Annals of Pure and Applied Logic 27 (1984) 253–264
North-Holland

RECURSIVE LINEAR ORDERS WITH RECURSIVE SUCCESSIVITIES*

Michael MOSES

Department of Mathematics, Western Illinois University, Macomb, IL 61455, USA

Communicated by A. Nerode
Received 1 July 1983

A *successivity* in a linear order is a pair of elements with no other elements between them. A recursive linear order with recursive successivities $\mathfrak{A}$ is *recursively categorical* if every recursive linear order with recursive successivities isomorphic to $\mathfrak{A}$ is in fact recursively isomorphic to $\mathfrak{A}$. We characterize those recursive linear orders with recursive successivities that are recursively categorical as precisely those with order type $k_1+g_1+k_2+g_2+\cdots+g_{n-1}+k_n$ where each k_i is a finite order type, non-empty for $i\in\{2,\ldots,n-1\}$ and each g_i is an order type from among $\{\omega, \omega^*, \omega+\omega^*\}\cup\{k\cdot\eta : k<\omega\}$.

1. Introduction

A structure $\mathfrak{A}$ is said to be *recursive* if it has a recursive universe A, and the atomic formulae uniformly denote recursive relations. Two such structures $\mathfrak{A}$, $\mathfrak{B}$ on recursive universes A, B respectively, are *recursively isomorphic* if there is a recursive function $f: A \to B$ which is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. A recursive structure $\mathfrak{A}$ is *recursively categorical* if every recursive structure isomorphic to $\mathfrak{A}$ is also recursively isomorphic to $\mathfrak{A}$.

Many results characterizing recursively categorical models of various theories have been obtained. Metakides and Nerode [7] considered algebraic closures of a given field; Boolean algebras were considered independently by LaRoche [5] and Goncharov [3] and Abelian p-groups by Smith [11]. In addition to linear orders [9], Remmel studied recursive Boolean algebras with recursive atoms [8] and together with Manaster in [6] dense two-dimensional partial orderings. Schwarz [10] characterized recursively categorical recursive linear orders with the block relation recursive; and Goncharov [4], structures with a language with only unary predicates.

In [2] Dzgoev and Goncharov introduce a property ('branching') satisfied, in a recursive structure, by certain formulae. Their main result is that if there is a universal formula that branches in a recursive structure, then the structure is not recursively categorical. In [2] and [4] this general result is used to obtain many of the above-mentioned characterizations.

* These results form part of the author's Ph.D. Thesis "Recursive properties of isomorphism types" presented at Monash University, Australia; under the supervision of John Crossley.

In [9] Remmel leaves open the problem of characterizing those recursive linear orders with recursive successivities that are recursively categorical. In Theorem 4 we characterize them as precisely those that can be partitioned, by a finite number of points, into intervals, each of which either has finitely many blocks or is of order type $k \cdot \eta$ for some finite k.

2. Intrinsically recursive relations

A relation R on a recursive structure $\mathfrak{A}$ is *intrinsically r.e.* on $\mathfrak{A}$ if for every recursive structure $\mathfrak{B}$ and isomorphism $g:\mathfrak{B}\cong\mathfrak{A}$, $g^{-1}(R)$ is r.e.

In [1] Ash and Nerode present a condition sufficient for a relation R to be intrinsically r.e. on a structure $\mathfrak{A}$. Assuming a certain amount of extra decidability of $\mathfrak{A}$, they show this condition to be also necessary. We present a condition necessary for R to be intrinsically r.e. on $\mathfrak{A}$ which does not require this decidability assumption. The condition is in general not sufficient, but will enable us to obtain results in certain cases.

The structure $\mathfrak{A}$ will have as its universe the recursive set $A=\{a_0, a_1, \ldots\}$. We write A_s for the set $\{a_0, \ldots, a_s\}$ and define $\mathfrak{A}_s$ to be the recursive structure obtained by restricting the relations of $\mathfrak{A}$ to the set A_s.

We use $\langle m, s\rangle$ to denote ordered pairs of integers.

Theorem 1. *If $\mathfrak{A}$ is a recursive structure with language consisting solely of a finite number of predicate symbols, and R is a recursive relation on $\mathfrak{A}$, then* (1) *implies* (2).

(1) *There is a recursive function f from $\mathbb{N}^2$ into $\mathbb{N}$ such that for every $m\in\mathbb{N}$ there is a sequence $\bar{a}\in R$ for which there are infinitely many $s\in\mathbb{N}$ with embeddings $\phi:\mathfrak{A}_s\to\mathfrak{A}_{f\langle m,s\rangle}$ with ϕ the identity on A_m and $\phi(\bar{a})\notin R$.*

(2) *R is not intrinsically r.e. on $\mathfrak{A}$.*

Proof. We shall construct a recursive structure $\mathfrak{B}$ and an isomorphism $g:\mathfrak{B}\cong\mathfrak{A}$ so that $g^{-1}(R)$ is not r.e. $\mathfrak{B}$ will have the recursive universe $B=\{b_0, b_1, \ldots\}$ and we write B_s for $\{b_0, b_1, \ldots, b_s\}$. At each stage s of the construction we shall define an integer $s'\geq(s-1)'$ and a bijection $g_s:B_{s'}\to A_{s'}$. To simplify notation we take R to be a one-place relation. Let $W_0, W_1, \ldots$ be a list of all r.e. subsets of B; that is, a list of all candidates for $g^{-1}(R)$. W_e^s is the part of W_e enumerated by stage s. Our construction will ensure that we meet the following list of requirements for $e=0, 1, 2, \ldots$.

$$Q_e: W_e\neq g^{-1}(R).$$

For some e and s we shall define $d_e^s\in B_{s'}\cap W_e^s$ with the intention that if $W_e\supseteq g^{-1}(R)$, then $d_e=\lim_s d_e^s$ will exist and lie in $W_e-g^{-1}(R)$.

The following phrases are used in the description of the construction.

Q_e *requires attention* at stage $s+1$ if $e \leq s+1$, $A_e \subseteq A_{s'}$, $B_e \subseteq B_{s'}$ and d_e^s is undefined.

Q_e *is injured at stage* $s+1$ if $g_{s+1}(d_e^s) \neq g_s(d_e^s)$. In this case we say d_e^{s+1} is undefined. Otherwise we define d_e^{s+1} to be d_e^s.

An element $b \in B_{s'}$ *may be used to attack* Q_e at stage $s+1$ if

(i) $b \in W_e^s$, and

(ii) there is an embedding $\phi : \mathfrak{A}_{s'} \to \mathfrak{A}_{f\langle m,s'\rangle}$ such that ϕ is the identity on A_m and $\phi(g_s(b)) \notin R$.

(Here m is the maximum of n_1, n_2 with n_1 = the least integer with $A_e \cup g_s(B_e) \subseteq A_{n_1}$, and n_2 = the last stage when a requirement $Q_{e'}$ with $e' < e$ was attacked.)

We now describe the construction.

Stage 0. Define $0' = 0$ and $g_0 : b_0 \to a_0$.

Stage $s+1$. Check if there is a Q_e requiring attention at stage $s+1$ which may be attacked at stage $s+1$. If there is no such Q_e define $(s+1)' = s'+1$, extend g_s to $g_{s+1} : B_{(s+1)'} \to A_{(s+1)'}$ in the obvious way, and go to the next stage. If there is such a Q_e, then choose the least one and the least b (in the listing of B) which may be used to attack this Q_e at stage $s+1$ and do so in the following manner. Define $(s+1)' = f\langle m, s'\rangle$ and g_{s+1} to be $g_s \circ \phi$ on $B_{s'}$, and extend it in any way to a bijection from $B_{(s+1)'}$ to $A_{(s+1)'}$. Define d_e^{s+1} to be b and go to the next stage.

This completes the description of the construction. Notice that if there is a stage after which no requirement from among $Q_0, Q_1, \ldots, Q_{e-1}$ is ever attacked, then after this stage the requirement Q_e is never injured. It follows, by induction on e, that each requirement Q_e is attacked at most finitely many times. This implies that $g = \lim_s g_s$ exists and is a bijection from B to A. If we consider $\mathfrak{B}_{s'}$ to be the recursive structure defined on $B_{s'}$ in such a way as to make $g_s : \mathfrak{B}_{s'} \to \mathfrak{A}_{s'}$ an isomorphism, we notice that the diagram of $\mathfrak{B}_{s'}$ is contained in that of $\mathfrak{B}_{(s+1)'}$ for every s. This ensures that $\mathfrak{B} = \lim_s \mathfrak{B}_s$ is a recursive structure and g is an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$.

We now prove that all the requirements Q_e are met. Suppose not. Let e be the least with $W_e = g^{-1}(R)$. Let s be a stage after which no requirement $Q_{e'}$ with $e' < e$ is attacked. Consider the A_m in the definition of "b may be used to attack Q_e". For Q_e this m remains fixed after stage s. Consider any element a of R satisfying the hypothesis of our theorem for this m. Go to a stage when g has taken on its final value on a; $g^{-1}(a) = b$, say. Since $W_e = g^{-1}(R)$, at some further stage b will enter W_e and it is clear that at some stage after this b may be used to attack Q_e. Thus Q_e would be attacked and never injured, contradicting the assumption that $W_e = g^{-1}(R)$.

3. 1-Recursive linear orders

In this section we consider the class of 1-recursive linear orders. A linear order is 1-*recursive* if it has a recursive universe and the existential formulae uniformly denote recursive relations. We begin by proving that a linear order is 1-recursive if and only if it is a recursive linear order with the successivity relation recursive.

We begin with some terminology. For elements a and b of a linear order, we write (a, b) for the open interval between a and b. That is, the set

$$\{x: a<x<b\} \cup \{x: b<x<a\}.$$

For $n<\omega$, $S_n(x, y)$ denotes the binary relation satisfied by any pair a, b with exactly n elements between them. That is

$$S_n(a, b) \Leftrightarrow (a \neq b \text{ and } |(a, b)| = n).$$

The relation $S_0(x, y)$ is therefore satisfied by any pair of elements between which there are no other elements. This relation will play a particularly important role in what is to follow. We write $S(x, y)$ for $S_0(x, y)$ and call it the *successivity relation.* We define the *block relation* $B(x, y)$ as that satisfied by any pair a, b between which there are only finitely many elements; that is, with $a \neq b$ and with (a, b) finite or empty. It is clear that the following equivalence holds in the theory of linear order.

$$B(x, y) \leftrightarrow \bigvee_{i=0,1,2,\ldots} S_i(x, y).$$

The *block containing* an element a is the set of elements separated from a by at most finitely many other elements; that is, the set

$$\{a\} \cup \{x: B(a, x)\}.$$

By an *arrangement* of the variables $x_1, \ldots, x_n$ we mean a finite conjunction of the form

$$\psi = \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_{n-1},$$

where $y_1, \ldots, y_n$ is some permutation of $x_1, \ldots, x_n$ and each formula θ_i is either $y_i < y_{i+1}$ or $y_i = y_{i+1}$. We first show that every quantifier-free formula $\sigma(x_1, \ldots, x_n)$ is (either trivial or) equivalent to the disjunction of a finite number of arrangements of the variables $x_1, \ldots, x_n$. It is not difficult to see that if ψ is an arrangement of $x_1, \ldots, x_n$, then one of the implications $\psi \rightarrow \sigma$ or $\psi \rightarrow \neg\sigma$ is a consequence of the theory of linear order. Notice that there are a finite number of arrangements of the variables $x_1, \ldots, x_n$. It follows that σ is equivalent to the disjunction of all those arrangements ψ for which the implication $\psi \rightarrow \sigma$ holds. Thus every quantifier-free formula $\sigma(x_1, \ldots, x_n)$ is equivalent to a disjunction $\psi_1 \vee \cdots \vee \psi_m$ of a finite number of arrangements of the variables $x_1, \ldots, x_n$.

Consider now an existential formula $\exists \bar{x}\, \sigma(\bar{x}, \bar{y})$. It follows that $\exists \bar{x}\, \sigma(\bar{x}, \bar{y})$ is equivalent to the finite disjunction

$$\exists \bar{x}\, \psi_1(\bar{x}, \bar{y}) \vee \cdots \vee \exists \bar{x}\, \psi_m(\bar{x}, \bar{y}),$$

where each ψ_i is an arrangement of the variables $\bar{x}$, $\bar{y}$. Consider any one of these formulae $\exists\bar{x}\,\psi(\bar{x}, \bar{y})$. It is not difficult to see that we can remove the variables $\bar{x}$ from the formula by replacing it with a finite conjunction of formulae which
4 represent some of the following statements (for some elements y_i, y_j of $\bar{y}$ and integers n).

(i) $y_i = y_j$.
(ii) $y_i < y_j$.
(iii) There are at least n elements less than y_i.
(iv) There are at least n elements greater than y_i.
(v) There are at least n elements between y_i *and* y_j.

All of the above may be done in an effective manner. Consider now a recursive linear order $\mathfrak{A}$ with the successivity relation recursive. It is clear that (i) and (ii) represent recursive relations in $\mathfrak{A}$. Since $\mathfrak{A}$ has at most one least and at most one greatest element and since $S(x, y)$ is recursive, (iii) and (iv) also represent recursive relations. The fact that $S(x, y)$ is recursive implies the same of (v). We therefore have the following result.

A linear order is 1-*recursive if and only if it has a recursive universe and the relations* $\leqslant$ *and* $S(x, y)$ *are both recursive.*

We define for a class A of order types, a class $\Delta(A)$ of order types as follows. A linear order has order type in $\Delta(A)$ if it has a finite number of points $p_0 < p_1 < \cdots < p_m$, such that each one of the intervals

$$(-\infty, p_0), (p_0, p_1), \ldots, (p_{m-1}, p_m), (p_m, \infty)$$

is either finite or has order type in A. By $(-\infty, a)$ and (a, ∞) we mean the open intervals to the left and right of a, respectively. $\mathfrak{A} \in \Delta(A)$ will mean the order type of $\mathfrak{A}$ is in $\Delta(A)$.

We use ω and ω^* to represent the order types of the positive and negative integers respectively. The order type of the rationals is η and for $k < \omega$, $k \cdot \eta$ is the order type of the structure obtained by replacing each point in the rationals with a block of length k.

Notice that $\Delta(\{\omega, \omega^*, \omega + \omega^*\})$ is the class of all order types with only finitely many blocks (each of which is finite or is ω or ω^*). A 1-recursive linear order in this class also has the block relation $B(x, y)$ recursive. This is because we can list the elements of the separate blocks by choosing an element from each block and then enumerating successivities.

For $k < \omega$ we write F_k for the class of all order types with all blocks of length $\leqslant k$. We write F for the class of all order types with no blocks of infinite length. Notice that $F \supseteq \bigcup_{k<\omega} F_k$, but they are not equal. We write $F_{<\omega}$ for $\bigcup_{k<\omega} F_k$.

Consider a linear order $\mathfrak{A}$ and a subset M of A. We say a block (of $\mathfrak{A}$) is *represented in* M if some element of this block is in M. We say $c_0, \ldots, c_m$ is the *ordering of* M *in* $\mathfrak{A}$ *if* $c_0 < c_1 < \cdots < c_m$ in $\mathfrak{A}$ and $M = \{c_0, \ldots, c_m\}$.

Theorem 2. *If $\mathfrak{A}$ is a 1-recursive linear order with the block relation recursive the following are equivalent.*

(1) $\mathfrak{A} \in \Delta\,(F_{<\omega} \cup \{\omega, \omega^*, \omega+\omega^*\})$.

(2) *Every 1-recursive linear order isomorphic to $\mathfrak{A}$ has the block relation recursive.*

Proof. It is easy to see that (1) implies (2). This follows from the fact that in a linear order in F_k, elements a and b are in separate blocks if and only if there are (at least) k elements between them. Consider a structure $\mathfrak{A}$ not in $\Delta(F_{<\omega} \cup \{\omega, \omega^*, \omega+\omega^*\})$. We shall use Theorem 1 to show that the relation $B(x, y)$ is not intrinsically recursive on $\mathfrak{A}$. The role of the relation R will be played by $\neg B(x, y)$. The function f from $\mathbb{N}^2$ into $\mathbb{N}$ is defined as follows. For $\langle m, s\rangle \in \mathbb{N}^2$, if $m \geqslant s$ define $f\langle m, s\rangle = s$. If $m < s$ let $c_0, c_1, \ldots, c_m$ be the ordering of A_m in $\mathfrak{A}$ and let c_{-1} and c_{m+1} denote $-\infty$ and ∞ respectively. For $i \in \{0, \ldots, m+1\}$ and $t \geqslant s$ we say that (c_{i-1}, c_i) is *large in* $\mathfrak{A}_t$ if there are at least four blocks represented in $(c_{i-1}, c_i) \cap A_t$. Define $f\langle m, s\rangle$ to be the least $t > s$ for which either

(i) there is an $i \in \{0, \ldots, m+1\}$ with (c_{i-1}, c_i) not large in $\mathfrak{A}_s$ but large in $\mathfrak{A}_t$; or

(ii) there is an $i \in \{0, \ldots, m+1\}$ with (c_{i-1}, c_i) large in $\mathfrak{A}_s$ and with $(c_{i-1}, c_i) \cap A_t$ containing a block of length at least $2s$.

We show that f is defined on every pair $\langle m, s\rangle \in \mathbb{N}^2$. Consider the ordering $c_0, \ldots, c_m$ of A_m in $\mathfrak{A}$. Under the assumption that statement (1) of our theorem is false it follows that there is an $i \in \{0, \ldots, m+1\}$ such that the interval (c_{i-1}, c_i) contains an infinite number of blocks of length greater than $2s$. This interval will then satisfy either (i) or (ii) and so define $f\langle m, s\rangle$. Each $f\langle m, s\rangle$ can be found effectively since $\mathfrak{A}$ is 1-recursive and has the block relation recursive. Thus f is a recursive function from $\mathbb{N}^2$ into $\mathbb{N}$. We now show that f satisfies the rest of the hypothesis of Theorem 1.

Consider any natural number m. For this m, $f\langle m, s\rangle$ is defined via (i) for only finitely many s; since if (c_{i-1}, c_i) is large in $\mathfrak{A}_t$, it is large in $\mathfrak{A}_{t+1}$ also. Thus $f\langle m, s\rangle$ is defined via (ii) for infinitely many s. It follows that there is an $i \in \{0, \ldots, m+1\}$ such that the interval (c_{i-1}, c_i) satisfies (ii) for infinitely many s (and $t = f\langle m, s\rangle$). Consider any pair a, b in this interval with c_{i-1}, a, b and c_i all from separate blocks. It is clear that for infinitely many s there is an embedding $\phi: \mathfrak{A}_s \to \mathfrak{A}_{f\langle m,s\rangle}$, with ϕ the identity on A_m and $\phi(a)$ and $\phi(b)$ in the same block. Thus f satisfies the hypothesis of Theorem 1. The relation $B(x, y)$ is therefore not intrinsically recursive on $\mathfrak{A}$; that is, there is a 1-recursive linear order isomorphic to $\mathfrak{A}$ with the block relation non-recursive. This proves our result.

In a linear order, the block containing an element a is the set $\{a\} \cup \{x : B(a, x)\}$. For $k < \omega$, we define the relation $M_k(x)$ as that satisfied by any element a with exactly k elements in the block containing a.

Theorem 3. *Let $\mathfrak{A} \in F$ be a 1-recursive linear order. If the relations $M_i(x)$ for $i < \omega$*

are uniformly recursive on $\mathfrak{A}$, *then the following are equivalent.*

(1) $\mathfrak{A} \in \Delta(\{k \cdot \eta : k < \omega\})$.

(2) *Every* 1-*recursive linear order isomorphic to* $\mathfrak{A}$ *has the relations* $M_i(x)$ *for* $i < \omega$, *uniformly recursive.*

Proof. Evidently (1) implies (2). We shall prove the converse by repeated applications of Theorem 1. Notice that if (2) is true, then since $\mathfrak{A} \in F$, every 1-recursive linear order isomorphic to $\mathfrak{A}$ has the block relation recursive. By Theorem 2, this implies that $\mathfrak{A} \in F_n$ for some finite n. Consider a structure $\mathfrak{A}$ not in $\Delta(F_{n-1} \cup \{n \cdot \eta\})$. We shall apply Theorem 1 to show that (2) is false.

The role of the relation R is played by $\neg M_n(x)$. We define a function f from $\mathbb{N}^2$ into $\mathbb{N}$ satisfying the hypothesis of Theorem 1. For $\langle m, s\rangle \in \mathbb{N}^2$, if $m \geq s$ define $f\langle m, s\rangle = s$. If $m < s$ consider $c_0, \ldots, c_m$, the ordering of A_m in $\mathfrak{A}$ and write c_{-1} and c_{m+1} for $-\infty$ and ∞ respectively. For $i \in \{0, \ldots, m+1\}$ and $t \geq s$ we say that the interval (c_{i-1}, c_i) is *large in* $\mathfrak{A}_t$ if there are at least three blocks of length less than n represented in $(c_{i-1}, c_i) \cap A_t$. Define $f\langle m, s\rangle$ to be the least $t > s$ for which either

(i) there is an $i \in \{0, \ldots, m+1\}$ with (c_{i-1}, c_i) not large in $\mathfrak{A}_s$ but large in $\mathfrak{A}_t$; or

(ii) there is an $i \in \{0, \ldots, m+1\}$ with (c_{i-1}, c_i) large in $\mathfrak{A}_s$ and with $(c_{i-1}, c_i) \cap A_t$ containing at least $2s$ blocks of length n.

Consider any $m \in \mathbb{N}$. It follows from the assumption that $\mathfrak{A} \notin \Delta(F_{n-1} \cup \{n \cdot \eta\})$ that there is an $i \in \{0, \ldots, m+1\}$ such that the interval (c_{i-1}, c_i) contains an infinite number of blocks of length n and an infinite number of length less than n. Arguing from this fact we can show, as in the previous theorem, that f is a recursive function from $\mathbb{N}^2$ into $\mathbb{N}$. Again as in the previous theorem, it follows that there is an interval (c_{i-1}, c_i) which satisfies (ii) for infinitely many s (and $t = f\langle m, s\rangle$). Any element b of this interval (c_{i-1}, c_i) with c_{i-1}, b and c_i in separate blocks and b in a block of length less than n, will satisfy the hypothesis of Theorem 1 for this m.

We have deduced from the assumption that the relation $M_n(x)$ is intrinsically recursive on $\mathfrak{A}$, that

$$\mathfrak{A} \in \Delta(F_{n-1} \cup \{n \cdot \eta\}).$$

This means that there are points $p_0 < p_1 < \cdots < p_m$ in $\mathfrak{A}$ such that each one of the intervals

$$(-\infty, p_0), (p_0, p_1), \ldots, (p_{m-1}, p_m), \quad (p_m, \infty)$$

has order type in $F_{n-1} \cup \{n \cdot \eta\}$. Consider the 1-recursive linear order $\mathfrak{A}'$ formed by removing those of order type $n \cdot \eta$. It is clear that the relations $M_1(x), \ldots, M_{n-1}(x)$ are intrinsically recursive on $\mathfrak{A}'$ (since they were on $\mathfrak{A}$). Using Theorem 1 we can show that since M_{n-1} is intrinsically recursive on $\mathfrak{A}'$ and $\mathfrak{A}' \in F_{n-1}$ that

$$\mathfrak{A}' \in \Delta(F_{n-2} \cup \{(n-1) \cdot \eta\});$$

and therefore that

$$\mathfrak{A} \in \Delta(F_{n-2} \cup \{(n-1)\cdot\eta, n\cdot\eta\}).$$

Repeated applications of this argument will prove that

$$\mathfrak{A} \in \Delta(\{k\cdot\eta : k<\omega\}).$$

We use the last few results to characterize by their order types 1-recursive linear orders that are recursively categorical. Notice that if a recursively categorical 1-recursive linear order has the block relation recursive, then it has the block relation *intrinsically* recursive. Theorem 2 would then enable us to describe to some extent the order type of this linear order. Unfortunately, this tactic does not work in general. In [8] Remmel shows that there is a 1-recursive linear order whose isomorphism type contains no copy with the block relation recursive. In Lemma 1 we present a weaker result that is sufficient for our needs.

Lemma 1. *If $\mathfrak{A}$ is a recursively categorical 1-recursive linear order, it has the block relation $B(x, y)$ intrinsically recursive.*

Proof. The tactic is to attempt to construct a 1-recursive linear order $\mathfrak{B}$ isomorphic but not recursively isomorphic to $\mathfrak{A}$. If $\mathfrak{A}$ is recursively categorical this construction will fail and we shall deduce from this that $\mathfrak{A}$ has the block relation recursive.

The recursive universe of $\mathfrak{B}$ will be $B=\{b_0, b_1, \ldots\}$. At each stage s we shall define an integer $s'>(s-1)'$ and a bijection g_s from $B_{s'}$ to $A_{s'}$. The structure $\mathfrak{B}_{s'}$ is defined on $B_{s'}$ in such a way so as to make g_s an isomorphism from $\mathfrak{B}_{s'}$ to $\mathfrak{A}_{s'}$. We shall arrange that the diagram of $\mathfrak{B}_{s'}$ is contained in that of $\mathfrak{B}_{(s+1)'}$ and therefore that $\mathfrak{B}=\lim_s \mathfrak{B}_s$ exists and is a 1-recursive linear order isomorphic to $\mathfrak{A}$. Let $f_0, f_1, \ldots$ be a list of all partial recursive functions from B to A and f_e^s the part of f_e enumerated by stage s. Our construction will attempt to meet the following list of requirements for $e=0, 1, 2, \ldots$.

Q_e: f_e is not an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$.

We define some phrases used in the description. We say a pair of elements a, b *is connected in* $\mathfrak{A}_s$ if there are (a finite number of) points $c_1<c_2<\cdots<c_n$ *in* A_s such that every pair c_i, c_{i+1} is a successivity in $\mathfrak{A}$ and $c_1=a$ and $c_n=b$ (or $c_1=b$ and $c_n=a$).

A_m *allows us to attack* Q_e at stage $s+1$ if there are elements c, d in $B_{s'}$ satisfying the following three properties.

(i) $g_s(c)$ is not connected in $\mathfrak{A}_{s'}$ to any element from among $g_s(d), a_0, \ldots, a_e$, $g_s(b_0), \ldots, g_s(b_e)$.

(ii) c and d are in $\text{dom}(f_e^s)$; $f_e(c)$ and $f_e(d)$ are in $A_{s'}$ and $f_e(c)$ is not connected to $f_e(d)$ in $\mathfrak{A}_{s'}$.

(iii) Either $g_s(c)$ is connected to $g_s(d)$ in $\mathfrak{A}_m$ or $f_e(c)$ is connected to $f_e(d)$ in $\mathfrak{A}_m$.

In case such a pair c, d exists we choose the least such pair and define $A(Q_e)$ to be A_n with n defined as follows. Property (iii) states that at least one of two pairs of elements is connected in $\mathfrak{A}_m$. Let x, y be a pair that is and p, g be the other. Then n is the least integer $\geqslant m$ such that either

(iv) p, g is connected in $\mathfrak{A}_n$; or

(v) $|(p, g) \cap A_n| \geqslant |(x, y) \cap A_n|$.

By (iii) and the fact that $\mathfrak{A}$ is 1-recursive, we can find $A(Q_e)$ effectively (from m). Notice that $A(Q_e)$ allows us to attack Q_e at stage $s+1$.

At each stage s the mapping g_s is defined as follows.

Stage 0. Define $0' = 0$ and $g_0 : b_0 \mapsto a_0$.

Stage $s+1$. Check whether $A_{s'+1}$ allows us to attack a Q_e with $e \leqslant s+1$ which has not yet been attacked. If not, define $(s+1)' = s'+1$ and extend g_s to g_{s+1} in the obvious way. If $A_{s'+1}$ does allow us to attack a Q_e with $e \leqslant s+1$ which has not yet been attacked, let Q_{e_1} be the least such (with respect to e). Find $A(Q_{e_1})$ and check whether $A(Q_{e_1})$ allows us to attack a Q_e with $e < e_1$ which has not yet been attacked. Let Q_{e_2} be the least such. Find $A(Q_{e_2})$ and check whether $A(Q_{e_2})$ allows us to attack a Q_e with $e < e_2$ which has not yet been attacked. Continue this process until we find a Q_e such that it is the least requirement not yet attacked which $A(Q_e)$ allows us to attack at this stage. Let c, d be the least pair satisfying (i), (ii) and (iii) for Q_e and $A(Q_e)$ and attack this Q_e as follows.

Define $A_{(s+1)'}$ to be $A(Q_e)$. If (iv) is false or if $|(p, g) \cap A_n| \neq |(x, y) \cap A_n|$ in (v), then extend g_s in any way to g_{s+1}, consider Q_e attacked and go to the next stage. Otherwise define g_{s+1} in such a way that

(1) $$|(g_s(c), g_s(d)) \cap A_{(s+1)'}| - 1 = |(g_{s+1}(c), g_{s+1}(d)) \cap A_{(s+1)'}|,$$

(2) g_{s+1} is the same as g_s on the points $a_0, \ldots, a_e$ and $b_0, \ldots, b_e$; and

(3) the diagram of $\mathfrak{B}_{s'}$ is contained in that of $\mathfrak{B}_{(s+1)'}$.

It is possible to define g_{s+1} in such a way because c, d satisfy properties (i) and (ii). Consider Q_e attacked and go to the next stage.

Notice that by attacking Q_e via a pair c, d we ensure that the intervals (c, d) and $(f_e(c), f_e(d))$ are of different lengths in $\mathfrak{B}$ and $\mathfrak{A}$ respectively and therefore that Q_e is met. Arguing as in Theorem 1 we see that (since each Q_e is attacked at most once), $g = \lim_s g_s$ exists and is an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$; and that $\mathfrak{B}$ is a 1-recursive linear order. We now show that $\mathfrak{A}$ has the block relation recursive.

$\mathfrak{A}$ is recursively categorical and therefore there is a least e with f_e an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. Consider a stage after which no requirement $Q_{e'}$ with $e' < e$ is ever attacked. After this stage g remains fixed on the elements $a_0, \ldots, a_e$ and $b_0, \ldots, b_e$. Consider a further stage t when all pairs from among $a_0, \ldots, a_e, g_s(b_0), \ldots, g_s(b_e)$ which share the same block in $\mathfrak{A}$ are connected in $\mathfrak{A}_{t'}$. Given elements c_1, c_2 in A we wish to decide whether or not the pair c_1, c_2 share a block in $\mathfrak{A}$. Let $f_e^{-1}(c_1) = d_1$ and $f_e^{-1}(c_1) = d_2$. Perform the construction up to the first stage $m \geqslant t$ when c_1 and c_2 are in $A_{m'} \cap \operatorname{ran}(f_e^m)$ and d_1 and d_2 are in

$B_{m'}$. We show that the pair c_1, c_2 share a block in $\mathfrak{A}$ if and only if either c_1 is connected to c_2 or $g_m(d_1)$ is connected to $g_m(d_2)$ in $\mathfrak{A}_{m'}$.

If either c_1 is connected to c_2 or $g_m(d_1)$ to $g_m(d_2)$ in $\mathfrak{A}_{m'}$ then since f_e and g are isomorphisms from $\mathfrak{B}$ to $\mathfrak{A}$, it is clear that c_1 and c_2 share the same block in $\mathfrak{A}$. If c_1 and c_2 share a block in $\mathfrak{A}$ but c_1 is not connected to c_2 and $g_m(d_1)$ is not connected to $g_m(d_2)$ in $\mathfrak{A}_{m'}$; then there is a *first* stage $s+1(>m)$ when either c_1 is connected to c_2 or $g_{s+1}(d_1)$ to $g_{s+1}(d_2)$ in $\mathfrak{A}_{(s+1)'}$. We show that $A_{(s+1)'}$ allows us to attack Q_e via the elements d_1, d_2 at this stage. By the choice of $s+1$ and the fact that $s \geqslant t$ we see that property (i) is satisfied. Again the choice of $s+1$ implies that (ii) and (iii) are satisfied. Therefore requirement Q_e would be attacked at stage $s+1$, contradicting the choice of e.

Once we have selected the stage t, we can for any pair c_1, c_2 in A effectively find the stage m. We therefore have an effective procedure for deciding whether or not a pair of elements in A satisfies the block relation in $\mathfrak{A}$.

Define the *predecessor relation* $P(x)$ to be that satisfied by any element a with an immediate predecessor b; that is, $b<a$ and $S(b, a)$. The *successor relation* $S(x)$ is satisfied by any element with an immediate successor.

Lemma 2. *Let $\mathfrak{A}$ be a recursive linear order with no blocks of infinite length. If $\mathfrak{A}$ has the block relation recursive, then the isomorphism type of $\mathfrak{A}$ contains*

(1) *a 1-recursive linear order with the relations $B(x, y)$ and $P(x)$ both recursive, and*

(2) *a 1-recursive linear order with the relations $B(x, y)$ and $S(x)$ both recursive.*

Proof. We construct a recursive linear order with recursive successivities, $\mathfrak{B}$, isomorphic to $\mathfrak{A}$ such that $\mathfrak{B}$ has the predecessor relation $P(x)$ recursive. $\mathfrak{B}$ will have recursive universe $B=\{b_0, b_1, \ldots\}$. At each stage s we shall define a bijection $g_s: B_s \to A_s$. We define on each B_s a structure $\mathfrak{B}_s$ with the relations $\leqslant$, $S(x, y)$ and $P(x)$ recursive as follows.

For a, b in B_s define:

(i) $a \leqslant b$ in $\mathfrak{B}_s$ if and only if $g_s(a) \leqslant g_s(b)$ in $\mathfrak{A}$,

(ii) $S(a, b)$ in $\mathfrak{B}_s$ if and only if there are no elements between a and b in $\mathfrak{B}_s$, and $\mathfrak{A} \models B(g_s(a), g_s(b))$,

(iii) $\neg P(a)$ in $\mathfrak{B}_s$ if and only if for every $c<a$ in $\mathfrak{B}_s$, $g_s(c)$ and $g_s(a)$ are in separate blocks in A.

These relations are clearly recursive. We ensure that the diagram of $\mathfrak{B}_s$ is contained in that of $\mathfrak{B}_{s+1}$, and therefore that $\mathfrak{B} = \lim_s \mathfrak{B}_s$ is a recursive structure. We define g_s as follows.

Stage 0. Define $g_0: b_0 \to a_0$.

Stage $s+1$. Consider D, the block containing a_{s+1}. If $D \cap A_s$ is empty, then define $g_{s+1}(b_{s+1}) = a_{s+1}$ and let g_{s+1} be the same as g_s on the elements of B_s.

If $D \cap A_s$ is non-empty, let $c_1 < c_2 < \cdots < c_n$ be the ordering of $g_s^{-1}(D \cap A_s)$ in

$\mathfrak{B}_s$ and let $d_1 < d_2 < \cdots < d_{n+1}$ be the ordering of $D \cap A_{s+1}$ in $\mathfrak{A}$. Define $g_{s+1}(c_i) = d_i$ for $i = 1, \ldots, n$, $g_{s+1}(b_{s+1}) = d_{n+1}$, and let g_{s+1} be the same as g_s on all other elements.

It is clear that the diagram of $\mathfrak{B}_s$ is contained in that of $\mathfrak{B}_{s+1}$. Notice that for every $a \in A$ the number of stages s at which $g_s^{-1}(a) \neq g_{s+1}^{-1}(a)$ is at most the size of the block containing a. Since $\mathfrak{A}$ has no blocks of infinite length, this implies that $g = \lim_s g_s$ exists, that $\mathfrak{B} = \lim_s \mathfrak{B}_s$ is a recursive linear order with recursive successivities and recursive predecessor relation $P(x)$ and that $g : \mathfrak{B} \to \mathfrak{A}$ is an isomorphism.

In like manner we can construct a recursive linear order with recursive successivities, $\mathfrak{C}$, isomorphic to $\mathfrak{A}$, such that $\mathfrak{C}$ has the successor relation $S(x)$ recursive.

Theorem 4. *For a* 1-*recursive linear order* $\mathfrak{A}$ *the following are equivalent.*

(1) $\mathfrak{A} \in \Delta(\{k \cdot \eta : k < \omega\} \cup \{\omega, \omega^*, \omega + \omega^*\})$.

(2) *Every* 1-*recursive linear order isomorphic to* $\mathfrak{A}$ *is recursively isomorphic to* $\mathfrak{A}$.

Proof. The usual back-and-forth argument may be used to show that (1) implies (2). We use the last few results to prove the converse. Let $\mathfrak{A}$ satisfy statement (2). It follows from Lemma 1 that $\mathfrak{A}$ has the block relation recursive. The block relation is preserved under isomorphism and is therefore intrinsically recursive on $\mathfrak{A}$. This means (Theorem 2) that the order type of $\mathfrak{A}$ is in

$$\Delta(F_{<\omega} \cup \{\omega, \omega^*, \omega + \omega^*\}).$$

It is therefore possible to partition $\mathfrak{A}$ by a finite number of points $p_0 < p_1 < \cdots < p_m$ so that each one of the intervals $(-\infty, p_0), \ldots, (p_m, \infty)$ has order type in $F_{<\omega} \cup \{\omega, \omega^*, \omega + \omega^*\}$. Consider $\mathfrak{A}'$ the 1-recursive linear order formed by removing those of order type ω, ω^* or $\omega + \omega^*$. It is clear that $\mathfrak{A}' \in F$ and that $\mathfrak{A}'$ is recursively categorical. Since $\mathfrak{A}$ has the block relation recursive, so has $\mathfrak{A}'$ and therefore by Lemma 2, the relations $B(x, y)$, $P(x)$ and $S(x)$ are all recursive in $\mathfrak{A}'$. Since $\mathfrak{A}'$ has no blocks of infinite length it follows that $\mathfrak{A}'$ has the relations $M_i(x)$ for $i < \omega$ uniformly recursive and hence intrinsically uniformly recursive. Therefore by Theorem 3,

$$\mathfrak{A}' \in \Delta(\{k \cdot \eta : k < \omega\}),$$

and so

$$\mathfrak{A} \in \Delta(\{k \cdot \eta : k < \omega\} \cup \{\omega, \omega^*, \omega + \omega^*\}).$$

References

[1] C. Ash and A. Nerode, Intrinsically recursive relations, in: J.N. Crossley, ed., Aspects of Effective Algebra (Upside Down A Book Co., Steel's Creek, Australia, 1981) 26–41.

[2] V.D. Dzgoev and S.S. Goncharov, Autostability of models, Algebra and Logic 19 (1980) 28–37.
[3] S.S. Goncharov, Some properties of the constructivization of Boolean algebras, Sibiriski Math. Zh. 16 (1975) 264–278.
[4] S.S. Goncharov, Autostability of models and Abelian groups, Algebra and Logic 19 (1980) 13–27.
[5] P.E. LaRoche, Recursively presented Boolean algebras, Notices Amer. Math. Soc. 24 (1977) A-552.
[6] A.B. Manaster and J.B. Remmel, Some recursion theoretic aspects of dense two dimensional partial orders, in: J.N. Crossley, ed., Aspects of Effective Algebra (Upside Down A Book Co., Steel's Creek, Australia, 1981) 161–188.
[7] G. Metakides and A. Nerode, Effective content of field theory, Ann. Math. Logic 17 (1979) 289–320.
[8] J.B. Remmel, Recursive Boolean algebras with recursive atoms, J. Symbolic Logic 46 (1981) 595–615.
[9] J.B. Remmel, Recursively categorical linear orderings, Proc. Amer. Math. Soc. 83 (1981) 379–386.
[10] S. Schwarz, Quotient lattices, index sets and recursive linear orders, Proc. Amer. Math. Soc. Summer Research Instit. on Recursion Theory, Cornell University (1982).
[11] R.L. Smith, Two theorems on autostability in p-groups, in: M. Lerman, J.H. Schmerl, and R.I. Soare, eds., Logic Year 1979–80, Lecture Notes in Math. 859 (Springer, Berlin) 302–311.