

ON CHOICE SETS AND STRONGLY NON-TRIVIAL SELF-EMBEDDINGS OF RECURSIVE LINEAR ORDERS

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In [1] DUSHNIK and MILLER prove that every countably infinite linear order has a non-trivial self-embedding. The first part of their argument considers linear orders with an interval of type ω or ω^* . Mapping x to its immediate successor (resp. predecessor) if x is in this interval and the interval is of type ω (resp. ω^*), and to itself otherwise, produces the required non-trivial self-embedding. The existence of an interval of type $\omega^* + \omega$ would similarly guarantee a non-trivial automorphism.

In [4] KIERSTEAD defines the maps described above to be *fairly trivial* in the sense that they are non-trivial but map every element x to an element y with $[x, y]$ finite. He defines a self-embedding to be *strongly non-trivial* if it is neither trivial nor fairly trivial.

The above-mentioned argument from DUSHNIK and MILLER [1] shows that every recursive linear order (i.e. with universe \mathbb{N} and $<$ a recursive relation) with a recursive subset consisting only of intervals of type ω , or only of intervals of type ω^* , has a fairly trivial Π_1 self-embedding. We use the fact that the successor relation $S(a, b)$ defined by

$$(\forall x) ((a < x < b) \vee (b < x < a))$$

is Π_1 in every recursive linear order. As before, if the recursive subset consisted only of intervals of type $\omega^* + \omega$, we are guaranteed a fairly trivial Π_1 automorphism. It follows that every recursive discrete linear order has a fairly trivial Π_1 automorphism. (A linear order is *discrete* if every element has both an immediate predecessor and an immediate successor, or equivalently, if the linear order is of type $(\omega^* + \omega) \cdot \tau$ for some order type τ .)

This contrasts with the main result of this paper:

Every recursive discrete linear order has a recursive copy with no strongly non-trivial Π_1 self-embedding.

This result proves the conjecture, stated in KIERSTEAD [4], that there is a recursive linear order of type $(\omega^* + \omega) \cdot \eta$ with no strongly non-trivial Π_1 automorphism.

Notice that if f is a non-trivial self-embedding of a linear order, then there is an element x with $x \neq f(x)$ and with $\{x, f(x), f^2(x), f^3(x), \dots\}$ defining a suborder of type ω or ω^* . If f is strongly non-trivial, then there is such a suborder with the elements all in separate blocks. (A *block* is an equivalence class $c_F(a) = \{b : [a, b] \text{ is finite}\}$; see ROSENSTEIN [6].) A *choice set* for a linear order is a subset consisting of precisely

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one element from each block of the linear order. If a strongly non-trivial Π_1 self-embedding exists, the above is an infinite Σ_2 subset of a choice set. We prove our result by proving

Theorem 1. *Every recursive discrete linear order has a recursive copy all of whose choice sets have no infinite Σ_2 subset.*

Theorem 1 extends the following result of LERMAN and ROSENSTEIN [5]:

There is a recursive linear order of type $(\omega^ + \omega) \cdot \eta$ with no Σ_2 dense subset.*

Indeed, our construction has many similarities with theirs. As is noted in LERMAN and ROSENSTEIN [5], every recursive linear order has a Π_2 choice set. This follows from the fact that the Π_2 relation

$$(\forall x < a) (\forall y) (\exists z > y) ((x < z < a) \vee (a < z < x))$$

defines a choice set. Here $<$ is the order relation of \mathbb{N} and $<$ that of the recursive linear order in question. We use this fact in our construction, it being less cumbersome to construct our recursive copy from a Π_2 choice set than to preserve an isomorphism between the discrete linear orders.

Any undefined terminology we use, and there is little of this, may be gleaned from ROSENSTEIN [6].

Proof of Theorem 1. Let \mathfrak{B} be a recursive discrete linear order and B a Π_2 choice set for \mathfrak{B} . Let τ be the order-type of the suborder of \mathfrak{B} defined by B . We construct a recursive linear order \mathfrak{A} of type $(\omega^* + \omega) \cdot \tau$; clearly $\mathfrak{A} \cong \mathfrak{B}$.

Since B is Π_2 there is a recursive binary relation R with

$$n \in B \Leftrightarrow QxR(n, x),$$

where Q is the quantifier "there are infinitely many". We assume that R has the additional property

$$(*) \quad (\forall n \in B) (Qx) (\forall k \leq n) (k \in B \Rightarrow R(k, x)).$$

A result in JOCKUSCH [3] allows such an assumption.

We construct \mathfrak{A} in stages on the universe \mathbb{N} . At each stage s we define a finite linear order \mathfrak{A}_s , and arrange that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \dots$. Defining $\mathfrak{A} = \lim \mathfrak{A}_s$ then guarantees that it is recursive.

Our basic strategy for arranging that \mathfrak{A} has order-type $(\omega^* + \omega) \cdot \tau$ is to define an interval $I(i)$ for each $i \in \mathbb{N}$, order these intervals as the i are in \mathfrak{B} and extend $I(i)$ at both ends at every stage s with $(i, s) \in R$, letting it lie fallow at other stages. This will produce an interval of type $\omega^* + \omega$ if and only if $i \in B$. If $j \notin B$, $I(j)$ will be finite (or empty).

The problem with this basic strategy is that it does not prevent the intervals $I(j)$ with $j \notin B$ from becoming separate blocks themselves. We must arrange that each such $I(j)$ becomes part of some interval $I(i)$ with $i \in B$. At each stage s with $(j, s) \notin R$ we must consider $I(j)$ part of the "nearest" $I(i)$ with $(i, s) \in R$. This means not only that we must not extend $I(j)$ at this stage but also that, while extending $I(i)$, we must not introduce elements between the present versions of $I(i)$ and $I(j)$. If, however, we reach a later stage t with $(j, t) \in R$, we must return to the building of $I(j)$, in case j turns out to be in B after all.

We have two problems with this strategy:

1. We may, at stage t , split $I(i)$ into two parts, since $I(i)$ may have been extended (on both sides of $I(j)$) between stages s and t .
2. $I(j)$ may not be ordered correctly with respect to intervals $I(k)$ that were introduced between stages s and t (for example with $j < k < i$ in \mathfrak{B}).

It is not enough to invoke the usual finite injury priority method of abandoning the present versions of those $I(k)$ with $j < k$ (in \mathbb{N}) and building them anew, elsewhere and ordered correctly with respect to $I(j)$. It may well happen that $j \in B$ and $I(j)$ lies fallow and returns to life infinitely often, causing these $I(k)$ never to be built.

We cope with these problems by building many versions of each $I(i)$, one for each of the 2^i possible guesses as to which $j \leq i$ (\leq in \mathbb{N}) are in B . For each such guess we build a version of $I(j)$ for each j guessed to be in B , ordering them correctly (i.e. as dictated by \mathfrak{B}) with respect to each other but not necessarily with respect to the intervals created for other guesses. In order that the only blocks produced are from intervals $I(i)$ with $i \in B$, and that, only from the version built for the "correct guess", we use the usual $2^{<\omega}$ tree A to introduce a priority ordering on our guesses. The nodes of A of odd length will correspond to our guesses and we define an interval $I(\sigma)$ for each such node σ ; the $I(\sigma)$ with $|\sigma| = 2i + 1$ will be the versions of $I(i)$. Property (*) ensures that the nodes corresponding to correct guesses will govern our construction at infinitely many stages. We write $I(\sigma, s)$ for the version of $I(\sigma)$ defined at stage s .

Our basic strategy to ensure that no choice set for \mathfrak{A} has an infinite Σ_2 subset is to arrange that each Σ_2 set T is either too small (in particular finite) or contains elements x, y with $[x, y]$ finite in \mathfrak{A} .

Let W_0, W_1, W_2, \dots be an effective enumeration of all r.e. binary relations. Consider the sets T_0, T_1, T_2, \dots defined by

$$n \in T_i \Leftrightarrow \neg(QxW_i(n, x)).$$

Every Σ_2 set is one of these T_i . We attempt to define elements x, y in each one of these T_i with $[x, y]$ finite in \mathfrak{A} . As the sets T_i may not be recursive, we may often need to re-define the pair x, y for T_i . We must ensure that if T_i is infinite, we do eventually settle on a pair x, y for T_i ; if T_i is finite, we may define new pairs x, y for T_i infinitely often.

To mesh this strategy with our earlier described block-building technique we select elements x, y for T_i that lie in the same "live" interval, i.e. some $I(\sigma)$ being extended at this stage. This may require us to unify previously separate live intervals, considering them separate no longer. However, when selecting x, y for T_i , we do not unify any live intervals $I(\alpha), I(\beta)$ with $|\alpha| \leq i$ and $|\beta| \leq i$. Since there may be as many as i such intervals, we attempt to select an x, y for T_i only if $|T_i| > i$.

To facilitate our construction we use the set $C = \{i: |T_i| \leq i\}$. C is a Π_2 set defined by the formula: "For every j , the j^{th} finite sequence either is of length $\leq i$ or contains an element (\leq its maximum element) that is not in T_i ". Consequently, there is a recursive binary relation S with

$$n \in C \Leftrightarrow QxS(n, x),$$

again, by JOCKUSCH [3], we assume that S has property

$$(*) \quad (\forall n \in C) (Qx) (\forall k \leq n) (k \in C \Rightarrow S(k, x)).$$

The nodes of A with even length will correspond to guesses as to what the elements of C are; σ with $|\sigma| = 2i + 2$ will guess which $j \leq i$ are in C . Nodes that guess $i \in C$ will not cause us to define a pair x, y for T_i . Once again, property $(*)$ ensures that the nodes corresponding to correct guesses will govern our construction at infinitely many stages.

In order that we do settle on a pair x, y for each infinite T_i , we always select the pair from the bottom of a list $\Gamma_{i,s}$. We will select the pair from among the least $i + 1$ members of $\Gamma_{i,s}$; the $i + 1$ elements will allow us not to unify any live intervals $I(\alpha)$, $I(\beta)$ with $|\alpha| \leq i$ and $|\beta| \leq i$. It is this requirement of $i + 1$ members of T_i that motivates our use of the set C . We arrange that the members listed earliest in $\Gamma_{i,s}$ are those "most likely" to be in T_i . If $i \notin C$, there will be a stage after which the least $i + 1$ members of $\Gamma_{i,s}$ will be members of T_i . In this manner we arrange that our construction along the "true path" of A is a finite injury priority construction meeting the requirements that, for each i , T_i is not an infinite subset of a choice set for \mathfrak{A} .

We now formally define the phrases used in the description of our construction.

A has as its nodes all finite $\{0, 1\}$ -sequences, we denote them with lower case Greek letters. We imagine A extending downward from the empty node λ with a path joining nodes α and β if and only if $\alpha < \beta$, i.e. if and only if (the sequence) α is an initial segment of (the sequence) β . We say α is *above* β and β *below* α if $\alpha < \beta$. For each α , $\alpha \smallfrown 0$ lies to the left $\alpha \smallfrown 1$, where \smallfrown is concatenation of sequences. With this picture in mind we may discuss when a node α (or the $\lambda - \alpha$ path) is *to the left* of a node β (or the $\lambda - \beta$ path), precisely when

$$(\exists \sigma \in A) (\sigma \smallfrown 0 \subseteq \alpha \text{ and } \sigma \smallfrown 1 \subseteq \beta).$$

We may also define an order relation on A by

$$\alpha < \beta \text{ if and only if } (\alpha < \beta \text{ or } \alpha \text{ is to the left of } \beta).$$

When extending an interval $I(\sigma)$ we will arrange the following:

1. Every interval $I(\alpha)$ with $\alpha < \sigma$ and $I(\alpha) \neq \emptyset$ will also be extended at this stage.
2. Every interval $I(\beta)$ with β to the left of σ will not be extended at this stage and will become part of some $I(\alpha)$ with $\alpha \subseteq \sigma$.
3. Every interval $I(\beta)$ with β to the right of σ will be abandoned at this stage.
4. The intervals being extended at this stage will partition the existing linear order and will be ordered as dictated by \mathfrak{B} . That is, if $I(\alpha_1)$ and $I(\alpha_2)$ are being extended and $|\alpha_1| = 2i + 1$ and $|\alpha_2| = 2j + 1$ and $i < j$ in \mathfrak{B} , then $I(\alpha_1) < I(\alpha_2)$.

Definition of $\Gamma_{i,s}$. For each i we define $\Gamma_{i,s}$ inductively:

$$\Gamma_{i,0} = \{0\}.$$

To form $\Gamma_{i,s+1}$ from $\Gamma_{i,s}$

- (1) add $s + 1$ to the end of $\Gamma_{i,s}$, and
- (2) acting in order for $j = 0, 1, 2, \dots, s$, move j to the end of the list if $(j, s) \in W_i$.

It is clear that the lists so defined have the property that if $i \notin C_s$, then there is a stage s such that the least $i + 1$ elements of $I_{i,s}$ (and of every $I_{i,t}$ with $t \geq s$) are actual members of T_i . The choice of a pair x, y for T_i from among these elements need therefore never be renewed.

For each node σ we define σ 's guess as follows: For each $i \in \mathbb{N}$,

- if $\sigma(2i + 1) = 0$, then σ guesses that $i \in B$,
- if $\sigma(2i + 1) = 1$, then σ guesses that $i \notin B$,
- if $\sigma(2i + 2) = 0$, then σ guesses that $i \in C$,
- if $\sigma(2i + 2) = 1$, then σ guesses that $i \notin C$.

We say σ 's guess seems correct at stage s if

- for each i with $2i + 1 \leq |\sigma|$, $(i, s) \in R \Leftrightarrow \sigma$ guesses that $i \in B$,
- for each i with $2i + 2 \leq |\sigma|$, $(i, s) \in S \Leftrightarrow \sigma$ guesses that $i \in C$.

Notice that if σ 's guess seems correct at stage s , then this is also true of every $\alpha \subset \sigma$ and of exactly one of σ^0, σ^1 . Notice also that if σ 's guess is correct, then it will seem correct at infinitely many stages s ; this because of property (*).

Definition of P_s and P . P_s is the unique path consisting of those nodes with length $\leq 2s$ whose guesses seem correct at stage s . Since R and S are recursive, so are the P_s , uniformly in s . P is the true path consisting of those nodes whose guesses are correct, i.e. $\sigma \in P$ if and only if

- for each i with $2i + 1 \leq |\sigma|$, $i \in B \Leftrightarrow \sigma$ guesses that $i \in B$,
- for each i with $2i + 2 \leq |\sigma|$, $i \in C \Leftrightarrow \sigma$ guesses that $i \in C$.

Since B and C are II_2 so is P . Because of property (*), every $\sigma \in P$ will appear on P_s at infinitely many stages s .

The construction may call for the introduction of a new element into the linear order, by this is meant the least element of \mathbb{N} not enumerated in \mathfrak{A} so far.

The Construction at stage s .

Find P_s . We describe the modification of \mathfrak{A}_{s-1} for the sake of P_s by describing it for each of the nodes of P_s in order, beginning with λ . Each of these successive modifications may define different versions of the interval $I(\sigma, s)$ for each node σ . So as not to clutter our notation further and at the risk of creating some confusion we call all these successive versions $I(\sigma, s)$. The "real" $I(\sigma, s)$ will be that defined when the modification on P_s is complete.

The modification for λ : Leave \mathfrak{A}_{s-1} as is. For each α on or to the left of P_s define $I(\alpha, s) = I(\alpha, s - 1)$. For each β below or to the right of P_s define $I(\beta, s) = \emptyset$.

Having modified for a node on P_s consider its successor σ on P_s .

Case 1: $|\sigma| = 2i + 1$ and σ guesses that $i \notin B$. Leave everything as is.

Case 2: $|\sigma| = 2i + 1$; σ guesses that $i \in B$ and $I(\sigma, s) = \emptyset$. The construction will guarantee that the intervals in $\{I(\alpha, s) : \alpha \subset \sigma\}$ either are all empty or form a partition of the existing linear order (see Remark 1). If they are all empty, define $I(\sigma, s)$ to be all of the existing linear order. Otherwise define $I(\sigma, s) = \{a\}$, where a is the new element, and place it among the intervals in $\{I(\alpha, s) : \alpha \subset \sigma\}$ as dictated by \mathfrak{B} . That

is, place $I(\sigma, s)$ to the left of such an $I(\alpha, s)$ if and only if $|\alpha| = 2j + 1$ and $i < j$ in \mathfrak{B} . Leave all else as is.

Case 3: $|\sigma| = 2i + 1$, σ guesses that $i \in B$ and $I(\sigma, s) \neq \emptyset$. Let D_1, D_2 be the intervals on either side of $I(\sigma, s)$ and (strictly) between it and all other intervals in $\{I(\alpha, s) : \alpha \in P_{s,j}\}$. Define $I(\sigma, s) = D_1 \cup I(\sigma, s) \cup D_2$ (where the $I(\sigma, s)$ on the right is the "old version"). Leave all else as is.

Case 4: $|\sigma| = 2i + 2$ and σ guesses the $i \in C$. Leave everything as is.

Case 5: $|\sigma| = 2i + 2$, σ guesses that $i \notin C$ and there are two elements of D lying in some interval $I(\beta, s)$ with $\beta \subset \sigma$; here D is the set consisting of the least $i + 1$ elements of $I_{i,s}$. Leave everything as is.

Case 6: $|\sigma| = 2i + 2$, σ guesses the $i \notin C$ and no two elements of D (defined above) lie in the same interval $I(\beta, s)$ with $\beta \subset \sigma$. Define $I(\alpha, s) = \emptyset$ for all $\alpha \supseteq \sigma$. Introduce into \mathfrak{A} any elements of D not already there, placing them at the extreme left of the linear order. Modify this linear order for P_s all over again.

(This time when we reach σ all elements will be in some $I(\beta, s)$ with $\beta \subset \sigma$, including the $i + 1$ elements of D . Consequently we will now be in Case 5. This is not strictly true but we may assume so without loss of generality, see Remark 2.)

This ends the modification for each node of P_s . Finally, add a new element at each end of each interval $I(\sigma, s)$ with $\sigma \in P_s$, σ of odd length and ending in 0.

This ends the construction at stage s .

In a series of remarks we will prove that for nodes σ of odd length

- if σ is to the left of P , then $I(\sigma)$ will be extended only finitely often;
- if σ is on P and ends in 0, then $I(\sigma)$ will be extended infinitely often;
- if σ is to the right of P , then $I(\sigma)$ will be abandoned infinitely often.

We prove also that every element of \mathfrak{A} eventually enters an $I(\sigma)$ with $\sigma \in P$ and stays in there permanently. The true path P will consequently produce an interval of type $\omega^* + \omega$ for each element of B and \mathfrak{A} will consist solely of these intervals. These intervals will be ordered in such a way that \mathfrak{A} has type $(\omega^* + \omega) \cdot \tau$.

For nodes σ of even length we will prove that

- if σ is to the left of P , then it will be on P_s at only finitely many stages s ;
- if σ is on P and ends in 1, then we will unify intervals $I(\alpha)$ and $I(\beta)$ because of σ only finitely often and then only if either $|\alpha| > |\sigma|$ or $|\beta| > |\sigma|$;
- if σ is to the right of P , any unification of intervals caused by σ will be subsequently forgotten.

Consequently, our action caused by these nodes of even length will not hinder our block-building. Our action for the nodes of even length that are on P and end in 1 will guarantee that no T_i is an infinite subset of a choice set for \mathfrak{A} .

Remarks. Unless otherwise stated, $I(\alpha, s)$ denotes the version defined when our construction at stage s is complete.

1. Addressing the comment in Case 2.

Case 1 guarantees that only the nodes α of odd length, ending in 0 and with $|\alpha| \leq 2s$ may have $I(\alpha, s) = \emptyset$. If α is of odd length and ends in 0 and $I(\beta, s) \neq \emptyset$ for some $\beta \supset \alpha$, then $I(\alpha, s) \neq \emptyset$ also. This is because α is on P_t whenever β is and Case 2 will guarantee that $I(\alpha, t) \neq \emptyset$. It follows that if our modification for $\sigma \in P_s$ puts us in Case 2, then, for every $\beta \supseteq \sigma$, $I(\beta, s) = \emptyset$ at the time of this modification and consequently the intervals in $\{I(\alpha, s) : \alpha \subset \sigma\}$ either are all empty or form a partition of the existing linear order (Cases 2 and 3 applied to these α guarantee this).

2. Addressing the comment in Case 6.

The comment is not strictly true as it may happen that every node above σ that is of odd length ends in 1. In this case there never would be a non-empty interval $I(\beta, s)$ with $\beta \subset \sigma$ for the elements of D to enter. There are many ways to rectify this situation and make the comment true. An obvious one is to arrange that $0 \in B$ (re-numbering two elements of \mathfrak{B} if necessary) and modifying for P_s only if P_s branches left at λ , i.e. only if all $\alpha \in P_s$ have $\alpha(1) = 0$.

3. The construction is effective and produces a recursive linear order \mathfrak{A} .

Remarks 4, 5 and 6 are true at every stage s .

4. If $I(\alpha, s) \cap I(\beta, s) \neq \emptyset$, then we must have $I(\alpha, s) \subset I(\beta, s)$ or $I(\beta, s) \subset I(\alpha, s)$ depending on whether $\alpha < \beta$ or $\beta < \alpha$ in \mathfrak{A} .

Proof. It is only in Case 3, when defining $I(\beta, s)$ say, that we could produce $I(\alpha, s) \cap I(\beta, s) \neq \emptyset$. In that case we must have α lying to the left of P_s ; consequently we define $I(\alpha, s) = I(\alpha, s-1)$ and (assuming Remark 4 is true at all stages previous to s) we would arrange that $I(\alpha, s) \subset I(\beta, s)$.

5. The intervals in $\{I(\sigma, s) : \sigma \in P_s\}$ either are all empty or they partition \mathfrak{A}_s and are ordered as dictated by \mathfrak{B} , i.e. for $\alpha, \beta \in P_s$, $I(\alpha, s) < I(\beta, s)$ in \mathfrak{A}_s if and only if $i < j$ in \mathfrak{B} and $|\alpha| = 2i + 1$ and $|\beta| = 2j + 1$.

Proof. The construction in Case 2 guarantees that the order is as stated. That of Case 3 guarantees the rest (see Remark 1).

6. Following directly from the previous remarks we have that each $I(\alpha, s)$ is an interval in \mathfrak{A}_s .

Remarks 7 and 8 follow from the definition of R and S ; property (*) is used in 8.

7. If α is to the left of P , then $(\exists s)(\forall t \geq s)(\alpha \notin P_t)$.

8. If $\alpha \in P$, then $Qs(\alpha \in P_s)$.

9. For every $\sigma \in P$ there is a stage s such that

$$(\forall \alpha \subseteq \sigma)(I(\alpha, s) \subseteq I(\alpha, s+1) \subseteq I(\alpha, s+2) \subseteq \dots).$$

Proof. Let s be any stage satisfying

- (i) $\sigma \in P_s$ (see Remark 8);
- (ii) for each of the nodes β to the left of σ we have $(\forall t \geq s)(\beta \notin P_t)$ (see Remark 7);
- (iii) for every node $\alpha \subseteq \sigma$ with α of even length and ending in 1, if $|\alpha| = 2i + 2$, then the least $i + 1$ elements of $\Gamma_{i,s}$ are truly elements of T_i (since $\alpha \in P$ and ends in 1; $i \notin C$).

Because of (iii) any construction via Case 6 at this stage will never be renewed; consequently Remark 9 holds with this choice of s .

10. For every $a \in |\mathfrak{A}| (= \mathbb{N})$ there is an $\alpha \in P$ and a stage s with

$$a \in I(\alpha, s) \subseteq I(\alpha, s + 1) \subseteq I(\alpha, s + 2) \subseteq \dots$$

Proof. Let s_0 be a stage when $a \in |\mathfrak{A}_{s_0}|$ and let $\sigma \in P$ be a node of odd length that ends in 0 but has $I(\sigma, s_0) = \emptyset$, one with length $> 2s_0$ for example. At every stage t with $I(\sigma, t - 1) = \emptyset$ but $I(\sigma, t) \neq \emptyset$ there will be an $\alpha \subseteq \sigma$ with $a \in I(\alpha, t)$ (see Remarks 1 and 5). Taking s to be any stage with $I(\sigma, s - 1) = \emptyset$ and $\emptyset \neq I(\sigma, s) \subseteq I(\sigma, s + 1) \subseteq I(\sigma, s + 2) \subseteq \dots$ (see Remark 9), we have Remark 10.

11. \mathfrak{A} has type $(\omega^* + \omega) \cdot \tau$.

Proof. Let $i \in B$ and let σ be the node on P with $|\sigma| = 2i + 1$ (σ ends in 0). Remarks 6, 8 and 9 and the fact that we extend $I(\sigma, s)$ at the ends only, guarantee that $I(\sigma)$ is an interval of type $\omega^* + \omega$. For every other node α , $I(\alpha)$ is finite or empty. (Remarks 7 and 8 guarantee that either $I(\alpha)$ stops being extended after a stage or is reduced to \emptyset infinitely often.) Remarks 5 and 10 guarantee that these intervals partition \mathfrak{A} and are ordered as dictated by \mathfrak{B} .

12. No T_i is an infinite subset of a choice set for \mathfrak{A} .

Proof. If $i \in C$, we have nothing to prove. Let $i \notin C$, let σ be the node on P with $|\sigma| = 2i + 2$ (σ ends in 1) and let s be the stage described in the proof of Remark 9. The construction at this stage will guarantee that there are two elements of T_i in the same $\omega^* + \omega$ block of \mathfrak{A} (see Remark 2).

This ends our proof of Theorem 1. \square

We observe that our construction may be relativized to prove:

Every Π_n discrete linear order has a Π_n copy, all of whose choice sets have no infinite Σ_{n+2} subset (consequently, the copy has no strongly non-trivial Π_{n+1} self-embedding).

We observe also that the construction may be easily modified to discuss linear orders of type $\omega \cdot \tau$ or $\omega^* \cdot \tau$.

We have shown that every recursive discrete linear order has a recursive copy with no strongly non-trivial Π_1 automorphism. The questions as to precisely which order types are such that all of their recursive copies have strongly non-trivial (fairly trivial) Π_1 -automorphisms (self-embeddings) remain open. In [4] KIERSTEAD discusses some of these questions offering conjectures as to their solutions. We support these with a few of our own:

Conjecture 1. (KIERSTEAD [4]). *Every recursive copy of order-type τ has a strongly non-trivial Π_1 automorphism if and only if τ has an interval of type η .*

This conjecture is supported by our Theorem 1 and that of KIERSTEAD [4] in which he constructs a recursive copy of $2 \cdot \eta$ with no (strongly) non-trivial Π_1 automorphism. We observe that this result may be strengthened to

If $f: \mathbb{Q} \rightarrow \mathbb{N} - \{0\}$ is a recursive function and the order-type $\tau = \sum\{f(q): q \in \mathbb{Q}\}$ has no interval of type η , then there is a recursive copy of τ which has no infinite Σ_2 subset consisting only of elements that have an immediate predecessor.

Every (strongly) non-trivial Π_1 automorphism would guarantee such a subset. If g is one such automorphism and x is an element with $x \neq g(x)$ then, since τ has no interval of type η , there must be an element y between x and $g(x)$ that has an immediate predecessor. Then $\{y, g(y), g^2(y), \dots\}$ is an infinite Σ_2 subset consisting only of elements that have an immediate predecessor. Notice that such a subset is not guaranteed by a self-embedding g .

The basic strategy to construct the recursive copy of τ is to define, at each stage s , blocks $I(0), I(1), \dots, I(s)$ of size $f(0), f(1), \dots, f(s)$ respectively and order them as $0, 1, \dots, s$ are in \mathbb{Q} . Meanwhile, diagonalize, as in Theorem 1, over all the candidates for Σ_2 sets T_0, T_1, T_2, \dots .

To deal with T_i we look for the least element of $T_{i,s}$, let it be a member of the block $I(s)$, say. If it is the initial element of $I(s)$, we do nothing. If not, we make it the initial element of some block by "destroying" $I(s)$ as follows. Find enough elements ($f(s)$ of them) that are ordered (in \mathbb{Q}) with respect to the elements $0, 1, \dots, s-1$ just as s is (\mathbb{Q} is dense). Build the block for each of these elements with size dictated by f and order by \mathbb{Q} , using the elements of $I(s)$ as the initial elements of these new blocks. We must of course ensure that when acting against T_i we do not destroy any block $I(j)$ with $j \leq i$. Using the $2^{<\omega}$ tree Λ and the set C as we did in Theorem 1 allows us to arrange that each T_i is either too small or includes an element that has no immediate predecessor.

This last result may be extended to cope with any Π_2 function $f: \mathbb{Q} \rightarrow \mathbb{N} - \{0\}$ by meshing, as in Theorem 1, the strategy described above with the block-building strategy used by FELLNER in [2] to construct a recursive copy of $\sum\{f(q): q \in \mathbb{Q}\}$ from any Π_2 function f . As this result is but a small step along the way to proving Conjecture 1 we prefer to leave the details of the construction to the reader.

As a matter of interest we observe that it is not known whether every recursive linear order of type $\sum\{h(q): q \in \mathbb{Q}\}$ (for some function $h: \mathbb{Q} \rightarrow \mathbb{N} - \{0\}$) is isomorphic to some $\sum\{f(q): q \in \mathbb{Q}\}$ with f a Π_2 function. It is known (FELLNER [2]) that there is always a Δ_3 function f that does the job; however, LERMAN and ROSENSTEIN show in [5] that there is a Δ_3 function $f: \mathbb{Q} \rightarrow \mathbb{N} - \{0\}$ such that $\sum\{f(q): q \in \mathbb{Q}\}$ has no recursive copy.

Conjecture 2. *Every recursive copy of τ has a fairly trivial Π_1 automorphism if and only if τ has elements x, y with (x, y) consisting only of blocks of type $\omega^* + \omega$.*

Conjecture 3. *Every recursive copy of τ has a strongly non-trivial Π_1 self-embedding if and only if τ has elements x, y with (x, y) an infinite interval consisting only of blocks of size $\leq k$ (for some fixed finite k).*

Conjecture 4. *Every recursive copy of τ has a fairly trivial Π_1 self-embedding if and only if τ has elements x, y with (x, y) consisting only of blocks of type ω or $\omega^* + \omega$ or only of blocks of type ω^* or $\omega^* + \omega$.*

References

- [1] DUSHNIK, B., and E. W. MILLER, Concerning similarity transformations of linearly ordered sets. Bull. Amer. Math. Soc. 46 (1940), 322-326.
- [2] FELLNER, S., Recursiveness and finite axiomatizability of linear orderings. Thesis, Rutgers University, N.J., 1976.

- [3] JOCKUSCH, JR., C. G., Reducibilities in recursive function theory. Thesis MIT, Cambridge, Mass., 1966.
- [4] KIERSTEAD, H. A., On Π_1 automorphisms of recursive linear orders. To appear.
- [5] LEBMAN, M., and J. G. ROSENSTEIN, Recursive linear orderings. In: Proceedings, Patras Logic Symposium (ed. G. METAKIDES), North-Holland Publ. Comp., Amsterdam 1980, pp. 123-136.
- [6] ROSENSTEIN, J. G., Linear Orderings. Academic Press, London-New York 1982.

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