M$ which are used to show that $Q$ (respectively $K$) is $s - 1$ (resp. $m - 1$ trivial). The delicate part is to distribute properly twists on $sat_Q K$ so they behave well under mirror images in balls. Essentially each mirrored crossing should have associated to it a twist of the opposite sign than the sign of the crossing. In this way, if $B^3$ is one of the balls chosen for $K$ in $M$ then after changing $K$ and $sat_Q K$ in $B^3$ we not only trivialize $K$ in $M$ but also $sat_Q K$; see Fig 2.4. □

If $Wh$ denotes de Whitehead knot in the solid torus (i.e. $sat_{Wh} K$ is the untwisted Whitehead double of $K$) then the Taniyama examples give, in the context of the V-G skein modules, the Lin’s observation that if $K$ is $m$-trivial then $sat_{Wh} K - T_1 \in C_{m+1}(S^3, R)$. One can formulate it more generally as follows (compare[Lin2, Lin3]):

**Theorem 2.9** Let $Sat : RK(S^1 \times D^2) \otimes RK(M) \to RK(M)$ be the linear extension of $sat$ (elements of $K(M)$ should be framed and $\partial(S^1 \times D^2)$ has a chosen longitude so $sat$ is well defined). Then:

(a) $Sat(C_s \otimes C_m) \subset C_{s+m}$.

(b) Assume that knots $Q$ and $K$ satisfy: $Q = T_1$ in $W_{s-1}(S^1 \times D^2)$ and $K = T_1$ in $W_{m-1}(M)$; $m \geq 1$, in particular $K$ is homotopic to the identity in $M$ so it has the natural framing. Then: $sat_Q K = sat_Q T_1$ in $W_{m+s-1}(M)$.

(c) In particular if $sat_Q T_1$ is the trivial knot in $M$ then $(sat_Q K - T_1) \in C_{m+s}(M)$.

Notice that if $Q$ lies in a ball in $S^1 \times D^2$ then $Sat_Q C_1(M, R) = 0$ in $RK(M)$. Theorem 2.9 for $s = 0$ (we use then the convention that $W_{-1} = RK/C_0 = 1$) should be compared with Theorem 3.5.

Proof:(Outline)

(a) We use the observation described after Lemma 2.7, and Theorem 2.9(a) follows almost immediately.

(b) Let $Q = T_1 + c_s$ and $K = T_1 + c_m$. Thus $sat_Q K = sat_Q T_1 + Sat_T c_m + Sat_c c_m = sat_Q T_1 + Sat_c c_m$. Therefore, by (a): $(sat_Q K - sat_Q T_1) \in C_{s+m}$ and (b) follows.

□

Gusarov extended constructions of [Oh, Ta, OO, Og] and found the remarkable fact:

**Lemma 2.10 (Gusarov [Gu3])** For any $m$ and a knot $K_1$ there is a knot $K_2$ such that $K_1 \# K_2$ is $m$-trivial; in particular knots form a group in $W_i(S^3, R)$.

We refer to [Gu3] for the proof of Lemma 2.10.

**Corollary 2.11** For any knot $K$, its inverse in the completion, $\overline{RK}$, is an infinite connected some of knots. More precisely, there is a sequence of knots: $K_1, K_2, K_3, \ldots$ such that $K_1, K_1 \# K_2, K_1 \# K_2 \# K_3, \ldots$ is a Cauchy sequence in $RK$ converging to $K^{-1}$ in $\overline{RK}$. 

Proof: It follows from Theorem 2.4 and the fact that $K$ is invertible in $\hat{\mathbb{R}K}$. $\square$

The main problem related to Vassiliev-Gusarov invariants of knots in $S^3$ is whether they classify knots. In our notation it is the question whether $\bigcap_{i=1}^{\infty} C_i = \{0\}$ or equivalently whether the topological module $\mathbb{R}K$ is a Hausdorff space. There are two competing conjectures:

Conjecture 2.12 (Vassiliev) \textit{V-G invariants classify oriented knots.}

Conjecture 2.13

(a) (Lin [Lin2]). Let $\text{or}(K)$ denote the knot obtained from $K$ by changing its orientation then no V-G invariant distinguishes $K$ from $\text{or}(K)$.

(b) (Bar-Natan [BN3]). V-G invariants are precisely as powerful as the skein (Homfly) and Kauffman polynomials of knots and all of their cablings (in particular (a) holds).

(c) The $R$-algebra $\bigcap C_i$ is the smallest subalgebra of $\mathbb{R}K$ containing expressions $K - \text{or}(K)$ and closed under operation of taking satellites.

The related question concerning $m$-trivial knots was asked before in [Oh]:

Question 2.14 ([Oh]) \textit{Is there a nontrivial knot which is $m$-trivial for any $m$.}

3 Periodic Links

A link $L$ in $S^3$ is called $n$-periodic if there is a $\mathbb{Z}_n$ action on $S^3$ with a circle as a fixed point set, which maps $L$ onto itself, and such that $L$ is disjoint from the fixed point set. Furthermore if $L$ is oriented, we assume that any element of $\mathbb{Z}_n$ preserves the orientation of $L$. By the positive solution to the Smith Conjecture [Sm, Th], the fixed point set of our $\mathbb{Z}_n$ action on $S^3$ is an unknotted circle and the action is conjugated to an orthogonal one. Therefore if we write $S^3$ as $R^3 \cup \infty$ we can assume that a fixed point set is a vertical axis with $\infty$ and our $\mathbb{Z}_n$-action is generated by the rotation $\phi$ given by the formula: $\phi(z, t) = (z^{2\pi i/n}, t)$ where $R^3 = \{z, t : z \text{ complex and } t \text{ real numbers}\}$. Each $n$-periodic link can be represented by a $\phi$-invariant diagram (also denoted by $L$); that is $\phi(L) = L$.

Theorem 3.1 Let $L$ be an $r$-periodic link in $S^3$ where $r$ is a prime number, then

(a) $L = \bar{L} \mod r$ in $W_n$ for $n < r$ where $\bar{L}$ is the mirror image of $L$.

(b) if $L$ is a knot of linking number $k$ with respect to the axis of rotation then $L = T(r, k) \mod r$ in $W_n$ for $n < r$, where $T(r, k)$ is the $(r, k)$ torus knot. In particular an $n$'th degree V-G invariant of $L$ is the same as that of $T(r, k)$ modulo $r$.

\footnote{More precisely we ask whether there are different knots $K_1$ and $K_2$ such that $(K_1 - K_2) \in \bigcap_{i=1}^{\infty} C_i$.}
Proof: Let $D_L$ denote an $r$-periodic diagram of $L$ and $p_1, p_2 = \phi(p_1), ..., p_r = \phi^{r-1}(p_1)$ be the orbit of a crossing $p_1$ under the $Z_r$-action. Let $D_{e_1}^{\varepsilon_1} ... D_{e_m}^{\varepsilon_m}$, where $e_1 = 1, -1$ or $0$ denote the link obtained from $D_L$ by changing it at any $p_i$ to positive or negative crossing or to a double point (depending on $e_i$). The link $D_{0}^{0} ... D_{0}^{0}$ has $r$ double points so it represents $0$ in $W_n$. Let us build the binary computational resolving tree of this singular link using all $r$ double points $p_i$ (any branching is using a relation of the type $L_{or} = L_+ - L_-$). Our resolving tree has $2^r$ leaves and the group $Z_r$ acts on the set of the leaves. There are exactly two fixed points of the action: the first when all $p_i$'s are changed to positive crossings and the second when all $p_i$'s are changed to negative crossings. All other orbits have the number of elements divisible by $r$ ($r$ is a prime number) and elements of a given orbit are isotopic one to another. Therefore, modulo $r$, we can omit these leaves. We have proved so far that we can change whole orbit of a crossing to its mirror image without changing the class in $W_n$ modulo $r$. Using this several times we can go from the given link to its mirror image, or in the case of knots, to the $T(r, k)$ torus knot. □

Example 3.2 A trefoil knot has no prime period $r \geq 5$.

Proof: $W_3(S^3, Z)$ is freely generated by $T_1$ (unknot), $T_{2,3}$ (right handed trefoil knot) and $T(4_1)$ (the figure eight knot). Furthermore the mirror image $\overline{T_{2,3}}$ of $T_{2,3}$ is equal to $-T_{2,3} + 4T_1 - 2T(4_1)$ in $W_3(S^3, Z)$. Therefore $T_{2,3} \neq \overline{T_{2,3}}$ mod $r$ in $W_3(S^3, Z)$ for $r \geq 5$ and a trefoil is not $r$-periodic. □

Remark 3.3 A $t_{2r}$ move on $L$ (see Fig. 3.1(a)) (respectively a $\overline{t_{2r}}$ move on $L$; see Fig. 3.1(b)) preserves V-G invariant of $n$-th degree modulo $r$ but can destroy $r$-periodicity.

Proof that a $t_{2r}$ move preserves an $n$'th degree V-G-invariant modulo $r$.

We can think of a $t_{2r}$ move as changing $r$ "consecutive" negative crossings into $r$ consecutive positive crossings, see Fig. 3.1(c).

If we build a binary resolving computational tree for $r$ positive crossings we get, as before, $2^r$ leaves but all except two reduce modulo $r$. Therefore we get: $L - t_{2r}(L) \equiv L_{(r \text{ double points})} \mod r$ in $RK$ (see Fig.3.2). $L_{(r \text{ double points})} = 0$ in $W_n (n < r)$ so we are done.
Figure 3.1

Figure 3.2

Similar proof works for a $t_{2k}$ move.
Remark 3.4 One would try to extend the applicability of Theorem 3.1 by using $r$-periodic satellites around a periodic knot $K$ (as for example in [Tr3]) and applying Theorem 3.1 to them. However one cannot strengthen Theorem 3.1 in such a straightforward manner. Namely we have the following result of Bar-Natan and Lin (which, for knots, is the case $s=0$ of Theorem 2.9):

Theorem 3.5 If $g$ is a $V$-$G$ invariant of degree $m$ and $\text{sat}_Q : K \rightarrow \mathcal{L}$ denotes the operation of forming satellites with the pattern $Q$, then $g(\text{sat}_Q)$ is a $V$-$G$ invariant of degree $m$. Equivalently if we extend $\text{sat}_Q$ to the $R$-modules homomorphism $\text{sat}_Q : RK \rightarrow RL$, then $\text{sat}_Q(C_m) \subset C_m^L$.

Corollary 3.6 Let $\text{sat}_Q L$ be a satellite of an $r$-periodic link $L$. Then the Jones, skein and Kauffman polynomials satisfy (compare [P2, Tr1]):

(a) $V_{\text{sat}_Q L}(t) \equiv V_{\text{sat}_Q L}(t) \mod (r, (t-1)^r)$ in $Z[t^{\pm 1/2}]$, 
(b) $P_{\text{sat}_Q L}(v, z) \equiv P_{\text{sat}_Q L}(v, z) \mod (r, (v^2-1, z)^r)$ in $Z[v^{\pm 1}, z, (v-1-v)/z]$, 
(c) $F_{\text{sat}_Q L}(a, z) \equiv F_{\text{sat}_Q L}(a, z) \mod (r, (a^2+1, z)^r)$ in $Z[a^{\pm 1}, z, (a+a^{-1})/z]$.

Proof: We have to combine Theorems 3.1, 3.5 and 4.1 (in particular Examples 4.2). □

The result similar to Theorem 3.1 and Corollary 3.6, holds also for $r^k$-periodic links where $r$ is a prime number.

My interest in periodic knots is stimulated, to great extend, by desire to understand relations between skein modules of coverings. Our analysis of periodic links gives the following result.

Let $p : \tilde{M} \rightarrow M$ be an $n$-fold covering of 3-manifolds. $p$ induces the transfer map $\tau : RL(M) \rightarrow RL(\tilde{M})$ given by $\tau(L) = p^{-1}(L)$.

Theorem 3.7 Let $Z_r$ acts freely on $\tilde{M}$ (yielding the regular covering $p : \tilde{M} \rightarrow M = \tilde{M}/Z_r$) in such a way that a generator of the action is isotopic to the identity (a Seifert fibered manifold gives the general example but $M = S^3, S^1 \times D^2$ or $S^1 \times S^2$, are still of considerable interest).

Then the transfer $\tau$ descends to the $R$-homomorphism of $V$-$G$ skein modules (modulo $r$): $\tau_{m-1} : W_{m-1}(M, Z_r) \rightarrow W_{m-1}(\tilde{M}, Z_r)$.

Proof: It suffices to show that $\tau(C_m)(M, Z_r) \subset C_{rm}(\tilde{M}, Z_r)$. Consider a singular link $c_m \in C_m$ with $m$ double points $p_1, p_2, ..., p_m$. $c_m = \epsilon_1, ..., \epsilon_m(K_{p_1}, ..., K_{p_m})$ where $\epsilon_i = 1$ or $-1$ and it denotes the change of $p_i$ to positive or negative crossing. Because $p^{-1}(c_m)$ has $rm$ double points (so $p^{-1}(c_m) \in C_{rm}$) therefore our goal is to show that $(p^{-1}(c_m) - \tau(c_m)) \in C_{rm}$. In fact we will show that $p^{-1}(c_m) \equiv \tau(c_m)$ modulo $r$ in $ZL(\tilde{M})$. The method is similar to that used in the proof of Theorem 3.1 so we give here only the outline:

Let $p^{-1}(p_1) = \{p_1, p_2, ..., p_r\}$ and we build the binary computational resolving tree using all $r$ double points $p_i$ (any branching is using a relation of the type $L_c = L_+ - L_-$). Our resolving tree has $2^r$ leaves and the group $Z_r$ acts on the
set of the leaves. There are exactly two fixed points of the action: the first when all \( p_i \)'s are changed to positive crossings and the second when all \( p_i \)'s are changed to negative crossings. All other orbits have the number of elements divisible by \( r \) (\( r \) is a prime number) and elements of a given orbit are isotopic one to another by the assumption on the \( Z_r \)-action. Therefore, modulo \( r \), we can omit these leaves. Applying our method to any double point \( p_j \), we obtain that modulo \( r \):

\[
p^{-1}(c_m) \equiv \Sigma c_{\epsilon_1,\ldots,\epsilon_r}(\prod_{i=1}^{m} p_i) p^{-1}(K_{\epsilon_1,\ldots,\epsilon_r}) = \tau(c_m)
\]

as required. \( \square \)

4 Modifications of V-G Skein Modules

We propose in this part some other relations which lead to V-G types of skein modules. First however we formulate in our language the Lin [Lin1] and Bar-Natan [BN3] theorem (see also [Bi]) that Jones type invariants of links (e.g. the skein and Kauffman polynomials of a link and its satellites) can be approximated by V-G invariants (for the Conway polynomial it was showed by Bar-Natan [BN1] and in different language by Ohyama[Oh] and for skein and Kauffman polynomials by Birman and Lin [BL] and Gusarov [Gu2]). We will demonstrate their result (for any 3-manifold) using skein modules. Our proof is in the spirit suggested by Problem 6 of Viro [Vi].

Theorem 4.1 Let \( S(M, R) \) be a skein module \( R \mathcal{L}/\mathcal{M} \) where \( \mathcal{L} \) is the set of ambient isotopy classes of oriented links in \( M \) and \( \mathcal{M} \) is a submodule of \( R \mathcal{L} \) such that for an ideal \( I \) of \( R \)

\[
C_{m+1}^\mathcal{L}(M, R)/\mathcal{M} \subset TC_{m+1}^\mathcal{L}(M, R)/\mathcal{M},
\]

then there is an epimorphism \( \phi_m : W_{m}(M, R) \to S(M, R) \otimes (R/I^{m+1}); \) equivalently \( C_{m+1}^\mathcal{L}(M, R) \) is equal to 0 in \( S(M, R) \otimes (R/I^{m+1}) \). Epimorphisms \( \phi_m \) yield the epimorphism of completions \( \phi : W_\infty(M) \to S(M) = \lim_{\to} S(M) \otimes R/I^{m} \), where the completion of \( S(M) \) is taken with respect to \( I \)-adic topology.

Proof: Our conditions give immediately that \( C_{m+1}^\mathcal{L}(M, R)/\mathcal{M} \subset I^{m+1} \mathcal{R}/\mathcal{M} \) and therefore \( C_{m+1}^\mathcal{L}(M, R) \) is equal to 0 in \( S(M, R) \otimes R/I^{m+1} \) as required. \( \square \)

Example 4.2

(a) (Jones polynomial) \( V(t) \mod (t - 1)^{m+1} \) is a V-G invariant of degree \( m \), that is, it defines a function from \( W_{m}(S^3, \mathbb{Z}) \) to \( \mathbb{Z}[t^{\pm 1/2}]/(t - 1)^{m+1} \). From the topological point of view, we can say that \( V(t) \) is a continuous function from \( Z \mathcal{L} \) with the topology yielded by filtration \( C_{i}^\mathcal{L} \) to \( Z[t^{\pm 1/2}] \) with \( (t - 1) \)-adic topology (and we can extend this function to completions).

(b) (skein (Homfly) polynomial) \( P_L(v, z) \mod (I^{m+1}) \) is a V-G invariant of degree \( m \), that is it defines a function from \( W_{m}(S^3, \mathbb{Z}) \) to \( \mathbb{Z}[v^{\pm 1}, z, (v^{-1} - v)/z]/I^{m+1} \) where \( I = (v^2 - 1, z) \) is an ideal in \( \mathbb{Z}[v^{\pm 1}, z, (v^{-1} - v)/z] \).
(c) (Kauffman polynomial) $F_L(a, z) \mod (I^{m+1})$ is a V-G invariant of degree $m$, that is it defines a function from $W_m(S^3, Z)$ to $Z[a^{\pm 1}, z, (a + a^{-1})/z]/I^{m+1}$, where $I = (a^2 + 1, z)$ is an ideal in $Z[a^{\pm 1}, z, (a + a^{-1})/z]$.

Proof of (a). It suffices to construct a natural epimorphism from $W_m(S^3, Z[t^{\pm 1}])$ to $Z[t^{\pm 1}]/(t-1)^{m+1}$. Let $L_+$ and $L_-$ be singular links with $m$ double points which are identical, except a small ball where they look as in Figure 0.1. Expressions $L_+ - L_-$ generate $C_{m+1}$ and $L_+ + L_- \equiv t^{-1}L_+ - tL_- - (t^{1/2} - t^{-1/2})L_0 \mod (t-1)C_m$. Therefore

$$C_m \equiv (t^{-1}D_+ - tD_- - (t^{1/2} - t^{-1/2})D_0)$$

$$C_{m+1} \equiv ((t^{-1}D_+ - tD_- - (t^{1/2} - t^{-1/2})D_0)$$

and we can use Theorem 4.1. Proofs of (b) and (c) are analogous.

Note here that there are equalities related to the fact that $W_2(S^3, Z)$ is 2-dimensional. Namely if, following the Jones’ notation [Jo], $W_K(t) = (V_K(t) - 1)/(1 - t)$, then $W_K(1) = a_2$, where $a_2$ is the second coefficient of the Conway polynomial of the knot $K$ [Mur]. Similarly for the skein polynomial (here the language of the partial derivatives is the most useful):

$$\frac{\partial F_K(a, z)}{\partial a}(1, 0) = 2a_2 \quad \frac{\partial F_K(a, z)}{\partial z}(1, 0) = -8a_2.$$  For the Kauffman polynomial we get:

$$\frac{\partial F_K(a, z)}{\partial a}(i, 0) = 0, \quad \frac{\partial F_K(a, z)}{\partial z}(i, 0) = 2a_2 \quad \frac{\partial F_K(a, z)}{\partial a^2}(i, 0) = 8a_2.$$

Example 4.3 Consider the skein module $S^3(M)$ described in Example 1.6. Then there is the natural map $\phi_m : W_m(M, Z) \rightarrow S^3(M) \otimes (Z[v^{\pm 1}, z]/(v^3 - 1, z)^{m+1})$ and the maps $\phi_m$’s yield the continuous map of completions, $\phi : W_\infty(M, Z) \rightarrow S^3(M)$, where $S^3(M)$ is equipped with $(v^2 - 1, z)$-adic topology.

Proof: We proceed as in the proof of Examples 4.2(a). Let $L_+$ and $L_-$ be singular links with $m$ double points which are identical, except a small ball where they look as on Figure 0.1. $L_+ + L_- \equiv vL_+ - vL_- - zL_0$ modulo $ICM(M, Z)[v^{\pm 1}, z]$. The existence of $\phi_m$ follows from Theorem 4.1 as in Examples 4.2. □

Theorem 4.1 demonstrates that $W_\infty(M, R)$ is a very powerful skein module and its “finite” parts can be often computed. It does not mean that it is not convenient sometimes to resolve singularity differently then putting $L_m = L_+ - L_-$ (in a mild way it is already done in [BN3] by considering framed knots). Essentially, whenever a definition of relations in a skein module involves only oriented 2-tangles, we can change relations of the skein module into relations resolving singularity. We will illustrate it by giving three crucial, I believe, examples.

Example 4.4 Let $M$ be an oriented 3-manifold, $R = Z[v^{\pm 1}, z]$, $L$ the set of ambient isotopy classes of links in $M$ and $RL$ the free $R$-module generated by $L$. We will interpret singular links in $M$ as elements of $RL$ using the following, resolving singularity, equation: $L_m = v^{-1}L_+ - vL_- - zL_0$. Let $C_n^L(M)$ be the submodule of $RL$ generated by singular links with $m$ double points. We define the skein module $W_m^L(M) = RL/C_n^{L+1}(M)$. Finally $W_m^L(M) = \lim_{m \rightarrow \infty} W_m^L(M)$ or equivalently it is the completion of $RL$ with respect to topology yielded by the filtration $\{C_n^L(M)\}$.
The advantage of $W^{\pm k}(M)$ over $W_\infty(M)$ is not that it distinguishes more knots (it should not - essentially because of Theorem 4.1) but because it distinguishes them much quicker. For example $W_i(S^3)$ contains whole skein (Homfly) polynomial and much more. Therefore it has potential of giving new information on the classical knot invariants.

Example 4.5 (Homotopy skein) Let $M$ be an oriented 3-manifold, $R = \mathbb{Z}[v^\pm 1, z]$, $L$ the set of ambient isotopy classes of links in $M$ and $RL$ the free $R$-module generated by $L$. We will interpret singular links in $M$ as elements of $RL$ using the following, resolving singularity, equation:

$$L_{cr} = \begin{cases} v^{-1}L_+ - vL_- & \text{in the case of a selfcrossing} \\ v^{-1}L_+ - vL_- - zL_0 & \text{otherwise} \end{cases}$$

The meaning of a selfcrossing may not be clear for singular links (and in fact there are several choices). Here we choose: a crossing is a selfcrossing if it is a selfcrossing after smoothing all singular points of the link.

Let $C^h_m(M)$ be the submodule of $RL$ generated by singular links with $m$ double points. We define the skein module $W^h_m(M) = RL/C^h_{m+1}(M)$. For $m = 0$ we get the homotopy skein module [HP², P², Tu²].

Example 4.6 (Anti-homotopy skein) Let $M$ be an oriented 3-manifold, $R = \mathbb{Z}[v^\pm 1, z]$, $L$ the set of ambient isotopy classes of links in $M$ and $RL$ the free $R$-module generated by $L$. We will interpret singular links in $M$ as elements of $RL$ using the following, resolving singularity, equation:

$$L_{cr} = \begin{cases} v^{-1}L_+ - vL_- - zL_0 & \text{in the case of a selfcrossing} \\ v^{-1}L_+ - vL_- & \text{otherwise} \end{cases}$$

Let $C^{ah}_m(M)$ be the submodule of $RL$ generated by singular links with $m$ double points. We define the skein module $W^{ah}_m(M) = RL/C^{ah}_{m+1}(M)$. For $m = 0$ we get the anti-homotopy skein module [HPS].

5 References


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[HK] Hoste, J.-Kidwell, W., Dichromatic link invariants, Trans. A.M.S. 321 (1), (1990), 197-229; see also the preliminary version of this paper: Invariants of colored links, Preprint (March 1987).


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