The Neptune-Pluto Orbital Resonance, from an Elementary Standpoint

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Introduction

The ninth planet Pluto orbits the Sun in 248 years. Its orbit is highly eccentric (e=.248), and at the current time (1993) it is near its perihelion, its closest approach to the Sun. In fact, it is actually slightly closer currently than the nominal eighth planet, Neptune, which orbits the Sun every 164 years in a nearly circular orbit (e=.01). At the current perihelion, Neptune is about 90° behind (in the counter-clockwise sense) Pluto's position, so Pluto is in no danger of collision or near-miss with it. What is remarkable, and was only discovered in 1965 ([??65]), is that at every perihelion of Pluto, Neptune will occupy a position either 90° behind or 90° ahead of it. This is basically true because the orbital period of Pluto is 1.668 times that of Neptune, and Pluto is probably in a 3/2 resonance with Neptune. Whether this resonance is stable over a long time is what this present paper explores.

After the discovery of Pluto by Tombaugh in 1930, its orbit was computed and found to be remarkably different from those of the other known planets. Its eccentricity is very high, and it is tilted from that of Neptune by about 15°. (Neptune's orbit is within 2° of the plane of the other planets.) At perihelion, Pluto's distance from the Sun is .753 times its semi-major axis. Since Neptune's near-circular orbit has a semi-major axis of .763 times that of Pluto, it was obvious very early on that the possibility of collision or near-collision existed between Neptune and Pluto. None of the other planets have intersecting orbits, however, and the long-term perturbation calculation methods used since Lagrange are not valid in such a case. The orbits of comets do frequently intersect those of planets, and methods have been created to compute the perturbation of comets by planets, but those methods are good for only a few (tens of) orbits into the future, and are also not applicable to the Neptune-Pluto system for long eons.

The paper [Hori & Giacaglia, 197?] computed the perturbations of the Neptune-Pluto system using Hamiltonian dynamics. It confirmed the stability of the current resonance, and confirmed the 20,000 year period first found by [Cohen & Hubbard, 1965]. Its methods are very advanced, though, and it will be the purpose of this paper to investigate the N-P resonance using only elementary vector calculus.

Fundamental Equations of Motion

Encke's Method of Special Perturbations [J.M.A. Danby, *Fundamentals of Celestial Mechanics*, Macmillan, 1962, p. 235] is often used to compute the orbit of comets. We apply it here to Pluto's orbit. The notation is a variation on Danby's.

The symbolism I use is that a name with an overhanging right arrow is a vector, with both direction and magnitude. The name bare is its magnitude, while the name covered by a circumflex is the unit vector in that direction. Overhanging dots mean derivatives with respect to time, while an equals sign topped with a triangle means definition. Thus, by these definitions,

$$\vec{r} \triangleq r\hat{r}$$

Newton's 2nd Law of Motion combined with Newton's Law of Gravity yields the differential equation of motion for Pluto's position vector, $\vec{r_P}$:

$$m_P \ddot{\vec{r_P}} = -\frac{G m_{Sun} m_P \vec{r_P}}{\left|\vec{r_P}\right|^3} + \vec{F}_{perturb}$$

where the perturbation force on Pluto by Neptune is just the gravitational attraction:

$$\vec{F}_{perturb} = -\frac{G m_N m_P (\vec{r_P} - \vec{r_N})}{\left|\vec{r_P} - \vec{r_N}\right|^3}$$

Clearly, the mass of Pluto, m_P, may be divided out.

Let the motion of Pluto be slightly perturbed from a reference orbit $\overrightarrow{r_0}$ by the minute position vector $\overrightarrow{\rho}$. That is, let

$$\overrightarrow{r_P} = \overrightarrow{r_0} + \overrightarrow{\rho}$$

The reference orbit is an elliptical path in the orbital plane of Neptune:

$$\overrightarrow{r_0} = a_P\left[(\cos E - e)\hat{x} + \sqrt{1-e^2}\sin E\hat{y}\right]$$

Pluto's eccentricity $e = 0.247$, while the parameter E is called the *eccentric anomaly*, and is connected to the *mean anomaly* M by Kepler's equation [Danby, p. ??]:

$$M = E - e\sin E$$

From now on, we will choose Pluto's orbital semi-major axis, a_P, as our unit of length. The x-axis $\hat{x}$ points in the direction of Pluto's perihelion; choose the most recent perihelion as the origin of time. Lastly, choose the unit of time such that one period of Pluto's orbital motion is exactly 2π; the mean anomaly M is now identical with the time.

Let Neptune follow a coplanar, circular orbit whose period is exactly $\frac{2}{3}$ that of Pluto:

$$\overrightarrow{r_N} = a_N\Big(\sin\frac{3M}{2}\hat{x} - \cos\frac{3M}{2}\hat{y}\Big)$$

By Kepler's 3rd Law of Motion, relating semi-major axes to periods, $a_N = (2/3)^{\frac{2}{3}}a_P \approx 0.76314$. Note that at the time origin $M = E = 0$, when Pluto is at perihelion and has no position component along the $\hat{y}$ axis, that Neptune will be 90° behind Pluto. (As usual, orbital motion is counterclockwise when seen from above.) At Pluto's next perihelion, when $M = E = 2\pi$, Neptune will be 90° ahead. In fact, at no point in its reference orbit will Pluto come closer than ??° to Neptune. Therefore, there is little chance of collision between the planets. (This was first noted by [Cohen & Hubbard, Astr. J., 1965].) But only if Pluto's orbit remains stable or oscillatory around this resonance configuration, will this situation hold over the long term.

Thus, the equations of motion become two: one for the reference orbit, and a second for the perturbed orbit:

$$\ddot{\overrightarrow{r_0}} = -\frac{Gm_{Sun}\overrightarrow{r_0}}{\left|\overrightarrow{r_0}\right|^3}$$

$$\ddot{\overrightarrow{r_0}} + \ddot{\overrightarrow{\rho}} = -\frac{Gm_{Sun}(\overrightarrow{r_0}+\overrightarrow{\rho})}{\left|\overrightarrow{r_0}+\overrightarrow{\rho}\right|^3} - \frac{Gm_N(\overrightarrow{r_0}+\overrightarrow{\rho}-\overrightarrow{r_N})}{\left|\overrightarrow{r_0}+\overrightarrow{\rho}-\overrightarrow{r_N}\right|^3}$$

From [Danby, p. 125], we have $Gm_{Sun} = a_P^3(2\pi/period_P)^2$. Due to our simplifying choice of units, we find that $Gm_{Sun} = 1$.

The size of the perturbation vector $\overrightarrow{\rho}$ is determined by the ratio of the two terms in the latter equation. In turn, that is dominated by the ratio of masses $m_N/m_{Sun} \approx 0.00005 \ll 1$. Therefore, $|\overrightarrow{r_0}|$ and $|\overrightarrow{r_N}|$ are both $O(1)$, while $|\overrightarrow{\rho}| = O(m_N/m_{Sun}) \ll |\overrightarrow{r_0}|$. In what follows, we will lump together all 2nd order small terms of $O(|\overrightarrow{\rho}|^2)$ as an error term.

Expand the -3 power of the length of the vector sum using the inner product and the Binomial Theorem (and here we diverge from Encke's Method):

$$\left|\overrightarrow{r_0}+\overrightarrow{\rho}\right|^{-3} = |\overrightarrow{r_0}|^{-3}\Big[1 + 2\frac{|\overrightarrow{\rho}|}{|\overrightarrow{r_0}|}(\hat{\rho}\cdot\hat{r_0}) + \frac{|\overrightarrow{\rho}|^2}{|\overrightarrow{r_0}|^2}\Big]^{-3/2}$$

$$= |\overrightarrow{r_0}|^{-3}\Big[1 - 3\frac{|\overrightarrow{\rho}|}{|\overrightarrow{r_0}|}(\hat{\rho}\cdot\hat{r_0}) + O\Big(\frac{|\overrightarrow{\rho}|^2}{|\overrightarrow{r_0}|^2}\Big)\Big]$$

Subtract the equation of motion for the reference orbit in $\overrightarrow{r_0}$ from the perturbed equation of motion and combine small 2nd order terms. Since the perturbing term has a coefficient of small order, any $O(|\overrightarrow{\rho}|)$ terms within it will therefore be of 2nd order smallness overall.

$$\ddot{\vec{\rho}} = \frac{-\vec{\rho} + 3(\vec{\rho} \cdot \hat{r_0})\hat{r_0}}{|\vec{r_0}|^3} - \vec{p_N} + O(|\vec{\rho}|^2)$$

where the middle term, representing the perturbation by Neptune, is

$$\vec{p_N} = \frac{(m_N/m_{Sun})(\vec{r_0} - \vec{r_N})}{\left|\vec{r_0} - \vec{r_N}\right|^3}$$

Note that $\vec{\rho}$ has dropped completely out of the perturbing term.

This is the differential equation which we must solve. Rather than solve a single vector equation, let us eliminate $\vec{\rho}$ in favor of its scalar components.

Perturbation Components Relative to the Reference Orbit

Separate the perturbation vector $\vec{\rho}$ into its *radial* and *tangential* (or *transverse*) components with respect to the reference orbit. That is, since the z-axis is perpendicular to the common orbital plane,

$$\vec{\rho} \stackrel{\triangle}{=} A(M)\vec{r_0} + B(M)\hat{z} \times \vec{r_0}$$

Note that the full vector $\vec{r_0}$ is used here, not the unit vector $\hat{r_0}$.

Since the scalars A and B are functions of time M, we wish to find their scalar differential equations, to replace the single vector differential equation for $\vec{\rho}$.

First, substitute this redefinition of $\vec{\rho}$ into the Newtonian perturbation equation on the right hand side:

$$\ddot{\vec{\rho}} = \frac{1}{|\vec{r_0}|^3}\left[(-A\vec{r_0} - B\hat{z} \times \vec{r_0}) + 3(Ar_0)\hat{r_0}\right] - \vec{p_N} + O(|\vec{\rho}|^2)$$

Next, on the left hand side of the perturbation equation, formally differentiate the redefinition of $\vec{\rho}$ twice with respect to time:

$$\ddot{\vec{\rho}} = \ddot{A}\vec{r_0} + 2\dot{A}\dot{\vec{r_0}} + A\ddot{\vec{r_0}} + \ddot{B}\hat{z} \times \vec{r_0} + 2\dot{B}\hat{z} \times \dot{\vec{r_0}} + B\hat{z} \times \ddot{\vec{r_0}}$$

In order to proceed further, we must now compute the time derivatives of the reference orbit position vector $\vec{r_0}$.

Time Derivatives of the Reference Orbit Vector

All of the following definitions of variables for the reference orbit of Pluto are adapted from [Danby, p. 125-7]. As stated above, M and E are connected by Kepler's equation:

$$M = E - e\sin E$$

Let us abbreviate an expression to be used frequently:

$$b \stackrel{\triangle}{=} \sqrt{1-e^2}$$

Since the semi-major axis of the orbit is $a_P = 1$, b is in fact the semi-minor axis. From above,

$$\overrightarrow{r_0} = (\cos E - e)\hat{x} + b\sin E\hat{y}$$

The length of this vector (i.e. the distance of Pluto from the Sun) is the square root of the inner product:

$$r_0 \stackrel{\triangle}{=} |\overrightarrow{r_0}| = 1 - e\cos E$$

We are now able to compute derivatives with respect to time. Differentiate Kepler's equation:

$$\dot{M} = 1 = \dot{E}(1 - e\cos E)$$

Thus:

$$\dot{E} = \frac{1}{r_0}$$

Differentiate the definition of r_0:

$$\dot{r_0} = e\sin E\dot{E} = \frac{e\sin E}{r_0}$$

Differentiate the definition of $\overrightarrow{r_0}$:

$$\begin{aligned}\dot{\overrightarrow{r_0}} &= (-\sin E\dot{E})\hat{x} + (b\cos E\dot{E})\hat{y} \\ &= \frac{1}{r_0}(-\sin E\hat{x} + b\cos E\hat{y})\end{aligned}$$

To determine the radial and tangential components of this formula, take its inner product and its vector product, respectively, with $\overrightarrow{r_0}$. The result is:

$$\dot{\overrightarrow{r_0}} = \frac{\dot{r_0}}{r_0}\overrightarrow{r_0} + \frac{b}{r_0^2}\hat{z} \times \overrightarrow{r_0}$$

As an aside, we may compute the angular momentum of Pluto in the reference orbit from its definition as a vector product:

$$\overrightarrow{r_0} \times m_P\dot{\overrightarrow{r_0}} = m_P b\hat{z}$$

As expected, the angular momentum is constant, altho our units of length and time have simplified the result slightly.

Return to the formula for $\dot{\overrightarrow{r_0}}$ in terms of $\hat{x}$ and $\hat{y}$, and differentiate it:

$$\ddot{\overrightarrow{r_0}} = \frac{-\dot{r_0}}{r_0^2}(-\sin E\hat{x} + b\cos E\hat{y}) + \frac{\dot{E}}{r_0}(-\cos E\hat{x} - b\sin E\hat{y})$$

This readily simplifies to

$$\ddot{\overrightarrow{r_0}} = -\frac{\overrightarrow{r_0}}{r_0^3}$$

which simply states that the reference orbit obeys Newton's Law of Gravity.

Perturbation Components Relative to the Reference Orbit (Continued)

Substitute the time derivatives found in the last section into the formal double derivative of $\overrightarrow{\rho}$:

$$\ddot{\overrightarrow{\rho}} = \Big(\ddot{A} + \frac{2\dot{r_0}}{r_0}\dot{A} - \frac{1}{r_0^3}A - \frac{2b}{r_0^2}\dot{B}\Big)\overrightarrow{r_0} + \Big(\frac{2b}{r_0^2}\dot{A} + \ddot{B} + \frac{2\dot{r_0}}{r_0}\dot{B} - \frac{1}{r_0^3}B\Big)\hat{z}\times\overrightarrow{r_0}$$

which equates to the right hand side of the perturbation equation:

$$= \frac{1}{r_0^3}\big(2A\overrightarrow{r_0} - B\hat{z}\times\overrightarrow{r_0}\big) - \overrightarrow{p_N} + O(|\overrightarrow{\rho}|^2)$$

Equating the two sides, we find that some groups of terms combine to form perfect derivatives:

$$\frac{1}{r_0^2}\Big[\frac{d}{dM}(r_0^2\dot{A}) - 2b\dot{B} - \frac{3}{r_0}A\Big]\overrightarrow{r_0} + \frac{1}{r_0^2}\Big[2b\dot{A} + \frac{d}{dM}(r_0^2\dot{B})\Big]\hat{z}\times\overrightarrow{r_0} = -\overrightarrow{p_N} + O(|\overrightarrow{\rho}|^2)$$

Now, if we can split the Neptune perturbation term $\overrightarrow{p_N}$ into radial and tangential components, we will have two scalar differential equations of motion.

Time-Averaged Perturbation

The term representing the perturbation by Neptune is

$$\overrightarrow{p_N} = \Big(\frac{m_N}{m_{Sun}}\Big)\frac{\overrightarrow{\Delta r}}{|\overrightarrow{\Delta r}|^3}$$

where the vector difference (Pluto's reference position minus Neptune's) is

$$\overrightarrow{\Delta r} = \big[(\cos E - e)\hat{x} + b\sin E\hat{y}\big] - a_N(\sin\frac{3M}{2}\hat{x} - \cos\frac{3M}{2}\hat{y})$$

The perturbation is small, but accumulates over many orbits. In fact, after every two orbits, Δr repeats its previous values. Therefore, let us replace the instantaneous perturbation by the perturbation averaged together over two consecutive Plutonian orbits, the 1st orbit between times $M = -\pi$ and π and the 2nd between times $M = \pi$ and 3π. Comparing the value at times M and $2\pi - M$ (or equivalently, at angles E and $2\pi - E$), it is readily seen that Δr has the same $\hat{x}$ component at those times, but a negated $\hat{y}$ component. Therefore, averaged over two orbits, the $\hat{y}$ components will cancel. We write the time-averaged perturbation term as:

$$\langle\overrightarrow{p_N}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} dM(\overrightarrow{p_N}\cdot\hat{x})\hat{x}$$

This integral is rather a handful. Since we know its direction, it may be sufficient to get a first approximation to its magnitude. We shall compute the value of the time-averaged perturbation of Neptune on a point mass in a *circular* orbit with Pluto's reference orbit size and perihelion. To do this, let $e \to 0$ (and therefore $E \to M$) in the vector difference above:

$$\overrightarrow{\Delta r} \to (\cos M\hat{x} + \sin M\hat{y}) - a_N(\sin\frac{3M}{2}\hat{x} - \cos\frac{3M}{2}\hat{y})$$

Time averaging:

$$\langle\overrightarrow{p_N}\rangle \approx \frac{1}{2\pi}\Big(\frac{m_N}{m_{Sun}}\Big)\hat{x}\int_{-\pi}^{\pi} dM\frac{\cos M - a_N\sin\frac{3M}{2}}{\big(1 + a_N^2 - 2a_N\sin\frac{M}{2}\big)^{3/2}}$$

This is a complete elliptic integral and we evaluate it exactly in the appendix. However, it is easy to compute an upper bound to its value, and this is all we need.

In absolute value, the numerator is never greater than $1 + a_N$ and the denominator is never less than $(1 + a_N^2 - 2a_N)^{3/2}$; therefore

$$\begin{aligned} \left|\langle\overrightarrow{p_N}\rangle\right| &\le \frac{1 + a_N}{(1 - a_N)^3}\left(\frac{m_N}{m_{Sun}}\right) \\ &= 132.69\left(\frac{m_N}{m_{Sun}}\right) \approx 0.007 \end{aligned}$$

More sharply, we may note that the largest contribution of the denominator occurs near the upper integration limit $M = \pi$; here the numerator is approximately $-(1 - a_N)$. The absolute magnitude is then reduced to $17.826(m_N/m_{Sun})$.

In fact, the exact calculation in the appendix finds an even smaller magnitude of $1.35(m_N/m_{Sun})$, but in any case, we have proven what was required–that the perturbation term has a small time-averaged magnitude.

Let us abbreviate this for later use:

$$\langle\overrightarrow{p_N}\rangle \approx -\alpha\hat{x} - \beta\hat{y}$$

where $0 < \alpha, \beta \ll 1$.

Initially, $\beta = 0$, but it will not be if Pluto gets ahead or behind the resonance position.

Orbit Components in Terms of the True Anomaly

Now, we must introduce another physical variable of the orbit, one which will simplify the solution to this equation of motion.

Let v be the *true anomaly*, the angle at the Sun between the current position vector $\overrightarrow{r_0}$ and its position at the time of perihelion. In terms of v,

$$\overrightarrow{r_0} = r_0\big(\cos v\hat{x} + \sin v\hat{y}\big)$$

By equating this to the definition of $\overrightarrow{r_0}$ in terms of E, taking the inner product, and solving the resulting quadratic equation, we have the length in terms of v:

$$r_0 = \frac{b^2}{1 + e\cos v}$$

Inverting:

$$1 + e\cos v = \frac{b^2}{r_0}$$

and differentiating:

$$-e\sin v\dot{v} = -\frac{b^2\dot{r_0}}{r_0^2} = -\frac{b^2 e\sin E}{r_0^3}$$

Equating the $\hat{y}$ components of $\overrightarrow{r_0}$,

$$r_0\sin v = b\sin E$$

so that

$$\dot{v} = \frac{b}{r_0^2}$$

This is a form of Kepler's 2nd Law of Motion, that the rate of area swept out by the position vector is constant (and is in fact a constant multiple of the angular momentum).

Rearranging the above differentiation, we have

$$\dot{r_0} = \frac{e\sin v}{b}$$

Equating the $\hat{x}$ components of $\overrightarrow{r_0}$:

$$r_0\cos v = \cos E - e$$

Therefore a direct connection between v and E is:

$$\begin{aligned} \tan\frac{v}{2} &= \frac{\sin v}{1 + \cos v} \\ &= \frac{b}{1 - e}\tan\frac{E}{2} \end{aligned}$$

Perturbation Components Relative to the Reference Orbit (Concluded)

Split $\hat{x}$ and $\hat{y}$ in the average perturbation term just found into radial and tangential components:

$$\langle \overrightarrow{p_N} \rangle \approx -\frac{\alpha}{r_0}\big(\cos v \overrightarrow{r_0} - \sin v \hat{z} \times \overrightarrow{r_0}\big) - \frac{\beta}{r_0}\big(\sin v \overrightarrow{r_0} + \cos v \hat{z} \times \overrightarrow{r_0}\big)$$

We can therefore write two differential equations of motion, in the scalar functions $A(M)$ and $B(M)$. The radial one is:

$$\frac{d}{dM}(r_0^2 \dot{A}) - 2b\dot{B} - \frac{3}{r_0}A = \alpha r_0 \cos v + \beta r_0 \sin v + O(|\overrightarrow{\rho}|^2)$$

And the tangential one is:

$$2b\dot{A} + \frac{d}{dM}(r_0^2 \dot{B}) = -\alpha r_0 \sin v + \beta r_0 \cos v + O(|\overrightarrow{\rho}|^2)$$

From now on, we will drop the explicit error term from our equations.

Solution of the Equations of Motion

Since $\sin v = b\dot{r_0}/e$, we can easily integrate the tangential equation once:

$$2bA + r_0^2 \dot{B} = -\frac{\alpha b}{2e} r_0^2 + \beta I_x + C_1$$

where

$$I_x \triangleq \int dM r_0 \cos v = \frac{1}{2}[-3eE + (2e^2 + 1 + r_0)\sin E]$$

and C_1 is a constant of integration, the 1st of the 4 that we must find.

Using this solution, we can eliminate $\dot{B}$ from the radial equation. In the resulting equation, we give the name $\mathcal{D}$ to the 2nd order differential operator on the left-hand side:

$$\mathcal{D}A \triangleq \frac{d}{dM}(r_0^2 \dot{A}) + \Big(-\frac{3}{r_0} + \frac{4b^2}{r_0^2}\Big)A$$

$$= \alpha r_0 \cos v - \frac{\alpha b^2}{e} + \frac{2\beta b}{r_0^2} I_x + \beta r_0 \sin v + \frac{2b}{r_0^2} C_1$$

$$= -\frac{\alpha r_0}{e} + \frac{2\beta b}{r_0^2} I_x + \beta r_0 \sin v + \frac{2b}{r_0^2} C_1$$

since $r_0 = b^2/(1 + e\cos v)$.

Integration of Orbital Functions

In solving the differential equations of motion, we will have to integrate r_0^n with respect to the orbital anomalies (i.e. angles); we outline a simple strategy for such integrations now rather than give details for each such integration later.

- If the variable of integration is M, replace it by $E - e\sin E$, and dM by $r_0 dE$.
- The integral $\int dE r_0^n$ for $n > 0$ is made elementary by replacing r_0 by $1 - e\cos E$.
- The integral $\int dv r_0^{-n}$ for $n > 0$ is made elementary by replacing r_0 by $b^2/(1 + e\cos v)$.
- The integral $\int dE r_0^{-n}$ for $n > 0$ should have a change of integration variable by $dE = dv(r_0/b)$, which reduces it to a case above.
- The integral $\int dv r_0^n$ for $n > 0$ should have a change of integration variable by $dv = dE(b/r_0)$, which reduces it to a case above.
- A factor $\cos E$ in the integrand should be replaced by either $(1 - r_0)/e$ or $e + r_0 \cos v$, depending on whether the integration variable is E or v.
- A factor $\sin E$ in the integrand should be replaced by either $(1/e)(dr_0/dE)$ or $(r_0/b)\sin v$, depending on whether the integration variable is E or v.
- A factor $\cos v$ in the integrand should be replaced by either $(\cos E - e)/r_0$ or $(b^2 - r_0)/(er_0)$, depending on whether the integration variable is E or v.
- A factor $\sin v$ in the integrand should be replaced by either $(b/r_0)\sin E$ or $[b^2/(er_0^2)](dr_0/dv)$, depending on whether the integration variable is E or v.
- A factor $\sin^2 v$ in the integrand should be replaced by $1 - \cos^2 v$, and then $\cos v$ should be replaced as per the case above.

E.g. using these rules, we readily find that

$$\int dv r_0 \cos v = \frac{b}{e}(bv - E)$$

Solution of the Equations of Motion (Concluded)

In terms of v, the differential operator is:

$$\mathcal{D} = \frac{b^2}{r_0^2}\Big[\frac{d^2}{dv^2} + \Big(4 - \frac{3r_0}{b^2}\Big)\Big]$$

which immediately yields this fruitful result:

$$\mathcal{D}[\exp(inv)] = \Big[\frac{(4 - n^2)b^2}{r_0^2} - \frac{3}{r_0}\Big]\exp(inv)$$

where $\exp(iv) = \cos v + i\sin v$ as usual.

Since $\mathcal{D}$ is linear (i.e. $\mathcal{D}(f + g) = \mathcal{D}f + \mathcal{D}g$), we may apply it to a sum of functions:

$$\mathcal{D}\Big[\frac{\exp(iv)b^2}{r_0}\Big] = \mathcal{D}\Big[\exp(iv)(1 + e\cos v)\Big] = \mathcal{D}\Big[\frac{e}{2}\exp(i2v) + \exp(iv) + \frac{e}{2}\Big]$$

$$= \frac{e}{2}\Big[\frac{(4 - 2^2)b^2}{r_0^2}\Big]\exp(i2v) + \Big[\frac{(4 - 1^2)b^2}{r_0^2}\Big]\exp(iv) + \frac{e}{2}\Big[\frac{(4 - 0^2)b^2}{r_0^2}\Big] - \frac{3}{r_0}\Big[\frac{e}{2}\exp(i2v) + \exp(iv) + \frac{e}{2}\Big]$$

$$= \frac{e}{2}(0)\exp(i2v) + \Big(\frac{3b^2}{r_0^2}\Big)\exp(iv) + \frac{2b^2 e}{r_0^2} - \frac{3}{r_0}\Big[\frac{\exp(iv)b^2}{r_0}\Big]$$

$$= \frac{2b^2 e}{r_0^2}$$

Give names to the real and imaginary parts of the differand here:

$$f + ig \stackrel{\triangle}{=} \frac{b^2}{r_0}(\cos v + i\sin v)$$

Taking the imaginary part of each side above, $\mathcal{D}g = 0$, so g is a homogeneous solution. Similarly, f is an inhomogeneous solution. Let us therefore guess a solution to the equation of motion in the form

$$A = \frac{C_1}{be} f + gh$$

for h an unknown function. By formal substitution:

$$\begin{aligned}\mathcal{D}A &= \frac{C_1}{be}(\mathcal{D}f) + (\mathcal{D}g)h + \Big[\frac{d}{dM}(r_0^2\dot{h})\Big]g + 2r_0^2\dot{g}\dot{h} \\ &= \frac{2bC_1}{r_0^2} + (0)h + \frac{1}{g}\frac{d}{dM}(r_0^2 g^2 \dot{h})\end{aligned}$$

Equating to the right-hand side of the equation of motion, we find a differential equation for function h:

$$\frac{1}{g}\frac{d}{dM}(r_0^2 g^2 \dot{h}) = -\frac{\alpha r_0}{e} + \frac{2\beta b}{r_0^2} I_x + \beta r_0 \sin v$$

Substituting the definition $g = (b^2/r_0)\sin v = (b^3/er_0)\dot{r_0}$ and integrating once:

$$b^4 \sin^2 v\dot{h} = -\frac{\alpha b^3}{e^2}\int dr_0 + \frac{2\beta b^4}{e}\int \frac{dr_0}{r_0^3} I_x + \frac{\beta b^3}{e}\int dr_0 \sin v$$

The α term integrates immediately, but the β terms must be integrated by parts. That is,

$$b^4 \sin^2 v\dot{h} = -\frac{\alpha b^3}{e^2}(r_0 + C_2) + \frac{2\beta b^4}{e}\Big(-\frac{1}{2r_0^2} I_x + \int dM \frac{1}{2r_0^2}\frac{dI_x}{dM}\Big) + \frac{\beta b^3}{e}\Big(r_0 \sin v - \int dv r_0 \cos v\Big)$$

where C_2 is the 2nd of the 4 constants of integration. (It absorbs the constants of integration from the β terms.)

From its definition $dI_x/dM = r_0 \cos v$ and $dM = (r_0^2/b)dv$, so the two indefinite integral terms cancel out, leaving

$$b^4 \sin^2 v\dot{h} = -\frac{\alpha b^3}{e^2}(r_0 + C_2) + \frac{\beta b^3}{e}\Big(-\frac{bI_x}{r_0^2} + r_0 \sin v\Big)$$

Integrate again and convert both M and v to E:

$$h = -\frac{\alpha}{b^3 e^2}\int dE \frac{(r_0^4 + C_2 r_0^3)}{\sin^2 E} + \frac{\beta}{b^2 e}\Big(-\int dE \frac{r_0 I_x}{\sin^2 E} + \int dE \frac{r_0^3}{\sin E}\Big)$$

Since $dE/\sin^2 E = d(-\cot E)$, we may integrate by parts:

$$h = -\frac{\alpha}{b^3 e^2}\Big[-(r_0^4 + C_2 r_0^3)\cot E + \int dE \cot E(4r_0^3 + 3C_2 r_0^2)(e \sin E)\Big] + \frac{\beta}{b^2 e}\Big[\cot E(r_0 I_x) - \int dE \cot E(e\sin E) I_x$$

$$- \int dE \cot E r_0 (r_0^2 \cos v) + \int dE \frac{r_0^3}{\sin E}\Big]$$

The last two indefinite integrals merge:

$$= \frac{\alpha}{b^3 e^2}\Big[(r_0^4 + C_2 r_0^3)\cot E - \int dE(4r_0^3 + 3C_2 r_0^2)(e\cos E)\Big] + \frac{\beta}{b^2 e}\Big(r_0 I_x \cot E - \int dE e I_x \cos E + \int dE r_0^2 \sin E\Big)$$

It is convenient to define h_α and h_β as the sums of all terms here which are multiplied by α and by β, respectively.

To compute h_α, it is useful to define a nonce integral:

$$J_n \triangleq (n+1)\int dE(e\cos E)r_0^n$$

Re-expressing,

$$h_\alpha = \frac{\alpha}{b^3e^2}\Big[(r_0^4 + C_2r_0^3)\cot E - J_3 - C_2J_2 + C_3\Big]$$

where C_3 is the 3rd of the 4 constants of integration, and absorbs the constants of integration from the β terms.

Integrate J_n by parts:

$$J_n = (n+1)r_0^n(e\sin E) - (n+1)\int dE\big(nr_0^{n-1}e\sin E\big)(e\sin E)$$

Replace $e^2\sin^2 E$ by $e^2 - (1-r_0)e\cos E$:

$$J_n = (n+1)r_0^n e\sin E - (n+1)ne^2\int dEr_0^{n-1} + (n+1)J_{n-1} - nJ_n$$

Rearranging,

$$J_n - J_{n-1} = r_0^n e\sin E - e^2\int dEnr_0^{n-1}$$

Sum this equation for values of n down to 0; the sums on the left telescope and $J_{-1} = 0$, hence

$$J_n = e\sin E\sum_{j=0}^{n} r_0^j - e^2\int dE\sum_{j=0}^{n} jr_0^{j-1}$$

The geometric sums here can be simplified to closed forms, but the latter one is not readily integrable. Instead, substitute $r_0 = 1 - e\cos E$, and perform the latter integral directly.

$$J_2 = (1 + r_0 + r_0^2 + 2e^2)e\sin E - 3e^2E$$

$$J_3 = \Big(1 + r_0 + r_0^2 + r_0^3 + \frac{13e^2 + 3e^2r_0}{2}\Big)e\sin E - e^2\Big(\frac{12 + 3e^2}{2}\Big)E$$

This completes h_α. To compute h_β, we integrate using the substitution $\sin E = (1/e)(dr_0/dE)$, as described in the previous section:

$$\begin{aligned} h_\beta &= \frac{\beta}{b^2e}\Big\{\Big[-\frac{3}{2}er_0E\cot E + \frac{1}{2}(2e^2r_0 + r_0 + r_0^2)\cos E\Big] \\ &+ \frac{3}{2}e^2(\cos E + E\sin E) + \frac{1}{2e}(r_0^3 + e^2r_0^2 - 2e^2r_0 - r_0)\Big\} \\ &= \frac{\beta}{2b^2}(-3bE\cot v + 3 - 3r_0 - r_0^2) \end{aligned}$$

Note to reviewers: I have double-checked the algebra thru here.

Therefore, combining all terms, including gh_β,

$$A = \frac{C_1b}{e}\frac{\cos v}{r_0} + \frac{\alpha}{2e^3}\Big[e^2r_0 + b^2e^2 + \frac{2b^4}{r_0} - \frac{2b^6}{r_0^2} + (2C_2 + 5 - b^2)\Big\{e^2 + \frac{(5b^2-6)}{r_0} - \frac{(2b^2-3)b^2}{r_0^2}$$

$$+ \frac{3e^3E\sin E}{r_0^2}\Big\}\Big] + \frac{\alpha C_3}{be^2}\frac{\sin v}{r_0} + \frac{\beta}{2r_0}[-3bE\cos v + (3 - 3r_0 - r_0^2)\sin v]$$

We can now integrate the solution to the tangential equation a 2nd, final, time:

$$B = \int dv\Big(\frac{C_1}{b} - 2A\Big) - \frac{\alpha b}{2e}M + C_4$$

where C_4 is the last of our 4 constants of the motion.

As time passes, M, E and v all become indefinitely large, each increasing by 2π after every orbit. Terms involving these angles outside of trigonometric functions will cause *secular* increase. In A, secular buildup is caused by the E term; however, its coefficient will be 0 if

$$2C_2 + 5 - b^2 = 0$$

which is the same as

$$C_2 \triangleq -2 - \frac{1}{2}e^2$$

Note to reviewers: Is this specification of a free constant permissible?

Replacing C_2 with this value, A is simplified:

$$A = \frac{C_1 b}{e}\frac{\cos v}{r_0} + \frac{\alpha}{2e^3}\Big(e^2 r_0 + b^2 e^2 + \frac{2b^4}{r_0} - \frac{2b^6}{r_0^2}\Big) + \frac{\alpha C_3}{be^2}\frac{\sin v}{r_0} + \frac{\beta}{2r_0}[-3bE\cos v + (3 - 3r_0 - r_0^2)\sin v]$$

Now the integration of A becomes:

$$\int dv\Big(\frac{C_1}{b} - 2A\Big) = \frac{C_1}{be}\int dv\Big(e - \frac{2b^2\cos v}{r_0}\Big) - \frac{\alpha}{e^3}\int dv\Big(e^2 r_0 + b^2 e^2 + \frac{2b^4}{r_0} - \frac{2b^6}{r_0^2}\Big) - \frac{2\alpha C_3}{be^2}\int dv\frac{\sin v}{r_0} + B_\beta$$

The computation splits naturally into evaluation of the three integrals shown, plus that of B_β, which is done afterward.

Substituting $r_0 = b^2/(1 + e\cos v)$, the 1st integral here becomes elementary.

$$\frac{C_1}{be}\int dv\Big(e - \frac{2b^2\cos v}{r_0}\Big) = -\frac{C_1}{be}(2\sin v + e\cos v\sin v)$$

Similarly, substitute for r_0 in the 2nd integral, except in the 1st term, where we substitute $dE = (r_0/b)dv$:

$$-\frac{\alpha}{e^3}\int dv\Big(e^2 r_0 + b^2 e^2 + \frac{2b^4}{r_0} - \frac{2b^6}{r_0^2}\Big) = -\frac{\alpha}{e^3}\Big[be^2 E - b^2\Big(1 + \frac{b^2}{r_0}\Big)e\sin v\Big]$$

From above, $dv = dr_0 b^2/(er_0^2\sin v)$, so we easily perform the 3rd integral:

$$\frac{2\alpha C_3}{be^2}\int dv\frac{\sin v}{r_0} = -\frac{\alpha b C_3}{e^3 r_0^2}$$

Summing up,

$$B = -\frac{\alpha b}{2e}(2E + M) + \Big(\frac{\alpha b^2}{e^2} - \frac{C_1}{be}\Big)\Big(1 + \frac{b^2}{r_0}\Big)\sin v - \frac{\alpha b C_3}{e^3 r_0^2} + B_\beta + C_4$$

Note to reviewers: Hmmm. The secular term isn't 0. I must recheck my calculations.

The contribution of the β term to B is therefore:

$$B_\beta \triangleq \int dv\Big\{-2\frac{\beta}{2r_0}[-3bE\cos v + (3 - 3r_0 - r_0^2)\sin v]\Big\}$$

Substitute for r_0 in the 1st term and $\sin v = (b^2/er_0^2)(dr_0/dv)$ in the latter term:

$$= \frac{3\beta}{b}\int E dv\cos v(1 + e\cos v) + \frac{\beta b^2}{2e}\Big(\frac{3}{r_0^2} - \frac{6}{r_0} + 2\ln r_0\Big)$$

Integrate the 1st term by parts, and then substitute for $\sin v$ in the resulting integral:

$$= \frac{3\beta}{b}\Big\{E\Big[\frac{e}{2}v + \sin v\Big(1+\frac{e}{2}\cos v\Big)\Big] - \int dE\Big[\frac{e}{2}v + \sin v\Big(1+\frac{e}{2}\cos v\Big)\Big]\Big\} + \frac{\beta b^2}{2e}\Big(\frac{3}{r_0^2} - \frac{6}{r_0} + 2\ln r_0\Big)$$

$$= \frac{3\beta}{2b}\Big[E\sin v\Big(1+\frac{b^2}{r_0}\Big) + e\int dvE + b\cos v - \frac{b}{e}\ln r_0 + \frac{b}{e}\ln b^2\Big] + \frac{\beta b^2}{2e}\Big(\frac{3}{r_0^2} - \frac{6}{r_0} + 2\ln r_0\Big)$$

The integral $\int dvE$ is a hard nut to crack. It can be expressed as:

$$\int dvE = \int dv\Big[2\tan^{-1}\Big(\frac{b}{1+e}\tan\frac{v}{2}\Big)\Big]$$

Perhaps we can differentiate the integrand with respect to e, perform the integral with respect to v, and then perform the integral with respect to e, so that

$$\begin{aligned}\int dvE &= \int de\int dv\frac{d}{de}\Big[2\tan^{-1}\Big(\frac{b}{1+e}\tan\frac{v}{2}\Big)\Big]\\ &= \int de\int dv\frac{-\sin v}{b(1+\cos v)}\\ &= \int de\frac{\ln(1+\cos v)}{be}\end{aligned}$$

Note to reviewers: This is as far as I've come. Can you help me finish?

Appendix 1. Computation of a Complete Elliptic Integral

Define I as the integral we seek

$$\langle\overrightarrow{p_N}\rangle \approx \Big(\frac{m_N}{m_{Sun}}\Big)I\hat{x}$$

where

$$I \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} dM\frac{\cos M - a_N\sin\frac{3M}{2}}{\big(1+a_N^2 - 2a_N\sin\frac{M}{2}\big)^{3/2}}$$

To convert it closer to a standard form, substitute $M = 4\theta - \pi$, so

$$I = -\frac{2}{\pi(1+a_N)^3}\int_0^{\frac{\pi}{2}} d\theta\frac{\cos 4\theta + a_N\cos 6\theta}{(1-m\sin^2\theta)^{3/2}}$$

The elliptic modulus $m \triangleq 4a_N/(1+a_N)^2$ has the value 0.98195 since a_N has the value 0.76314.

A handy formula for the trigonometric function of a multiple angle argument is:

$$\cos 2n\theta = \sum_{l=0}^{n}\binom{n+l}{2l}\frac{n}{n+l}(-4\sin^2\theta)^l$$

which is obtained by expanding first deMoivre's Formula and then $\cos^2\theta = 1 - \sin^2\theta$ by the Binomial Theorem, exchanging summations and performing the inner one.

In particular,

$$\cos 4\theta = 1 - 8\sin^2\theta + 8\sin^4\theta$$

and

$$\cos 6\theta = 1 - 18\sin^2\theta + 48\sin^4\theta - 32\sin^6\theta$$

For nonce use, define a sequence of elliptic integrals:

$$I_n(m) \triangleq \int_0^{\frac{\pi}{2}} d\theta Z^{\frac{n}{2}}$$

where

$$Z = 1 - m\sin^2\theta$$

Make these trigonometric substitutions, replace $\sin^2\theta$ everywhere by $(1-Z)/m$ and integrate:

$$I = -\frac{2}{\pi(1+a_N)^3}\Big\{32a_N I_3(m) + \big[48a_N(m-2)+8m\big]I_1(m) + \big[6a_N(3m^2-16m+16)+8m(m-2)\big]I_{-1}(m)$$
$$+\big[a_N(m^3-18m^2+48m-32)+(m^3-8m^2+8m)\big]I_{-3}(m)\Big\}$$

Two standard elliptic integrals are $E(m)\triangleq I_1(m)$ and $K(m)\triangleq I_{-1}(m)$. To compute the others, we differentiate an expression similar to the integrand of $I_n(m)$:

$$\frac{d}{d\theta}\big[\sin\theta\cos\theta Z^{\frac{n}{2}}\big] = \big(\cos^2\theta - \sin^2\theta\big)Z^{\frac{n}{2}} - nm\big(\sin^2\theta\cos^2\theta\big)Z^{\frac{n-2}{2}}$$
$$= \frac{1}{m}\big[(n+2)Z^{\frac{n+2}{2}} - (n+1)(2-m)Z^{\frac{n}{2}} + n(1-m)Z^{\frac{n-2}{2}}\big]$$

Integrating this between limits 0 and $\pi/2$ yields 0 on the left-hand side, so that we have proven a recurrence relation among our nonce integrals:

$$0 = (n+2)I_{n+2}(m) - (n+1)(2-m)I_n(m) + n(1-m)I_{n-2}(m)$$

In particular, for $n=1$ and -1:

$$I_3(m) = \frac{2}{3}(2-m)I_1(m) - \frac{1}{3}(1-m)I_{-1}(m)$$

and

$$I_{-3}(m) = \frac{1}{1-m}I_1(m)$$

It is clear that, numerically, integral $I_{-3}(m)$ contributes the most to I, since $1-m$ in the denominator is so small. In fact, it dominates the upper bound found in the main part of this paper.

Using these relations:

$$I = -\frac{2}{\pi(1+a_N)^3}\Big\{\frac{1}{3(1-m)}\Big[a_N(3m^3-134m^2+384m-256)+3m(m^2-16m+16)\Big]E(m)$$
$$+\Big[\frac{2}{3}a_N(27m^2-128m+128)-8m(2-m)\Big]K(m)\Big\}$$

Since m is very close to 1, let us re-express I in terms of the small quantity $m_1\triangleq 1-m \approx 0.01805$:

$$I = -\frac{2}{\pi(1-m_1)^3(1+a_N)^3}\Big\{-\frac{1}{3m_1}\Big[a_N(3+125m_1+125m_1^2+3m_1^3)-3(1-m_1)(1+14m_1+m_1^2)\Big]E(1-m_1)$$
$$+\Big[\frac{2}{3}a_N(27+74m_1+27m_1^2)-8(1-m_1^2)\Big]K(1-m_1)\Big\}$$

Substituting numerical values for a_N and m_1:

$$I \approx -0.75314E(0.98195) + 0.74605K(0.98195)$$

For values of the elliptic modulus approaching 1 [Abramowitz & Stegun, *Handbook of Mathematical Functions*, Washington DC 1964, p. 591]

$$K(1-m_1) \approx \ln\frac{4}{\sqrt{m_1}}$$
$$E(1-m_1) \approx \frac{\pi}{2}\Big(1-\frac{m_1}{4}\Big)$$

so $I \approx 1.3542$ and

$$\langle\overrightarrow{p_N}\rangle \approx -1.3542\Big(\frac{m_N}{m_{Sun}}\Big)$$

As expected, the exact computation of integral I gives a value well inside the upper bounds estimated in the main part of this paper.

Appendix 2. Alternate Solution When Perturbation has a Y-Component

Let us abbreviate this for later use:

$$\langle \overrightarrow{p_N} \rangle \approx -\alpha\hat{x} - \beta\hat{y}$$

where $0 < \alpha, \beta \ll 1$.

We can therefore write two differential equations of motion, in the scalar functions $A(M)$ and $B(M)$. The radial one is:

$$\frac{d}{dM}(r_0^2\dot{A}) - 2b\dot{B} - \frac{3}{r_0}A = \alpha r_0 \cos v + \beta r_0 \sin v$$

And the tangential one is:

$$2b\dot{A} + \frac{d}{dM}(r_0^2\dot{B}) = -\alpha r_0 \sin v + \beta r_0 \cos v$$

Since $\sin v = b\dot{r_0}/e$, we can easily integrate the tangential equation once:

$$2bA + r_0^2\dot{B} = -\frac{\alpha b}{2e}r_0^2 + \beta I_x + C_1$$

where

$$I_x \triangleq \int dM r_0 \cos v = \frac{1}{2}[-3eE + (2e^2 + 1 + r_0)\sin E]$$

and C_1 is a constant of integration, the 1st of the 4 that we must find.

Using this solution, we can eliminate $\dot{B}$ from the radial equation. In the resulting equation, we give the name $\mathcal{D}$ to the 2nd order differential operator on the left-hand side:

$$\mathcal{D}A \triangleq \frac{d}{dM}(r_0^2\dot{A}) + \Big(-\frac{3}{r_0} + \frac{4b^2}{r_0^2}\Big)A$$

$$= \alpha r_0 \cos v - \frac{\alpha b^2}{e} + \frac{2\beta b}{r_0^2}I_x + \beta r_0 \sin v + \frac{2b}{r_0^2}C_1$$

$$= -\frac{\alpha r_0}{e} + \frac{2\beta b}{r_0^2}I_x + \beta r_0 \sin v + \frac{2b}{r_0^2}C_1$$

since $r_0 = b^2/(1 + e\cos v)$.

Equating to the right-hand side of the equation of motion, we find a differential equation for function h:

$$\frac{1}{g}\frac{d}{dM}(r_0^2 g^2 \dot{h}) = -\frac{\alpha r_0}{e} + \frac{2\beta b}{r_0^2}I_x + \beta r_0 \sin v$$

Substituting the definition $g = (b^2/r_0)\sin v = (b^3/er_0)\dot{r_0}$ and integrating once:

$$b^4 \sin^2 v\dot{h} = -\frac{\alpha b^3}{e^2}\int dr_0 + \frac{2\beta b^4}{e}\int \frac{dr_0}{r_0^3}I_x + \frac{\beta b^3}{e}\int dr_0 \sin v$$

The α term integrates immediately, but the β terms must be integrated by parts. Hence

$$b^4 \sin^2 v\dot{h} = -\frac{\alpha b^3}{e^2}(r_0 + C_2) + \frac{2\beta b^4}{e}\Big(-\frac{1}{2r_0^2}I_x + \int dM \frac{1}{2r_0^2}\frac{dI_x}{dM}\Big) + \frac{\beta b^3}{e}\Big(r_0 \sin v - \int dv r_0 \cos v\Big)$$

where C_2 is the 2nd of the 4 constants of integration. (It absorbs the constants of integration from the β terms.)

From its definition $dI_x/dM = r_0 \cos v$ and $dM = (r_0^2/b)dv$, so the two indefinite integral terms cancel out, leaving

$$b^4 \sin^2 v \dot{h} = -\frac{\alpha b^3}{e^2}(r_0 + C_2) + \frac{\beta b^3}{e}\Big(- \frac{bI_x}{r_0^2} + r_0 \sin v\Big)$$

Integrate again and convert both M and v to E:

$$h = -\frac{\alpha}{b^3 e^2}\int dE \frac{(r_0^4 + C_2 r_0^3)}{\sin^2 E} + \frac{\beta}{b^2 e}\Big(- \int dE \frac{r_0 I_x}{\sin^2 E} + \int dE \frac{r_0^3}{\sin E}\Big)$$

Since $dE/\sin^2 E = d(-\cot E)$, we may integrate by parts:

$$h = -\frac{\alpha}{b^3 e^2}\Big[-(r_0^4 + C_2 r_0^3)\cot E + \int dE \cot E(4r_0^3 + 3C_2 r_0^2)(e \sin E)\Big] + \frac{\beta}{b^2 e}\Big[\cot E(r_0 I_x) - \int dE \cot E(e \sin E) I_x$$

$$- \int dE \cot E r_0 (r_0^2 \cos v) + \int dE \frac{r_0^3}{\sin E}\Big]$$

The last two indefinite integrals combine:

$$= -\frac{\alpha}{b^3 e^2}\Big[-(r_0^4 + C_2 r_0^3)\cot E + \int dE (4r_0^3 + 3C_2 r_0^2) e \cos E\Big] + \frac{\beta}{b^2 e}\Big(r_0 I_x \cot E - \int dE e I_x \cos E + \int dE r_0^2 \sin E\Big)$$

The integrals in the β term are readily done using $\sin E = (1/e)(dr_0/dE)$, so the β term in h is:

$$h_\beta \triangleq \frac{\beta}{b^2 e}\Big\{\Big[- \frac{3}{2} e r_0 E \cot E + \frac{1}{2}(2e^2 r_0 + r_0 + r_0^2)\cos E\Big] + \frac{3}{2}e^2(\cos E + E \sin E) + \frac{1}{2e}(r_0^3 + e^2 r_0^2 - 2e^2 r_0 - r_0)\Big\}$$

$$= \frac{\beta}{2b^2}(-3bE \cot v + 3 - 3r_0 - r_0^2)$$

I have double-checked the algebra thru here.

Therefore, combining all terms, including gh_β,

$$A = \frac{C_1 b}{e}\frac{\cos v}{r_0} + \frac{\alpha}{2e^3}\Big[e^2 r_0 + b^2 e^2 + \frac{2b^4}{r_0} - \frac{2b^6}{r_0^2} + (2C_2 + 5 - b^2)\Big\{e^2 + \frac{(5b^2 - 6)}{r_0} - \frac{(2b^2 - 3)b^2}{r_0^2}$$

$$+ \frac{3e^3 E \sin E}{r_0^2}\Big\}\Big] + \frac{\alpha C_3}{be^2}\frac{\sin v}{r_0} + \frac{\beta}{2r_0}[-3bE \cos v + (3 - 3r_0 - r_0^2)\sin v]$$

We can now integrate the solution to the tangential equation a 2nd, final, time:

$$B = \int dv\Big(\frac{C_1}{b} - 2A\Big) - \frac{\alpha b}{2e}M + C_4$$

where C_4 is the last of our 4 constants of the motion.

The contribution of the β term to B is therefore:

$$B_\beta \triangleq \int dv\Big\{ - 2\frac{\beta}{2r_0}[-3bE \cos v + (3 - 3r_0 - r_0^2)\sin v]\Big\}$$

Substitute for r_0 in the 1st term and $\sin v = (b^2/er_0^2)(dr_0/dv)$ in the latter term:

$$= \frac{3\beta}{b}\int E dv \cos v(1 + e \cos v) + \frac{\beta b^2}{2e}\Big(\frac{3}{r_0^2} - \frac{6}{r_0} + 2\ln r_0\Big)$$

Integrate the 1st term by parts, and then substitute for $\sin v$ in the resulting integral:

$$= \frac{3\beta}{b}\left\{E\left[\frac{e}{2}v + \sin v\left(1 + \frac{e}{2}\cos v\right)\right] - \int dE\left[\frac{e}{2}v + \sin v\left(1 + \frac{e}{2}\cos v\right)\right]\right\} + \frac{\beta b^2}{2e}\left(\frac{3}{r_0^2} - \frac{6}{r_0} + 2\ln r_0\right)$$

$$= \frac{3\beta}{2b}\left[E\sin v\left(1 + \frac{b^2}{r_0}\right) + e\int dvE + b\cos v - \frac{b}{e}\ln r_0 + \frac{b}{e}\ln b^2\right] + \frac{\beta b^2}{2e}\left(\frac{3}{r_0^2} - \frac{6}{r_0} + 2\ln r_0\right)$$

The integral $\int dvE$ is a hard nut to crack. It can be expressed as:

$$\int dvE = \int dv\left[2\tan^{-1}\left(\frac{b}{1+e}\tan\frac{v}{2}\right)\right]$$

Perhaps we can differentiate the integrand with respect to e, perform the integral with respect to v, and then perform the integral with respect to e, so that

$$\begin{aligned}\int dvE &= \int de\int dv\frac{d}{de}\left[2\tan^{-1}\left(\frac{b}{1+e}\tan\frac{v}{2}\right)\right]\\ &= \int de\int dv\frac{-\sin v}{b(1+\cos v)}\\ &= \int de\frac{\ln(1+\cos v)}{be}\end{aligned}$$