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ON APPROXIMATION OF GROUPS, GROUP ACTIONS, AND HOPF ALGEBRAS

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We give new examples and criteria in the theory of approximation of groups by finite groups. Bibliography: 17 *titles.*

Introduction

In [15], a class of groups locally embeddable into the class of finite groups (LEF-groups) was introduced and thoroughly investigated. This class arises in a natural way in various approximation problems. First of all, the definition of local embeddability into the class of finite groups is a particular case (for groups with the discrete topology) of the definition of approximability of topological groups by finite ones that was introduced and investigated by the third author in connection with the problems of approximation of operators in function spaces on topological groups. In particular, it was this definition that allowed him to study the convergence of the approximation of the Fourier transform in the Hilbert space of functions on a locally compact group by a finite Fourier transform [4] and to construct and investigate approximations of operators in the function spaces on such groups. These approximations were based on the discretization of their symbols [2].

In A. M. Vershik's paper [16], the notion of "closeness" of infinite-dimensional groups to finite ones is understood as the possibility of approximating their group algebras by finite-dimensional algebras in a sense, and it is shown that the existence of such an approximation sometimes implies the existence of an invariant mean on the group. The problem on the possibility of approximating group algebras by finite-dimensional ones is also interesting for the problems of approximation of operators in function spaces on groups. In [15], a question is raised about the relation of local embeddability into the class of finite groups and the possibility of approximating the group algebra. This problem seems to be rather hard in such a general setting. In the present paper, we introduce the notion of the possibility of approximating bialgebras, which is a natural generalization of the definition of approximability of group algebras from [15], and show that a group is an LEF-group if and only if its Hopf algebra is approximable, as a bialgebra, by finite-dimensional algebras.

In connection with the study of approximability of compact groups, we give a definition of approximability of commutative normed Hopf algebras by finite-dimensional bialgebras and show that a compact group is approximable by finite groups if and only if its commutative Hopf algebra can be approximated by finite-dimensional commutative Hopf algebras. Moreover, we give a negative answer to the question from [15] about the possibility of approximating the group SO(3) by finite groups.

Another class of problems where the appearance of the class of LEF-groups is natural is ergodic theory. Namely, we mean the study of the possibility of approximating group actions

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in probability measure spaces. In particular, in [11] and [15] it is shown that the class of countable groups admitting quasiinvariant freely approximable action is exactly the class of LEF-groups. It is known that every group that admits a free and approximable (but not necessarily freely approximable) invariant action is amenable. To the authors' knowledge, the question of whether the inverse statement is true is open. In this connection, it is interesting to study the relation between the properties of local embeddability into finite groups and amenability. In [15], it is proved that these classes are in generic position: both non-amenable LEF and amenable non-LEF groups exist. This makes the question of singling out amenable groups in the class of LEF groups rather pressing. In [15], a conjecture was proposed that this is connected with the asymptotic estimation of some function ψ that gives a quantitative characteristic of the approximability of the group by finite groups. Here we show, with examples of some concrete groups, that this conjecture is not true and give a sufficient condition of amenability of LEF-groups that is based on the asymptotic properties of another quantitative characteristic of approximability by finite groups.

It follows from the preceding paragraph that the class of groups that can be approximated by amenable groups, or, to be more precise, by locally embeddable into the class of amenable groups (LEA-*groups*) is wider than the classes of LEF-groups and amenable groups. The next question that thus arises is the question on the existence of non-LEA-groups, which was stated by V. Bergelson. We show here that infinite simple finitely presented groups do not admit approximation by amenable groups. It is not known if such groups exist. We also give a new notion of approximability by finite groups that introduces a class containing all LEA-groups, but the question of coincidence of the two classes is still open.

The method for the study of approximation properties of group and group actions proposed in this paper is based upon the exact coincidence of the class of LEF-groups and the class of ultra-products of finite groups. The last statement is a particular case of the general theorem by A. I. Maltsev on the local embedding property of algebraic systems [10]. On the other hand, the construction of Loeb spaces [9] (see also [1] and [4]) is well known in nonstandard analysis. In classical terms, Loeb spaces are measure spaces which are just ultraproducts of finite sets. We construct here some special action of an amenable group G on a Loeb space associated with a sequence of Følner sets in G. It turns out that this action coincides, on a space of total measure, with an action of some subgroup of the ultraproduct of permutations of Følner sets. Thus, every amenable group is an "almost" LEF-group, i.e., it is almost a subgroup of the ultraproduct of finite groups associated with the action above. Another example of applications of this action is a simple proof of the well-known Furstenberg's correspondence principle applied in his ergodic proof of the Szemerédi theorem on the existence of arbitrarily long arithmetic progressions in sets of positive density (see the review [3] treating this subject).

The ultraproduct method makes it possible to obtain a generalization of the Følner characterization on which the proof of the above-mentioned sufficient condition of the amenability of LEF-groups is based.

To make the presentation self-contained, we also give basic facts about the theory of ultraproducts and describe a construction of Loeb spaces that does not involve nonstandard analysis.

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1. Ultraproducts and Loeb spaces

In this paper, we extensively use the "poorly known, even now in Russia, notions of filter and ultrafilter" [13]. That is why we recall here the definitions and the main properties of these notions.

Definition 1.1. *A family $\mathfrak{F}$ of subsets of the set I is called a filter (over I) if the following conditions are satisfied:*

(1) $\emptyset \notin \mathfrak{F}$;
(2) $A, B \in \mathfrak{F} \Longrightarrow A \cap B \in \mathfrak{F}$;
(3) $A \in \mathfrak{F},\ A \subseteq B \Longrightarrow B \in \mathfrak{F}$.

Example 1.2. The family of all cofinite subsets of the set I is a filter. This filter is called the *Fréchet filter* over I.

Example 1.3. Let $i_0 \in I$ and $\mathfrak{F}(i_0) = \{A \subseteq I \mid A \ni i_0\}$. Then $\mathfrak{F}(i_0)$ is evidently a filter. This filter is called the *principal filter* (generated by the element i_0). The Fréchet filter is an example of a filter that is not principal.

Definition 1.4. *Let $\alpha : I \to \mathbb{R}$ be an arbitrary function and $\mathfrak{F}$ a filter over I. The number a is called the limit of the function α along the filter $\mathfrak{F}$ (we write $a = \lim\limits_{\mathfrak{F}} \alpha$) if for every $\varepsilon > 0$ the set $\{i \in I \mid |\alpha(i) - a| < \varepsilon\}$ belongs to $\mathfrak{F}$.*

Infinite limits along a filter and the limits of functions taking value in an arbitrary topological space are defined similarly. Moreover, if the space is Hausdorff, then every function can have at most one limit.

Example 1.5. Let x_n be a sequence whose elements are taken from an arbitrary topological space. Then $a = \lim\limits_{n\to\infty} x_n$ if and only if a is the limit of x_n along the Fréchet filter over $\mathbb{N}$.

A filter $\mathfrak{F}$ over I is called an *ultrafilter* if it is a maximal (relative to inclusion) filter. It is a direct consequence of the Zorn lemma that every filter is a subset of some ultrafilter. The following statement is easily proved.

Proposition 1.6. *A filter $\mathfrak{F}$ is an ultrafilter if and only if*

$$\forall A \subseteq I\ (A \in \mathfrak{F} \vee I \setminus A \in \mathfrak{F}).$$

It is evident now that the principal filter $\mathfrak{F}(i_0)$ from Example 1.3 is an ultrafilter. We call a filter *free* if it is not principal. It is easy to see that a filter $\mathfrak{F}$ is free if and only if it contains the Fréchet filter.

Proposition 1.7 (properties of a limit along an ultrafilter)**.**

(1) *Let X be a Hausdorff topological space and a function $\alpha : I \to X$ be such that its range is relatively compact. Then α has a limit along an arbitrary ultrafilter over I. In particular, a bounded sequence has a limit along an arbitrary ultrafilter.*
(2) *$a = \lim\limits_{n\to\infty} x_n$ if and only if $a = \lim\limits_{\mathfrak{F}} x_n$ for any free ultrafilter $\mathfrak{F}$ over $\mathbb{N}$.*
(3) *A number a is a limit point of the sequence x_n if and only if $a = \lim\limits_{\mathfrak{F}} x_n$ for every free ultrafilter $\mathfrak{F}$ over $\mathbb{N}$.*

Proof. Statement (1) easily follows from the fact that the closure of the range of α is compact. Statements (2) and (3) are immediate consequences of the following claim (see, e.g., Example 1.5).

Let n_k be an increasing sequence of positive integers and A the set of its values. Then $\lim\limits_{k\to\infty} x_{n_k} = a$ if and only if $a = \lim\limits_{\mathfrak{F}} x_n$ for every free ultrafilter $\mathfrak{F}$ over $\mathbb{N}$ that contains A. ☐

In the sequel, we need only ultrafilters over $\mathbb{N}$, which is why our considerations refer only to this case, though they can be extended without much effort to the case of filters over an arbitrary infinite set I.

Definition 1.8. *Let an ultrafilter $\mathfrak{F}$ and a sequence $\{K_i\}_{i\in\mathbb{N}}$ of sets be given. We define an equivalence relation on the Cartesian product $\underset{i}{\times} K_i$ by*

$$\langle x_i\rangle \sim \langle y_i\rangle \qquad \overset{\text{def}}{\Longleftrightarrow} \qquad \{i \mid x_i = y_i\} \in \mathfrak{F}.$$

The set of all equivalence classes thus introduced on the product $\underset{i}{\times} K_i$ is called the ultraproduct of the sets K_i over $\mathfrak{F}$ and denoted by $\underset{i}{\times} K_i/\mathfrak{F}$. In the case where $K_i = K$ for all i, this ultraproduct is called the ultrapower of the set K over $\mathfrak{F}$ and is denoted by $K^{\mathfrak{F}}$. There is a canonical embedding $\iota : K \to K^{\mathfrak{F}}$ defined by the relation $\iota(k) = \langle k_i\rangle_{\mathfrak{F}}$. Here $k_i = k$ for every $i \in \mathbb{N}$ and $\langle k_i\rangle_{\mathfrak{F}}$ means the class of the element $\langle k_i\rangle \in \underset{i}{\times} K_i$ in $\underset{i}{\times} K_i/\mathfrak{F}$.

If the set of indices i for which the element $k_i \in K_i$ has a certain property $\mathcal{P}$ is a subset belonging to the ultrafilter $\mathfrak{F}$, we say that the property $\mathcal{P}$ is fulfilled for almost all k_i relative to $\mathfrak{F}$.

If every set K_i is a group relative to an operation $\odot_i$, then the ultraproduct $\underset{i}{\times} K_i/\mathfrak{F}$ is also a group relative to the operation $\odot_{\mathfrak{F}}$ (called the *ultraproduct of the operations* $\odot_i$) defined by

$$\langle x_i\rangle_{\mathfrak{F}} \odot_{\mathfrak{F}} \langle y_i\rangle_{\mathfrak{F}} = \langle x_i \odot_i y_i\rangle_{\mathfrak{F}}.$$

Moreover, if K is a group, then the canonical embedding $\iota : K \to K^{\mathfrak{F}}$ is an injective homomorphism.

Similarly, if $\varphi_i : A_i \to B_i$ is a sequence of mappings, then we call the *ultraproduct of maps* φ_i the mapping

$$\underset{i}{\times}\varphi_i/\mathfrak{F} : \underset{i}{\times} A_i/\mathfrak{F} \to \underset{i}{\times} B_i/\mathfrak{F}, \text{ where } \underset{i}{\times}\, \varphi\left(\langle a_i\rangle_{\mathfrak{F}}\right) = \langle \varphi_i(a_i)\rangle_{\mathfrak{F}}.$$

The following statement is evident.

Proposition 1.9. *If all the mappings φ_i are injective, then their ultraproduct $\underset{i}{\times}\varphi_i/\mathfrak{F}$ is also injective.*

Definition 1.10. *A subset of the ultraproduct $\underset{i}{\times} K_i/\mathfrak{F}$ of the sets K_i is called internal if it is an ultraproduct of some subsets $A_i \subseteq K_i$.*

Proposition 1.11. *The family $\mathfrak{K}^{\mathfrak{F}}$ of all internal subsets of $\times_i K_i/\mathfrak{F}$ is an algebra. Moreover, we have $\mathfrak{A} = \times_i A_i/\mathfrak{F} \subseteq \mathfrak{B} = \times_i B_i/\mathfrak{F}$ if and only if $\{i \in \mathbb{N} \mid A_i \subseteq B_i\} \in \mathfrak{F}$.*

We omit the simple proof.

The following theorem is a well-known fact from model theory. Its more general formulation and a proof can be found in [8]. For the sake of completeness of the exposition, we give the proof of this theorem in the case of ultrafilters over $\mathbb{N}$.

Theorem 1.12. *The algebra of internal sets is saturated, i.e., for every sequence $\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \ldots \supset \mathfrak{A}_n \supset \ldots$ of internal sets*

$$\bigcap_n \mathfrak{A}_n = \emptyset \qquad \Longleftrightarrow \qquad \exists N \quad \forall n \geqslant N \quad \mathfrak{A}_n = \emptyset. \tag{1}$$

Proof. We shall prove the nontrivial part by reductio ad absurdum. Let $\mathfrak{A}_n = \times_i A_i^n/\mathfrak{F} \neq \emptyset$ for all $n \in \mathbb{N}$. Without loss of generality, we can assume that $A_i^n \neq \emptyset$ for every i and n. Put, for $m \in \mathbb{N}$,

$$I_m = \{i \geqslant m \mid A_i^0 \supset A_i^1 \supset \ldots \supset A_i^m \neq \emptyset\}.$$

As is easily seen,

(1) $I_0 = \mathbb{N}$;
(2) $I_m \in \mathfrak{F}$;
(3) $I_m \supset I_{m+1} \quad \forall m \in \mathbb{N}$;
(4) $\bigcap_{m \in \mathbb{N}} I_m = \emptyset$.

For every $i \in \mathbb{N}$, we put

$$m(i) = \max\{m \mid i \in I_m\}.$$

It follows from the definition of I_m that $m(i) \leq i$. Choose an element $a_i \in A_i^{m(i)}$ for each i. We show that $\langle a_i \rangle_{\mathfrak{F}} \in \mathfrak{A}_m$ for every $m \in \mathbb{N}$, i.e., that $\{i \in \mathbb{N} \mid a_i \in A_i^m\} \in \mathfrak{F}$. But to do this it suffices to check the inclusion $I_m \subset \{i \in \mathbb{N} \mid a_i \in A_i^m\}$. The latter is clear because for $i \in I_m$ we have $m(i) \geqslant m$, i.e., $A_i^{m(i)} \subseteq A_i^m$. □

Let $\langle X_i, \Omega_i, \mu_i \rangle$, $i \in \mathbb{N}$, be a sequence of spaces with finitely additive probability measures, i.e., we assume that Ω_i is an algebra of subsets of X_i and that $\mu_i(X_i) = 1$. If $\mathfrak{X} = \times_i X_i/\mathfrak{F}$, then the family Ω of all internal subsets of $\mathfrak{X}$ having the form $\times_i A_i/\mathfrak{F}$, where $A_i \in \Omega_i$, is evidently an algebra. The formula

$$\times_i \mu_i/\mathfrak{F}(\times_i A_i/\mathfrak{F}) = \lim_{\mathfrak{F}} \mu_i(A_i)$$

defines a finitely additive measure $\times_i \mu_i/\mathfrak{F}$ on this algebra.

Proposition 1.13. *The measure $\times_i \mu_i/\mathfrak{F}$ can be extended to a countably additive measure μ_L on the $\sigma-$algebra $\sigma(\Omega)$. Moreover, for every set $B \in \sigma(\Omega)$ there exists a set $U \in \Omega$ such that $\mu_L(B \Delta U) = 0$.*

The spaces $(\mathfrak{X}, \sigma(\Omega), \mu_L)$ are called the Loeb spaces and the measures μ_L the Loeb measures.

Proof. If the sequence $\mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \ldots \supset \mathfrak{A}_n \supset \ldots$ of sets from Ω has an empty intersection, then, by Theorem 1.12, $\lim\limits_{n\to\infty} \underset{i}{\times}\mu_i/\mathfrak{F}(\mathfrak{A}_n) = 0$, and, by the Carathéodory property, the measure $\underset{i}{\times}\mu_i/\mathfrak{F}$ extends to a σ-algebra. If $B \in \sigma(\Omega)$, then it easily follows from the construction of the extension that there are sequences of sets $\mathfrak{A}_n \in \Omega$ and $\mathfrak{B}_n \in \Omega$ such that $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1} \subseteq B \subseteq \mathfrak{B}_{n+1} \subseteq \mathfrak{B}_n$ and $\underset{i}{\times}\mu(\mathfrak{A}_n) + \frac{1}{n} > \mu_L(B) > \underset{i}{\times}\mu(\mathfrak{B}_n) - \frac{1}{n}$. Let $\mathfrak{A}_n = \underset{i}{\times}A_i^{(n)}/\mathfrak{F}$, $\mathfrak{B}_n = \underset{i}{\times}B_i^{(n)}/\mathfrak{F}$. Without loss of generality, we can assume that $A_i^{(n)} \subseteq A_i^{(n+1)}$, $B_i^{(n)} \supseteq B_i^{(n+1)}$ for every n and $i \in \mathbb{N}$, and also $A_i^{(n)} \subseteq B_i^{(m)}$ for every m. Put $U = \underset{i}{\times} A_i^{(i)}/\mathfrak{F}$. Since the set $\{i \mid A_i^{(i)} \supseteq A_i^{(m)}\} \supseteq \{i \mid i > m\} \in \mathfrak{F}$, we have $U \supseteq \mathfrak{A}_m$ for every m. Similarly, $U \subseteq \mathfrak{B}_m$. Now it is clear that U satisfies the claimed condition. □

Definition 1.14. *A set is called ultrafinite if it is an ultraproduct of finite sets.*

In the case where $\mathfrak{X}$ is an ultrafinite set, i.e., all X_i are finite, we shall always assume in the sequel that the Loeb spaces carry uniform probability measures, i.e., Ω_i is the algebra of all subsets of X_i and $\mu_i(A_i) = \frac{|A_i|}{|X_i|}$. Such a space will be called a *canonical Loeb space* related to X_i.

2. Approximability and local embeddability

In the sequel, we use the abbreviations $\forall^f$ meaning "for every finite ... " and $\exists^f$ for "there exist a finite ... ." We shall also use the symbol $\rightarrowtail$ to denote injective maps.

The next definition and theorem are particular cases of the general notion of local embeddability of an algebraic system into some class of algebraic systems of the same signature and of the corresponding theorem for general algebraic systems from [10]. We give a proof for the sake of completeness. All the groups considered are assumed to be countable, but this assumption is not restrictive.

Definition 2.1. *A group G is called an approximable group of class $\mathfrak{K}$ (locally embeddable into the class $\mathfrak{K}$) if for every finite subset $H \subset G$ there exists a group $\langle K, \odot\rangle \in \mathfrak{K}$ and a map $\phi : H \rightarrowtail K$ such that for every $h_1, h_2 \in H$ we have*

$$h_1h_2 \in H \Longrightarrow \varphi(h_1h_2) = \varphi(h_1) \odot \varphi(h_2). \tag{2}$$

Theorem 2.2. *A group G is locally embeddable into the class $\mathfrak{K}$ of groups if and only if there exists a sequence of groups $K_i \in \mathfrak{K}$, $i \in \mathbb{N}$, a free ultrafilter $\mathfrak{F}$ over $\mathbb{N}$, and an injective homomorphism $\psi : G \to \underset{i}{\times} K_i/\mathfrak{F}$.*

Proof. Let the group G be locally embeddable into the class of groups $\mathfrak{K}$. We fix an increasing sequence $\{H_i \mid i \in \mathbb{N}\}$ of finite subsets of G such that $\bigcup\limits_{i\in\mathbb{N}} H_i = G$ (we remind the reader that we have restricted ourselves to the case of countable groups) and construct, for every H_i, a group $\langle K_i, \odot_i\rangle$ and a map $\varphi_i : H_i \to K_i$ that satisfy the hypothesis of the preceding definition.

Fix an arbitrary free ultrafilter $\mathfrak{F}$ on $\mathbb{N}$. We now define an embedding $\psi : G \to \underset{i}{\times} K_i/\mathfrak{F}$ as follows. Let $g \in G$. Then we define $k_i \in K_i$ by the formula

$$k_i = \begin{cases} \varphi_i(k), & k \in H_i \\ e_i, & k \notin H_i \end{cases},$$

where e_i is the unit in the group K_i, and set $\psi(g) = \langle k_i \rangle_{\mathfrak{F}}$. Since $k_i = \varphi_i(k)$ for almost all i, and all φ_i are injective, the map ψ is also injective (see Proposition 1.9). Let $g_1, g_2 \in G$ and $g_1 g_2 = g_3$. For almost all $i \in \mathbb{N}$ we have $g_1, g_2, g_3 \in H_i$, i.e., $\varphi_i(g_1 g_2) = \varphi_i(g_1) \odot_i \varphi_(g_2)$. This means that $\psi(g_1 g_2) = \psi(g_1) \odot_{\mathfrak{F}} \psi(g_2)$, so ψ is a homomorphism.

To prove the converse, assume that $\psi : G \to \underset{i}{\times} K_i/\mathfrak{F}$ is an injective homomorphism, $K_i \in \mathfrak{K}$ for all $i \in \mathbb{N}$. Consider an arbitrary finite set $H \subseteq G$. For every $h \in H$, we fix a representative $\langle k_i^{(h)} \rangle \in \underset{i}{\times} K_i$ of the element $\psi(h)$. For any $h_1 \neq h_2 \in H$, let $A_{h_1,h_2} = \{i \in \mathbb{N} \mid k_i^{(h_1)} \neq k_i^{(h_2)}\}$. Then $A_{h_1,h_2} \in \mathfrak{F}$ by the injectivity of ψ. Assume now that $h_1, h_2, h_3 \in H$ are such that $h_1 h_2 = h_3$. We set $B_{h_1,h_2,h_3} = \{i \in \mathbb{N} \mid k_i^{(h_1)} \odot_i k_i^{(h_2)} = k_i^{(h_3)}\}$. Since ψ is a homomorphism, $B_{h_1,h_2,h_3} \in \mathfrak{F}$. It follows that

$$C = \bigcap_{\substack{h_1,h_2 \in H \\ h_1 \neq h_2}} A_{h_1,h_2} \cap \bigcap_{\substack{h_1,h_2,h_3 \in H \\ h_1 h_2 = h_3}} B_{h_1,h_2,h_3} \neq \emptyset.$$

Let $i_0 \in C$. We define $\varphi : H \to K_{i_0}$ by setting $\varphi(h) = k_{i_0}^{(h)}$. It is clear that K_{i_0} and φ satisfy the definition of local embeddability. □

In [15], a class of groups locally embeddable into the class of finite groups (LEF-groups) is thoroughly studied. This class also implicitly arises in [11]. In these papers it is stated, in particular, that a finitely presentable group is an LEF-group if and only if it is finitely approximable, i.e., the intersection of all its normal subgroups of finite index contains a single element, the unity. We extend this statement to the case of local embeddability into an arbitrary class $\mathfrak{K}$. Recall that a class $\mathfrak{K}$ is said to have an *inheritance property* if, along with every group $G \in \mathfrak{K}$, $\mathfrak{K}$ contains also all subgroups of G.

Proposition 2.3. *Let a finitely presentable group G be locally embeddable into a class $\mathfrak{K}$ with the inheritance property. Then for every finite subset $H \subseteq G$ there exists a group $K \in \mathfrak{K}$ and a surjective homomorphism $\varphi : G \to K$ which is injective on H.*

Proof. Let $g_1, \ldots, g_n$ be generators of the group G and $\alpha_1, \ldots, \alpha_k$ determining relations, i.e., some words in the alphabet $g_1^{\pm 1}, \ldots, g_n^{\pm 1}$. We fix some system of words $\beta_1, \ldots, \beta_s$ that define elements of a finite set H. Let H_1 denote a finite set of elements of the group G which consists of the generators and of the elements represented by various subwords of the words $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_s$; then take a group $K_1 \in \mathfrak{K}$ and an injective map $\psi : H_1 \to K_1$ as in Definition 2.1. Consider a subgroup K of K_1 generated by the elements $\psi(g_1^{\pm 1}), \ldots, \psi(g_n^{\pm 1})$. The inheritance property of the class $\mathfrak{K}$ implies that $K \in \mathfrak{K}$. It follows from condition (2) that the relations $\alpha_1, \ldots, \alpha_k$ (or, to be more exact, $\psi(\alpha_1), \ldots, \psi(\alpha_k)$) are fulfilled in the group K. This means that the map ψ, which is injective on H, extends to a homomorphism $\varphi : G \to K$. □

Remark 2.4. It is clear that any group G satisfying the statement of Proposition 2.3 is locally embeddable into the class $\mathfrak{K}$. As is known, the groups with this property, in the case where $\mathfrak{K}$ is the class of finite groups, are called finitely approximable. Thus, all finitely approximable groups are LEF-groups [15].

We give a sufficient condition for local embeddability of a group G into a class $\mathfrak{K}$. The formulation of Theorem 2.14 below is due to A. M. Vershik.

Let $\mathcal{G}$ be a group with metric ρ such that

$$\forall \gamma \in \mathcal{G} \forall \varepsilon > 0 \exists \delta > 0 \, \forall \xi\eta \in \mathcal{G}(\rho(\xi,\eta)) < \delta \Longrightarrow \max\{\rho(\gamma\xi,\gamma\eta),\rho(\xi\gamma,\eta\gamma)\} < \varepsilon). \tag{3}$$

Here are some examples of groups with natural metric satisfying this property.

Example 2.5. Every invariant metric ρ on the group $\mathcal{G}$ clearly satisfies condition (3).

Example 2.6. Let (X,μ) be a probability measure space and $\mathcal{G}$ a group of quasiinvariant transformations of the space X (i.e., transformations leaving invariant the ideal of the sets of zero measure). We introduce a metric ρ on $\mathcal{G}$ by the formula

$$\rho(\xi,\eta) = \mu(\{x \in X \mid \xi x \neq \eta x\}) \tag{4}$$

(to be exact, it is a metric if one identifies transformations that coincide almost everywhere). We show that ρ satisfies (3). Since every transformation $\gamma \in \mathcal{G}$ is invertible, for $x \in X$ we have $\xi x = \eta x \Longleftrightarrow \gamma\xi x = \gamma\eta x$, i.e., $\rho(\xi,\eta) = \rho(\gamma\xi,\gamma\eta)$ for all $\xi,\eta \in \mathcal{G}$. If we multiply by γ from the right side, the analogous equality is not valid any more. But since $\gamma \in \mathcal{G}$ leaves invariant the ideal of the sets of measure zero, it follows that the measure μ^γ that is defined by the relation $\mu^\gamma(A) = \mu(\gamma^{-1}A)$, $A \subseteq X$, is absolutely continuous relative to the measure μ. This means that

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall A \subseteq X \, (\mu(A) < \delta \Longrightarrow \mu^\gamma(A) < \varepsilon). \tag{5}$$

Let $A = \{x \in X \mid \xi x \neq \eta x\}$. Then $\gamma^{-1}A = \{x \in X \mid \xi\gamma x \neq \eta\gamma x\}$, and (3) follows immediately from (5).

Example 2.7. If $\mathcal{G}$ is a group of unitary operators in a Hilbert space $\mathfrak{H}$ and $\rho(U,V) = \|U - V\|$, then the inequalities $\rho(UW,VW) \leq \rho(U,V)$ and $\rho(WU,WV) \leq \rho(U,V)$ are obviously true, and (3) immediately follows.

Example 2.8. Let $\mathcal{G}$ be again a group of unitary operators in the Hilbert space $\mathfrak{H}$. Consider a metric ρ that defines a strong operator topology on $\mathcal{G}$. To this end, we take some orthonormal basis $\{e_i \mid i \in \mathbb{N}\}$ and set

$$\rho(U,V) = \sum_{i=1}^{\infty} \frac{\|Ue_i - Ve_i\|}{2^i}. \tag{6}$$

Now again $\rho(WU,WV) \leq \rho(U,V)$. The analogous inequality is no longer true for the multiplication by W from the right. But we can regard $\rho(UW,VW)$, a function of U and V for W fixed, as a metric that is defined in the same way with the help of the basis $\{We_i \mid i \in \mathbb{N}\}$. This metric also defines the strong operator topology, and consequently it is equivalent to the first one and (3) follows immediately.

Remark 2.9. The metrics considered in Examples 2.6–2.8 are bounded, i.e., $\exists a > 0\, \forall \xi, \eta\ \rho(\xi, \eta) < a$.

Definition 2.10. *Let $q > 0$ and assume that H is a subgroup of the group $\mathcal{G}$ with metric ρ. We call it q-discrete if $\rho(h, e) \geq q$ for all $h \in H$, $h \neq e$, where e is the unit of the group $\mathcal{G}$.*

Example 2.11. In the setting of Example 2.6, the group H of quasiinvariant transformations of the space (X, μ) is discrete if and only if it is acting freely on X. The latter means, as usual, that for all $\xi \in H$, $\xi \neq e$, $\mu(\{x \in X \mid \xi x = x\}) = 0$. By analogy, we say that H is acting *q-freely* on X if for all $\xi \in H$, $\xi \neq e$, $\mu(\{x \in X \mid \xi x = x\}) \leq 1 - q$.

Example 2.12. Consider the left regular representation of the group G. We take $\mathfrak{H} = l_2(G)$ and identify G with the group of operators in this representation. Here $\mathcal{G}$ is the group of all unitary operators in the space $l_2(G)$. Choose the basis $\{e_g \mid g \in G\}$ in $\mathfrak{H}$ consisting of operators $e_g(h) = \delta_{gh}$. If we define ρ by formula (6), after an arbitrary numeration of the elements in G, then we evidently get $\rho(g_1, g_2) = \sqrt{2}$ for all g_1, g_2, i.e., G is a $\sqrt{2}$-discrete subgroup of $\mathcal{G}$.

Definition 2.13. *We call a subgroup G of the group $\mathcal{G}$ with metric ρ q-discretely approximable by groups of class $\mathfrak{K}$ if for every $\varepsilon > 0$ and every finite set $\{g_1, \ldots, g_n\} \subseteq G$ there exist $h_1, \ldots, h_n \in \mathcal{G}$ such that $\rho(g_i, h_i) < \varepsilon$, $i = 1 \ldots n$, and the subgroup H of the group G generated by the elements $h_1, \ldots, h_n$ is q-discrete and belongs to the class $\mathfrak{K}$.*

It is clear that if a subgroup H is q-discretely approximable by some groups, then it is q'-discrete itself for any $q' < q$.

Theorem 2.14. *Let a subgroup G of the group $\mathcal{G}$ with metric ρ satisfying (3) be q-discretely approximable by groups of class $\mathfrak{K}$ for some $q > 0$. Then G is locally embeddable into the class $\mathfrak{K}$.*

Proof. Let $G = \{g_1, \ldots, g_n, \ldots\}$ and put $G_n = \{g_1, \ldots, g_n\}$. Taking a sequence $\varepsilon_n \searrow 0$, we construct a sequence of groups $H_n \in \mathfrak{K}$ that satisfy the hypothesis of the theorem. Denote by $\varphi : G_n \to H_n$ an injective map such that $\varphi_n(g_i) = h_i$. We consider the ultraproduct $\mathcal{G}^{\mathfrak{F}}$ for some free ultrafilter $\mathfrak{F}$ on $\mathbf{N}$ and define the metric $\rho^{\mathfrak{F}}$ on this group by

$$\rho^{\mathfrak{F}}(\alpha, \beta) = \lim_{\mathfrak{F}} \rho(a_n, b_n),$$

where $\langle a_n \rangle$ and $\langle b_n \rangle$ are representatives of the classes α and β respectively. The condition (3) ensures that $K = \{\alpha \mid \rho^{\mathfrak{F}}(\alpha, e) = 0\}$ is a normal subgroup in $\mathcal{G}^{\mathfrak{F}}$. We set $\mathcal{G}^{\#} = \mathcal{G}^{\mathfrak{F}}/K$. Then $\mathcal{H} = \prod_n H_n/\mathfrak{F}$ is a subgroup of $\mathcal{G}^{\mathfrak{F}}$. Moreover, by the q-discreteness condition of the approximation, $\mathcal{H} \cap K = \varnothing$, i.e., $\mathcal{H}$ is embedded in $\mathcal{G}^{\#}$ injectively. Next we define an embedding $\Psi : G \to \mathcal{H}$, taking for $\Psi(g)$ the class of the sequence $\varphi_n(g)$ (for a finite set of n such that $g \notin G_n$, the corresponding component can be defined arbitrarily). By the approximability condition, $\rho^{\mathfrak{F}}(\iota(g), \Psi(g)) = 0$, i.e., the images of these elements in $\mathcal{G}^{\#}$ coincide. This proves the injective embedding property of G into the ultraproduct $\mathcal{H}$ of groups of class $\mathfrak{K}$. □

Definition 2.15. *We say that a group G of quasiinvariant transformations of a probability measure space (X, μ) is q-freely approximable by elements of class $\mathfrak{K}$ if for every $\varepsilon > 0$ and any finite set $\{g_1, \ldots, g_n\} \subseteq \mathcal{G}$ there exist quasiinvariant transformations $h_1, \ldots, h_n$ of the space (X, μ) such that*

$$\mu(\{x \in X \mid h_i x = g_i x,\ i = 1, \ldots, n\}) \geq 1 - \varepsilon$$

and the subgroup H they generate belongs to the class $\mathfrak{K}$ and is q-freely acting on X (see Example 2.11).

The inspection of Examples 2.6 and 2.11 shows that if G is q-freely approximable by groups of class $\mathfrak{K}$, then it is q-discretely approximable by groups of this class (if one considers the metric (4) on the group $\mathcal{G}$). The following statement is a direct consequence of Theorem 2.14.

Corollary 2.16. *If the group G of quasiinvariant transformations of a probability measure space (X, μ) is q-freely approximable by groups of class $\mathfrak{K}$, then it is locally embeddable in the class $\mathfrak{K}$.*

The last corollary was obtained in [11] and [15] in the particular case where the class $\mathfrak{K}$ is the class of finite groups and the group G is *freely* approximable by finite groups. The reverse statement is also established there: every LEF-group can act in a freely-approximable way on some probability measure space (X, μ).

For LEF-groups the statement of our theorem can be inverted also in the case considered in Example 2.12.

Proposition 2.17. *Let G be an LEF-group. Then, under the assumptions of Example 2.12, G is q-discretely approximable by finite groups for some $q > 0$.*

Proof. Without loss of generality, we can assume that $G = \{1, 2, \dots\}$ with 1 as the unit of the group G. Then the metric in $\mathcal{G}$ (see Example 2.12) can be defined by (6). For $m \in \mathbb{N}$, set $\mathbb{N}_m = \{1, \dots, m\}$. We define the quasimetric ρ_m in $\mathcal{G}$ by the formula

$$\rho_m(U, V) = \sum_{i=1}^{m} \frac{\|Ue_i - Ve_i\}}{2^i}.$$

Take $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $|\rho - \rho_m| < \varepsilon$ for any U and V.

Fix arbitrary $g_1, \dots, g_n \in G$ and set $F = \{g_1, \dots, g_n\} \cup \mathbb{N}_m$, $H = F^2$. Since F contains the unit, we have $F \subseteq H$. For H, we construct a finite group $(K, \odot)$ that satisfies Definition 2.1. Without loss of generality, we can assume that $K \subseteq G$, as a set, and that ψ is the identity map, i.e., $H \subseteq K$. Consider the metric

$$\rho_K(U, V) = \sum_{i \in K} \frac{\|Ue_i - Ve_i\|}{2^i}$$

on $\mathcal{G}$. Since $N_m \subseteq K$, we have $|\rho - \rho_K| < \varepsilon/2$ for all U and V. Let $h_1, \dots, h_n$ be the operators that act on the elements e_i as operators of the left regular representation of the group $(K, \odot)$ for $i \in K$ and leave e_i fixed for $i \notin K$. It follows from Definition 2.1 that $\rho_m(g_j, h_j) = 0$ for $j \leq n$, i.e., $\rho_K(g_j, h_j) < \varepsilon$. By definition, the group generated by the operators $h_1, \dots, h_n$ is isomorphic to the subgroup of K, so it is finite. Applying the considerations of Example 2.12 to the group K, we see that K is $\sqrt{2}/2$-discrete in the group of all unitary operators of the space $l_2(K)$. This means that for every element $h \neq e$ of this group we have $\rho_K(h, e) > \sqrt{2}/2$ and, consequently, $\rho(h, e) > \sqrt{2}/2 - \varepsilon$. It suffices to choose ε smaller than, say, $\sqrt{2}/4$ to get what was to be proved. $\square$

3. Approximability and amenability

To begin, we recall the definition of an amenable group.

Definition 3.1. *A discrete group G is called amenable if there exists a nontrivial finitely additive finite left-invariant (right-invariant) measure μ on the set $\mathcal{P}(G)$ of all its subsets.*

An important characterization of amenability is the Følner condition (see, for example, [5]).

Theorem 3.2 (the Følner condition)**.** *A group G is amenable if and only if for every $\epsilon > 0$ and for every finite set $K \subset G$ there is a finite set $U \subset G$ such that for all $x \in K$*

$$\frac{|xU \triangle U|}{|U|} < \epsilon. \tag{7}$$

The following statement is a straightforward consequence of this theorem.

Corollary 3.3. *A countable group G is amenable if and only if there exists a sequence of subsets $U_1 \subset U_2 \subset \ldots \subset U_n \subset \ldots \subset G$ exhausting G such that for all $x \in G$*

$$\lim_{n\to\infty} \frac{|xU_n \cap U_n|}{|U_n|} = 1. \tag{8}$$

We call such a sequence a Følner sequence in what follows.

In [15], it is shown that the class of amenable groups is in a generic position with the class of LEF-groups. There exist LEF-groups that are not Abelian (for example, free groups with a finite number of generators are among these since they are finitely approximable; see the remark in the preceding section). There also exist amenable non-LEF-groups, but it is much more difficult to construct such an example; see [15].

We call groups that are locally embeddable into the class of amenable groups *LEA-groups*. It follows from what has just been said above that the class of LEA-groups is wider than the class of LEF-groups. However, the following fact holds true.

Proposition 3.4. *Finitely presented non-amenable simple groups (if they exist) are not embeddable into the class of amenable groups.*

Proof. Let G be a finitely presented non-amenable simple group. Assume that G is an LEA-group. Then, by Proposition 2.3, for any $a \in G$ distinct from the unit there exists an amenable group K and a surjective homomorphism $\varphi : G \to K$ such that $\varphi(a) \neq e$. This means that the group K is nontrivial. Since G is simple, φ is injective, i.e., an isomorphism. Consequently, the group G is amenable, which is not true. □

In [14], the problem of amenability of R. Thompson's groups is discussed, but it is only proved there that these groups are not elementary amenable.

We give here another definition of approximability by finite groups that covers the classes of amenable and LEF-groups.

Definition 3.5. *A group G is called weakly approximable by finite groups if for all $\epsilon > 0$ and finite $H \subset G$ there exist (K, A), where K is a finite group acting on a finite set A, and a map $\varphi : H \rightarrowtail K$ such that for all $h_1, h_2 \in H$ with $h_1h_2 \in H$ the inequality*

$$\frac{|\{x \in A \mid \varphi(h_1h_2)x = \varphi(h_1)\varphi(h_2)x\}|}{|A|} \geqslant 1 - \epsilon \tag{9}$$

holds.

It is not hard to see that an LEF-group is weakly approximable by finite groups. To see this, it suffices to put $A = K$ and to consider the action of K on itself by left shifts.

Proposition 3.6. *Let G be an amenable group. Then G is weakly approximable by finite groups.*

Proof. Let H be a finite subset of G. We may assume that for all $h \in H$, also $h^{-1} \in H$. By the Følner condition, for any $\epsilon > 0$ there exists a finite set $U \subset G$ such that for all $h \in H$

$$\frac{|hU \cap U|}{|U|} \geqslant 1 - \epsilon/2.$$

Take $A = U$ and $K = S(U)$. For every $h \in H$ we fix an arbitrary bijection $\psi_h : (U \setminus h^{-1}U) \longrightarrow (U \setminus hU)$ and define the map $\varphi : H \rightarrowtail K$, setting

$$\varphi(h)u = \begin{cases} hu & \text{if} \quad hu \in U; \\ \psi_h(u) & \text{if} \quad hu \notin U. \end{cases}$$

It is easily seen that

$$\{x \in A \mid \varphi(h_1h_2)x = \varphi(h_1)\varphi(h_2)x\} \supset h_2^{-1}U \cap (h_1h_2)^{-1}U \cap U. \tag{10}$$

For $B = h_2^{-1}U \cap U$ and $C = (h_1h_2)^{-1}U \cap U$ we have

$$1 \geqslant \frac{|B \cup C|}{|U|} = \frac{|B|}{|U|} + \frac{|C|}{|U|} - \frac{|B \cap C|}{|U|} \geqslant (1 - \epsilon/2) + (1 - \epsilon/2) - \frac{|B \cap C|}{|U|}.$$

It follows that

$$\frac{|h_2^{-1}U \cap (h_1h_2)^{-1}U \cap U|}{|U|} = \frac{|B \cap C|}{|U|} \geqslant 1 - \epsilon,$$

and, due to (10), the inequality (9) is proved. □

Proposition 3.7. *Let G be an LEA-group. Then G is weakly approximable by finite groups.*

Proof. Since G is approximable by amenable groups, for every finite set $H \subset G$ there exist an amenable group M and a map $\psi : H \rightarrowtail M$ such that for all $h_1, h_2 \in H$

$$h_1h_2 \in H \Longrightarrow \varphi(h_1h_2) = \varphi(h_1)\varphi(h_2).$$

By Proposition 3.6, the group M is weakly approximable by finite groups. This means, in particular, that for any $\epsilon > 0$ there exist a finite group K acting on a set A and a map $\theta : \psi(H) \rightarrowtail K$ such that for all $m_1, m_2 \in \psi(H)$

$$m_1 m_2 \in \psi(H) \Longrightarrow \frac{|\{x \in A \mid \theta(m_1 m_2)x = \theta(m_1)\theta(m_2)x\}|}{|A|} \geqslant 1 - \epsilon.$$

Then $\varphi = \theta \circ \psi$ obviously satisfies (9). □

Let G be an amenable group. Choosing some Følner sequence $\{U_i \mid i \in \mathbb{N}\}$ of subsets in G and a free ultrafilter $\mathfrak{F}$ on $\mathbb{N}$, take the canonical Loeb space constructed according to U_i. Then $\mathcal{U} = \underset{i}{\times} U_i/\mathfrak{F}$, as a set, is a subset of the ultrapower $G^{\mathfrak{F}}$. As was shown in Sec. 2, the group G is canonically embedded into the group $G^{\mathfrak{F}}$, which is why in the sequel we shall assume, for simplicity, that G is a subgroup of $G^{\mathfrak{F}}$. Put $\mathcal{X} = \bigcap\limits_{g \in G} g\mathcal{U}$.

Proposition 3.8.

(1) $\mu_L(\mathcal{X}) = 1$, *where* μ_L *is the Loeb measure.*
(2) *The set* $\mathcal{X}$ *is invariant under the action of the group* G *by left shifts in* $G^{\mathfrak{F}}$.
(3) *The action of* G *in* $\mathcal{X}$ *is free and leaves the Lebesgue measure invariant.*

Proof. To prove (1) it suffices to show that $\mu_L(\mathcal{U} \cap g\mathcal{U}) = 1$. But this is clear because from (8) and Proposition 1.7 (2) it follows that $\lim\limits_{\mathfrak{F}} \frac{|gU_i \cap U_i|}{|U_i|} = 1$.

The second statement directly follows from the definition of the set $\mathcal{X}$.

The action of G on $G^{\mathfrak{F}}$ is obviously free since a group action on itself by left shifts is free. To prove (3) it remains to show that for every inner set $\mathcal{D} = \underset{i}{\times} D_i/\mathfrak{F} \subseteq \mathcal{U}$ one has $\mu_L(\mathcal{X} \cap \mathcal{D}) = \mu_L(g(\mathcal{X} \cap \mathcal{D}))$. It is clear that $\mu_L(g(\mathcal{X} \cap \mathcal{D})) = \mu_L(\mathcal{X} \cap g\mathcal{D}))$. Further, we have $|D_i| = |gD_i| = |gD_i \cap gU_i|$ for almost all i since $D_i \subseteq U_i$ for almost all i. It follows that

$$|\, |D_i| - |gD_i \cap U_i|\, | = |\, |gD_i \cap gU_i| - |gD_i \cap U_i|\, | \leq |U_i \Delta gU_i|.$$

By (8), we have $\lim\limits_{\mathfrak{F}} \frac{|U_i \Delta gU_i|}{|U_i|} = 0$, i.e., $\lim\limits_{\mathfrak{F}} \frac{|D_i|}{|U_i|} = \lim\limits_{\mathfrak{F}} \frac{|gD_i \cap U_i|}{|U_i|}$. Since the set $\mathcal{X}$ has total measure in $\mathcal{U}$, we are done. □

Remark 3.9. For arbitrary $A \subseteq G$, put $\nu(A) = \mu_L(A^{\mathfrak{F}} \cap \mathcal{X})$. It is a direct consequence of Proposition 3.8 that ν is an invariant finitely additive measure on $\mathcal{P}(G)$, and, clearly, $\nu(G) = 1$. Thus, the proof of the sufficiency of the Følner condition immediately follows from Proposition 3.8.

The construction given in the last remark can be slightly modified to give the known Furstenberg's construction that makes it possible to assign to each subset of positive density in $\mathbb{N}$ some measure space in such a way that the shift by one in $\mathbb{N}$ goes to a measure-preserving transformation of that space. This construction was used in the "ergodic" proof of the Szemerédi theorem on the existence of arbitrarily long arithmetic progressions in an arbitrary set of positive density in $\mathbb{N}$.

Definition 3.10. *Assume that $A \subseteq G$. We call the number*

$$d_U(A) = \overline{\lim_{i\to\infty}} \frac{|A \cap U_i|}{|U_i|}$$

the density of the set A relative to a Følner sequence $U = \{U_i \mid i \in \mathbb{N}\}$.

Following [3], we call the following statement the *Furstenberg correspondence principle.*

Proposition 3.11. *Let G be a countable amenable group and $U = \{U_n \mid n \in \mathbb{N}\}$ a sequence of Følner subsets of G. Then for any $A \subset G$ such that $d_U(A) > 0$ there exists a probability measure space $(X, \mathfrak{B}, \mu)$, a measure-preserving action T of the group G on X, and a homomorphism of the Boolean algebras $\varphi : 2^G \longrightarrow \mathfrak{B}$ such that*

(i) $\forall B \subset G \quad \mu(\varphi(B)) \leqslant d_U(B)$;
(ii) $\forall B \subset G \quad \varphi(gB) = T_g(\varphi(B))$;
(iii) $\mu(\varphi(A)) = d_U(A)$.

Proof. By Proposition 1.7, for every free ultrafilter $\mathfrak{F}$ on $\mathbb{N}$ there holds an inequality $\lim\limits_{\mathfrak{F}} \frac{|A\cap U_i|}{|U_i|} \leq d_U(A)$, whereas there exists an ultrafilter $\mathfrak{F}_0$ on $\mathbb{N}$ such that $\lim\limits_{\mathfrak{F}_0} \frac{|A\cap U_i|}{|U_i|} = d_U(A)$. We take a Loeb space $\mathcal{X}$ constructed for this $\mathfrak{F}_0$ as in Proposition 3.8 and define an action of G in the same way. Then for any $B \subseteq G$ we set $\varphi(B) = B^{\mathfrak{F}} \cap \mathcal{X}$. Properties (i)–(iii) are clearly fulfilled. □

Slightly modifying the construction of $\mathcal{X}$, we can obtain the following strengthening of the Følner condition.

Theorem 3.12. *A group G is amenable if and only if*

$$\exists q > 0 \quad \forall^f H \subset G \quad \exists^f U \subset G \quad \frac{|\{x \in U \mid \forall h \in H \ hx \in U\}|}{|U|} \geqslant q. \tag{11}$$

Proof. (1) Sufficiency. Consider an ascending sequence of subsets $H_i \subset H_{i+1} \subset \ldots \subset G$ exhausting G and a sequence U_i that corresponds to it. Set $\mathfrak{H} = \mathop{\times}\limits_i H_i/\mathfrak{F}$, $\mathfrak{U} = \mathop{\times}\limits_i U_i/\mathfrak{F}$. It is clear that $G \subset \mathfrak{H} \subset G^{\mathfrak{F}}$. Consider the set $V = \bigcap\limits_{g\in G} g\mathfrak{U}$ in the space $(\mathfrak{U}, \mu_L)$. It is not hard to see that $V \supset \mathop{\times}\limits_i \{x \in U_i \mid \forall h \in H_i \ hx \in U_i\}/\mathfrak{F}$, and consequently $\mu_L(V) \geqslant q$. We define a measure $\mu(A) = \mu_L(A^{\mathfrak{F}} \cap V)$ on the set 2^G. First we check that it is left-invariant. Let C be any set on which the group G is acting. For $D \subset C$ and $B \subset G$ we write $B \star D \stackrel{\text{def}}{=} \bigcap\limits_{g\in B} gD$. It is clear that $V = \lim\limits_{B\in\rho(G)} B \star \mathfrak{U}$, where $\rho(G) = \{B_\alpha\}$ is an arbitrary directed family of finite subsets of G expanding to G. Fix any $g \in G$. The directed family $g\rho(G) = \{gB_\alpha\}$ is also expanding to G, so $V = \lim\limits_{B\in g\rho(G)} B \star \mathfrak{U}$. Now we have

$$\mu(A) = \mu_L(A^{\mathfrak{F}} \cap V) = \lim_{B\in\rho(G)} \mu_L(A^{\mathfrak{F}} \cap B \star \mathfrak{U}) = \lim_{B\in\rho(G)} \lim_{\mathfrak{F}} \frac{|A \cap B \star U_i|}{|U_i|}$$

$$= \lim_{B\in\rho(G)} \lim_{\mathfrak{F}} \frac{|gA \cap gB \star U_i|}{|U_i|} = \lim_{B\in\rho(G)} \mu_L(gA^{\mathfrak{F}} \cap gB \star \mathfrak{U})$$

$$= \lim_{B\in g\rho(G)} \mu_L(gA^{\mathfrak{F}} \cap B \star \mathfrak{U}) = \mu_L(gA^{\mathfrak{F}} \cap V) = \mu(gA).$$

(2) Necessity. Let G be an amenable group. Then it satisfies the Følner condition, which may be written, in particular, in the form

$$\forall^f H \subset G \; \exists^f U \subset G \; \forall h \in H \quad \frac{|U \setminus h^{-1}U|}{|U|} \leqslant \frac{1}{2|H|}.$$

On the other hand, it is clear that

$$\{x \in U \mid \forall h \in H \; hx \in U\} = \bigcap_{h\in H} h^{-1}U \cap U = U \setminus \bigcup_{h\in H} (U \setminus h^{-1}U).$$

It follows that

$$\frac{|\{x \in U \mid \forall h \in H \; hx \in U\}|}{|U|} \geqslant 1 - \frac{|\bigcup_{h\in H} (U \setminus h^{-1}U)|}{|U|}$$

$$\geqslant 1 - \sum_{h\in H} \frac{|U \setminus h^{-1}U|}{|U|} \geqslant 1 - \sum_{h\in H} \frac{1}{2|H|} = 1 - \frac{1}{2} = \frac{1}{2},$$

and thus condition (11) is satisfied for $q = \frac{1}{2}$. □

Corollary 3.13. *Let G be an LEF-group and*

$$\exists q > 0 \quad \forall^f H \subset G \quad \exists^f (K, \odot), H \subset K \subset G \mid$$

$$\forall h_1, h_2 \in H \, (h_1h_2 \in H \Longrightarrow h_1h_2 = h_1 \odot h_2)$$

$$\&$$

$$\frac{|\{k \in K \mid \forall h \in H \; hk = h \odot k\}|}{|K|} \geqslant q.$$

Then the group G is amenable.

Proof. Since the inclusion

$$\{k \in K \mid \forall h \in H \; hk \in K\} \supset \{k \in K \mid \forall h \in H \; hk = h \odot k\}$$

is true, condition (11) is satisfied for $U = K$. □

In [15], a question on how one can single amenable groups out of the class of LEF-groups was raised and a conjecture that amenability is connected to some quantitative characteristics of approximation of the group G by finite ones was proposed. The above corollary shows that it is possible to consider the following function defined for all finite subsets of G as such a characteristic.

Definition 3.14. *We say that a finite group $(K, \odot)$ H-approximates the group G if the inclusion $H \subseteq K \subseteq G$ of the sets takes place and*

$$\forall h_1, h_2 \in H \, (h_1 h_2 \in H \Longrightarrow h_1 h_2 = h_1 \odot h_2) \, .$$

In this case we write $K \in Ap_H(G)$.

Let us define the function λ_G on finite subsets of G by

$$\lambda_G(H) \stackrel{\text{def}}{=} \sup_{K \in Ap_H(G)} \frac{|\{k \in K \mid \forall h \in H \; hk = h \odot k\}|}{|K|}.$$

Then the above corollary may be reformulated as follows.

Proposition 3.15. *If an LEF-group G satisfies the condition*

$$\inf_{H \subseteq G, \; |H| < \infty} \lambda_G(H) > 0,$$

then G is amenable.

The question of whether the reverse is true remains open.

In [15], it was proved that G is an LEF-group if and only if there exists an expanding sequence of finite subsets L_n of G with union G such that there is an operation $\odot_n$ on every L_n that makes it a group and the following condition is satisfied:

$$\forall a, b \in L_n \; (a \cdot b \in L_n \Longrightarrow a \odot_n b \in L_n).$$

We shall call the sequence $(L_n, \odot_n)$ the *approximating sequence of the group G*.

If the approximating sequence L_n of the group G is also a Følner sequence of subsets, then it is easily seen that

$$\forall \, 0 < q < 1 \; \forall^f H \subset G \; \exists N \in \mathbb{N} \; \forall n > N$$

$$\frac{|\{x \in L_n \mid \forall h \in H \; hx = h \odot_n x\}|}{|L_n|} \geqslant q$$

and thus the hypothesis of Proposition 3.15 is fulfilled.

It was conjectured in [15] that there exists a connection between the amenability of the group G and the asymptotic behavior of the function

$$\psi(n) = \frac{|\{< a, b > \in L_n^2 \mid a \cdot b \in L_n\}|}{|L_n|^2}, \tag{12}$$

where L_n is its approximating sequence. We consider below two examples of amenable LEF-groups (in the second case the group is even commutative) where the approximating sequence L_n is Følner, i.e., the hypothesis of Proposition 3.15 is satisfied, but $\lim\limits_{n \to \infty} \psi(n) = 0$.

3.1 The group $\mathrm{Symm}(\mathbb{Z}) \leftthreetimes \mathbb{Z}$.

This group was considered in [15] as an example of an LEF-group that is not locally finitely approximable.

Let $\mathrm{Symm}(\mathbb{Z})$ be the group of finite substitutions of $\mathbb{Z}$. The action of $\mathbb{Z}$ on itself by shifts generates the action of $\mathbb{Z}$ on $\mathrm{Symm}(\mathbb{Z})$ by the automorphisms $\pi \mapsto \pi^{(m)}$, $\pi \in \mathrm{Symm}(\mathbb{Z})$, $m \in \mathbb{Z}$, where $\pi^{(m)}$ is defined by the formula

$$\pi^{(m)}(k) \stackrel{\text{def}}{=} \pi(k+m) - m.$$

Thus, a semidirect product $\mathrm{Symm}(\mathbb{Z}) \leftthreetimes \mathbb{Z}$ is defined with the operation

$$(m, \pi) \cdot (k, \sigma) = (m + k, \pi^{(-k)}\sigma).$$

The group $\mathrm{Symm}(\mathbb{Z}) \leftthreetimes \mathbb{Z}$ is amenable as a semidirect product of amenable groups.

Let $\mathbb{Z}_{2n+1} \stackrel{\text{def}}{=} \{-n, \dots, n\}$ be the group of residues modulo $2n+1$, and

$$S_{2n+1} \stackrel{\text{def}}{=} \{\pi \in \mathrm{Symm}(\mathbb{Z}) \mid \pi|_{\mathbb{Z}\setminus\mathbb{Z}_{2n+1}} = Id\}.$$

We show that the subsets $U_n = \mathbb{Z}_{2n+1} \times S_{2n+1} \subset \mathrm{Symm}(\mathbb{Z}) \leftthreetimes \mathbb{Z}$ form a Følner sequence.

Take an element $x = (m, \pi) \in \mathrm{Symm}(\mathbb{Z}) \leftthreetimes \mathbb{Z}$. It is easily seen that there exists an element n_0 such that $m \in \mathbb{Z}_{2n_0+1}$ and $\pi \in S_{2n_0+1}$. We can assume that $n \geqslant n_0$. Now we count the number of elements $(k, \sigma) \in U_n$ such that

$$(m, \pi) \cdot (k, \sigma) \in U_n$$

or, what is the same,

$$m + k \in \mathbb{Z}_{2n+1}, \tag{13}$$

$$\pi^{(-k)}\sigma \in S_{2n+1}. \tag{14}$$

These conditions are satisfied if $|k| \leqslant n - n_0$. Indeed, in this case $|k+m| \leqslant |k| + |m| \leqslant (n - n_0) + n_0 = n$, and (13) is satisfied. Assume that $|t| > n$. Then $\sigma(t) = t$ since $\sigma \in S_{2n+1}$. As $|t-k| \geqslant |t| - |k| > n - (n - n_0) = n_0$, we have $\pi^{(-k)}(t) = t$. Thus, $\pi^{(-k)}\sigma(t) = t$ for $|t| > n$, which is equivalent to (14).

As for $|k| \leqslant n - n_0$ the choice of $\sigma \in S_{2n+1}$ is unrestricted, $|U_n \cap xU_n| \geqslant (2(n-n_0)+1)(2n+1)!$. Consequently,

$$\frac{|U_n \bigcap xU_n|}{|U_n|} \geqslant \frac{(2(n-n_0)+1)(2n+1)!}{(2n+1)(2n+1)!} = 1 - \frac{2n_0}{2n+1} \underset{n\to\infty}{\longrightarrow} 1,$$

which proves (8).

In [15], it is shown that $\mathrm{Symm}(\mathbb{Z}) \leftthreetimes \mathbb{Z}$ is an LEF-group. Moreover, the sets U_n with the operation $\odot_n$ defined below can play the role of elements of the approximating sequence L_n.

The action of $\mathbb{Z}_{2n+1}$ on itself by shifts modulo $2n+1$ induces the action of $\mathbb{Z}_{2n+1}$ on S_{2n+1} by the automorphisms $\pi \mapsto \pi^{[m]}$, $\pi \in S_{2n+1}$, $m \in \mathbb{Z}_{2n+1}$, given by the formula

$$\pi^{[m]}(k) \stackrel{\text{def}}{=} \begin{cases} \pi(k \oplus_n m) \ominus_n m, & k \in \mathbb{Z}_{2n+1}; \\ k, & k \notin \mathbb{Z}_{2n+1}, \end{cases}$$

where $\oplus_n$ and $\ominus_n$ are addition and subtraction modulo $2n+1$.

Now it is clear that $L_n = U_n$ is the support of the group $S_{2n+1} \leftthreetimes \mathbb{Z}_{2n+1}$, i.e., the operation in L_n is defined by the formula

$$(m, \pi) \odot_n (k, \sigma) = (m \oplus_n k, \pi^{[-k]}\sigma).$$

The groups $(L_n, \odot_n)$ form an approximating sequence of the group $G = \text{Symm}\,(\mathbb{Z}) \leftthreetimes \mathbb{Z}$.

To study the behavior of function (12) for the sequence $(L_n, \odot_n)$, we are going to count the number of pairs of elements that satisfy the relation

$$(m, \pi) \odot_n (k, \sigma) = (m, \pi) \cdot (k, \sigma), \tag{15}$$

or, which is the same, the two relations

$$m \oplus_n k = m + k, \tag{16}$$

$$\pi^{[-k]}\sigma(t) = \pi^{(-k)}\sigma(t). \tag{17}$$

Fix $k \in \mathbb{Z}_{2n+1} \subset \mathbb{Z}$. Assume, for the sake of definiteness, that $k > 0$. Then to satisfy (16) it is necessary that m be one of the numbers $\{-n, \dots, n-k\}$, i.e., it can be chosen in $(2n+1-k)$ ways.

Next we choose an arbitrary $\sigma \in S_{2n+1}$ (in one of the $(2n+1)!$ ways). For every $t \in \mathbb{Z}$, (17) is true. This is trivial if $t > n+k$ or $t < -n$, since $\sigma, \pi, \pi^{[-k]} \in S_{2n+1}$. In the case $n < t \leq n+k$, relation (17) is equivalent to $t - k = \pi(t-k)$. The latter means that π is the identity on $\{n+1-k, ..., n\}$. For $t \in \mathbb{Z}_{2n+1}$, (17) can be rewritten in the form

$$\pi(\sigma(t) \ominus_n k) \oplus_n k = \pi(\sigma(t) - k) + k.$$

We set $z = \sigma(t)$. Since σ is bijective on $\mathbb{Z}_{2n+1}$, z runs over $\mathbb{Z}_{2n+1}$ when t does the same. Therefore, the last relation is equivalent to

$$\forall z \in Z_{2n+1} \quad \pi(z \ominus_n k) \oplus_n k = \pi(z-k) + k.$$

For $z \in \{-n+k, ..., n\}$ this relation is clearly true, and for $z \in \{-n, ..., -n+k-1\}$ it is equivalent to

$$\pi(z \ominus_n k) = z \ominus_n k,$$

which also means that π is identity on $\{n+1-k, ..., n\}$. Thus π may be chosen in $(2n+1-k)!$ ways.

Now we see that the number of pairs satisfying (15) with fixed $k > 0$ equals $(2n+1-k)(2n+1)!(2n+1-k)!$. Similar arguments show that if $k \leqslant 0$, then $(2n+1+k)(2n+1)!(2n+1+k)!$. Finally, the number of pairs satisfying (15) is equal to

$$\sum_{k=-n}^{n} (2n+1-|k|)(2n+1)!(2n+1-|k|)!$$

$$= 2\sum_{k=1}^{n}(2n+1-k)(2n+1)!(2n+1-k)! + (2n+1)(2n+1)!(2n+1)!$$

$$= (2n+1)!\left[2\sum_{t=n+1}^{2n} t\ \ t! + (2n+1)(2n+1)!\right]$$

$$= (2n+1)!\left[2\sum_{t=n+1}^{2n} ((t+1)-1)t! + (2n+1)(2n+1)!\right]$$

$$= (2n+1)!\left[2\left(\sum_{t=n+2}^{2n+1} t! - \sum_{t=n+1}^{2n} t!\right) + (2n+1)(2n+1)!\right]$$

$$= (2n+1)!\left[2\left((2n+1)! - (n+1)!\right) + (2n+1)(2n+1)!\right]$$

$$= (2n+1)!\left[(2n+3)(2n+1)! - 2(n+1)!\right].$$

Correspondingly,

$$\psi(n) = \frac{(2n+1)!\left[(2n+3)(2n+1)! - 2(n+1)!\right]}{\left[(2n+1)(2n+1)!\right]^2}$$

$$= \frac{2n+3}{(2n+1)^2} - \frac{2(n+1)!}{(2n+1)^2(2n+1)!} \sim \frac{1}{2n} \underset{n\to\infty}{\longrightarrow} 0.$$

3.2. The group $(\mathbb{Q}_+, \cdot)$.

Consider the multiplicative group $(\mathbb{Q}_+, \cdot)$ of positive rationals. Being commutative, it is an amenable LEF-group. Every number $x \in \mathbb{Q}_+$ is uniquely representable as an infinite product $x = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_n^{k_n} \cdot \ldots$, where p_n are prime numbers taken in natural order and all but a finite number of $k_n \in \mathbb{Z}$ are zero. Thus, $\mathbb{Q}_+$ is an infinite direct sum $\overset{\infty}{\underset{n=1}{\oplus}} \mathbb{Z}^{(n)}$. This presentation gives a “natural” approximation of $\mathbb{Q}_+$.

Now we put $L_n = \overset{n}{\underset{i=1}{\oplus}} \mathbb{Z}_{2n+1}^{(i)}$, where $Z_{2n+1} = \{-n, -n+1, \ldots, n\}$ is the group of residues modulo $2n+1$. It is clear that the sequence L_n is an approximating sequence of the group $(\mathbb{Q}_+, \cdot)$.

Since for any $k \in \mathbb{Z}$ and sufficiently large n

$$|(k + \mathbb{Z}_{2n+1}) \cap \mathbb{Z}_{2n+1}| = (2n+1) - |k|,$$

for any $x = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_m^{k_m} \in \mathbb{Q}_+$ we have

$$|xL_n \cap L_n| = (2n+1-|k_1|)(2n+1-|k_2|)\ldots(2n+1-|k_m|)(2n+1)^{n-m}$$

and, therefore,

$$\frac{|(xL_n) \triangle L_n|}{|L_n|} = 2\frac{(2n+1)^n - (2n+1-|k_1|)(2n+1-|k_2|)\dots(2n+1-|k_m|)(2n+1)^{n-m}}{(2n+1)^n}$$

$$= 2\left(1 - \frac{2n+1-|k_1|}{2n+1}\frac{2n+1-|k_2|}{2n+1}\dots\frac{2n+1-|k_m|}{2n+1}\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Thus L_n is a Følner sequence, i.e., the hypothesis of Proposition 3.15 is fulfilled.

It is not hard to see that the number of pairs of $\mathbb{Z}_{2n+1}$ whose sums (in the conventional sense) are in the same set is equal to $(2n+1)^2 - n(n+1) = 3n^2 + 3n + 1$. Now it is easy to calculate function (12) for the obtained approximation:

$$\psi(n) = \frac{(3n^2+3n+1)^n}{(2n+1)^{2n}} = \frac{(3n^2+3n+1)^n}{(4n^2+4n+1)^n} \sim \left(\frac{3}{4}\right)^n \underset{n\to\infty}{\longrightarrow} 0.$$

4. Approximability of discrete groups and their Hopf algebras

Definition 4.1. *We say that the bialgebra $(A, \cdot, \Delta)$ is approximated by finite-dimensional bialgebras $(A_n, \odot_n, \Delta_n)$ if*

(1) *A_n is a linear subspace of A_{n+1} and $A = \cup_n A_n$;*
(2) *$\forall a, b \in A\ \exists n\ \forall m > n\ \ a \cdot b = a \odot_m b$;*
(3) *$\forall a \in A\ \ \exists n\ \ \forall m > n\ \ \Delta_m(a) = \Delta(a)$.*

We remind the reader of the definition of the cocommutative Hopf algebra $A(G)$ of a discrete group G (over field $\mathbb{C}$). As a linear space,

$$A(G) = \{\sum_{i=1}^{n} a_i g_i\ ;\ a_i \in \mathbb{C},\ g_i \in G,\ n < \infty\}.$$

It suffices to define multiplication $(\cdot)$, comultiplication (Δ), antipode S, and counit ϵ only on group elements because they can be extended further by linearity. Multiplication is the same as the group one, and for $g \in G$ we put $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$, and $\epsilon(g) = 1$.

Theorem 4.2. *The cocommutative Hopf algebra of a countable discrete group G is approximable (as a bialgebra) by finite-dimensional bialgebras if and only if G is an LEF-group.*

For the proof, recall the definition of the group-like element of a bialgebra.

Definition 4.3. *A group-like element of the bialgebra A is an element $a \in A$ such that $a \neq 0$ and $\Delta(a) = a \otimes a$.*

We also need two almost evident properties of group-like elements.

Proposition 4.4. *Group-like elements of a bialgebra are linearly independent.*

Proof. Let $a = \lambda_1 g_1 + \lambda_2 g_2 + ... + \lambda_n g_n$ be a group-like element with $\lambda_i \neq 0$ and assume that $g_1, g_2, ..., g_n$ are linearly independent group-like elements. Then $x = a \otimes a - \Delta(a) = 0$ and we have

$$x = \sum_{\substack{i=1\\j=1}}^{n} \lambda_i\lambda_j g_i \otimes g_j - \sum_{i=1}^{n} \lambda_i g_i \otimes g_i = 0,$$

which is not true since, for example, $\delta_i \otimes \delta_j(x) = \lambda_i\lambda_j$, where $\delta_k(g_i) = 1$ for $k = i$ and $\delta_k(g_i) = 0$ for $k \neq i$. $\square$

Proposition 4.5. *A product of group-like elements of a bialgebra is either a group-like element or* 0.

Proof. $\Delta(g_1 \cdot g_2) = \Delta(g_1) \cdot \Delta(g_2) = g_1 g_2 \otimes g_1 g_2$ □

Proof of the theorem. Let $A(G)$ be an approximable commutative Hopf algebra of a discrete group G and $(A_n, \odot_n, \Delta_n)$ the approximating bialgebras. For any set of elements $\{g_1, g_2, ..., g_n\} \subset G$, we construct a group $(H, \odot)$, $\{g_1, g_2, ..., g_n\} \subset H \subset G$, such that

(1) $(H, \odot)$ is a finite group;
(2) $g_i^{\pm 1} \cdot g_j^{\pm 1} = g_i^{\pm 1} \odot g_j^{\pm 1}$.

This will mean exactly that G is an LEF-group. Choose m such that

(i) $g_1^{\pm 1}, g_2^{\pm 1}, ..., g_n^{\pm 1} \in A_m$;
(ii) $g_i^{\pm 1} \cdot g_j^{\pm 1} = g_i^{\pm 1} \odot_m g_j^{\pm 1}$, $i, j = 1, ..., n$;
(iii) $\Delta_m(g_i^{\pm 1}) = g_i^{\pm 1} \otimes g_i^{\pm 1}$, $i = 1, ..., n$.

We consider a subgroup H in $(A_m, \odot_m)$ generated by the elements $\{g_i^{\pm 1}\ ,\ i = 1, ..., n\}$. By property (ii), the generating set of this group contains the inverse (with respect to $\odot_m$) of each of its elements. Consequently, H is a group. By Proposition 4.5 and property (iii), the group H consists of group-like elements of the bialgebra A_m. By Proposition 4.4, all elements of H are linearly independent in A_m. Hence, since A_m is finite-dimensional, the group H is finite.

Conversely, let G be an LEF-group. Consider its approximating sequence $(K_n, \odot_n)$ (see the definition of approximating sequence in Sec. 3 after Proposition 3.15). Then bialgebras $A(K_n)$ approximate $A(G)$. Indeed, let $a = \sum_{i=1}^{k} a_i g_i$ and $b = \sum_{i=1}^{k'} b_i h_i$, where $b_i, h_i \in \mathbb{C}$ and $h_i, g_i \in G$. Choose n such that $h_i, g_j, h_i \cdot g_j \in K_n$, $i = 1, ..., k'$, $j = 1, ..., k$. Then $a \odot_m b = a \cdot b$ for any $m \geq n$ and $\Delta_m(a) = \Delta(a)$ for any m, since the linear space $A(K_m)$ is a subspace of $A(G)$ for every m. □

5. Approximability of compact groups and their Hopf algebras

In this section, we consider the correlation between the approximability of compact groups and of their commutative Hopf algebras.

Let G be a compact group and $A(G)$ its commutative Hopf algebra, i.e., $A(G)$ is an algebra (over $\mathbb{C}$) of matrix elements of finite-dimensional representations of G. The product $(\cdot)$, coproduct (Δ), antipode S, and counit ϵ are defined as follows. For $f, h \in A(G)$ and $g, g' \in G$ we put $(f \cdot h)(g) = f(g)h(g)$, $(\Delta f)(g, g') = f(gg')$, $(Sf)(g) = f(g^{-1})$, and $\epsilon(f) = f(e)$, where e is the unit of the group G. We assume that the norm on $A(G)$ is defined by $\|f(\cdot)\| = \sup\limits_{g \in G} |f(g)|$.

Definition 5.1. *We say that a normed bialgebra* $(A, \|\ \ \|, \cdot, \Delta)$ *is approximable by finite-dimensional normed bialgebras* $(A_n, \|\ \ \|_n, \odot_n, \Delta_n)$ *if there exist maps* $\psi_n : A \to A_n$ *such that*

(1) $\psi_n(\cdot)$ *is a continuous homomorphism, i.e.,* $\psi_n(ab) = \psi_n(a) \odot_n \psi_n(b)$ *and* $\|\psi_n(a)\|_n < C_n\|a\|$;
(2) $\exists 0 < \epsilon < C\ \forall a \in A\ \exists m \in \mathbb{N}\ \forall n > m\ \ \epsilon\|a\| < \|\psi_n(a)\| < C\|a\|$;
(3) $\forall a \in A\ \|\psi_n \otimes \psi_n \Delta a - \Delta_n \psi_n(a)\|_{n \times n} \to 0$ *as* $n \to \infty$, *where the norm on* $A_n \otimes A_n$ *is defined by*

$$\|r\|_{n \times n} = \sup_{\substack{\alpha, \beta \in A_n^* \\ \|\alpha\|_* = \|\beta\|_* = 1}} | < \alpha \otimes \beta, r > |.$$

The following definition of the approximability of compact groups by finite ones is equivalent to the corresponding definition from [4] (see also [15]).

Definition 5.2. *We say that a compact group G is approximable by finite groups $(G_n, *_n)$ if there exist maps $\phi_n : G_n \to G$ such that*

(1) *for all open $U \subset G$ $\exists m \in \mathbb{N}$ $\forall n > m$ $\phi_n(G_n) \cap U \neq \emptyset$*
(2) *$\forall U_e$ $\exists m \in \mathbb{N}$ $\forall n > m$ $\forall g_1, g_2 \in G_n$ $\phi_n^{-1}(g_1 *_n g_2)\phi_n(g_1)\phi_n(g_2) \in U_e$, where U_e is a neighborhood of the unity in G.*

The following statement follows immediately from Definition 5.2.

Proposition 5.3. *Assume that a compact group G is approximable by finite groups G_n and $\phi_n : G_n \to G$ are the maps involved in Definition 5.2. Then*

(3) *$\forall U_e$ $\exists m$ $\forall n > m$ $\phi_n(e_n) \in U_e$, where e_n is the unity in G_n;*
(4) *$\forall U_e$ $\exists m$ $\forall n > m \forall g \in G_n$ $\phi_n(g)\phi_n(g^{-1}) \in U_e$.*

Proof. Taking $g_2 = e_n$ in item (2) of Definition 5,2, we get $\phi(e_n) \in U_e$, which proves (3).

Taking $g_1 = g$ and $g_2 = g^{-1}$, and also replacing U_e with U'_e in item (2) of Definition 5.2, we get $\phi_n^{-1}(e_n)\phi_n(g)\phi_n(g^{-1}) \in U'_e$. Arguing as in the proof of the preceding item, we obtain $\phi_n(g)\phi_n(g^{-1}) \in \{xy;\ \ x, y \in U'_e\}$. It remains to choose numbers that ensure the inclusion $\{xy;\ \ x, y \in U'_e\} \subset U_e$. □

Theorem 5.4. *A commutative Hopf algebra of the compact group G is approximable (as a normed bialgebra) by finite-dimensional commutative semisimple normed Hopf algebras if and only if the group G is approximable by finite groups.*

Proof. $\Leftarrow$. Assume that the group G is approximable by finite groups G_n and ϕ_n are the maps from Definition 5.2. Then, as direct computations show, the algebra $A(G)$ is approximable by the algebras $A(G_n)$ (if we introduce the norm $\|f\| = \sup_{g\in G_n} |f(g)|$ for $f \in A(G_n)$). The maps ψ_n from Definition 5.1 are given by the relation $\psi_n(f)(g) = f(\phi_n(g))$, where $f \in A(G)$ and $g \in G_n$.

$\Rightarrow$. Let $\mathcal{X}$ be the space of continuous characters of the algebra $A = A(G)$ with multiplication $\chi_1\chi_2(a) = \chi_1 \otimes \chi_2(\Delta a)$ and $*$-weak topology. It is known that $\mathcal{X}$ is a compact group isomorphic to the group G. Analogously, the character spaces $\mathcal{X}_n$ (with multiplication $\chi_1 *_n \chi_2(a) = \chi_1 \otimes \chi_2(\Delta_n a)$) approximating the algebras A_n are finite groups.[1] Since the mappings ψ_n in Definition 5.2 are homomorphisms, $\psi_n^*(\mathcal{X}_n) \subset \mathcal{X}$. Let $G_n = \mathcal{X}_n$, $G = \mathcal{X}$. We are going to check that $\phi_n = \psi_n^*$ satisfy properties (1) and (2) of Definition 5.2. For an arbitrary neighborhood $U \subset G$, choose a function $f \in C(G)$ such that $f \not\equiv 0$ and $f(G\backslash U) = \{0\}$. Since A is dense in $C(G)$, the homomorphisms ψ_n can be extended to $C(G)$, preserving property (2) in Definition 5.2, i.e., $\psi_n(f) \neq 0$ for n large enough, and it follows from the semisimplicity of A_n that there exists $\chi_n \in G_n$ such that $< \psi_n(f), \chi_n > \neq 0$, and consequently $\psi_n^*(\chi_n) \in U$ (here we identify $\mathcal{X}$ and G). Thus, we have proved the statement of item (1) of Definition 5.2.

Let $\chi_0 \in \mathcal{X}$, $a_1, a_2, ..., a_r \in A$, $\epsilon \in \mathbb{R}^+$. We set

$$U_{\chi_0}(a_1, a_2, ..., a_r, \epsilon) = \{\chi \in \mathcal{X}\ :\ |\chi_0(a_i) - \chi(a_i)| < \epsilon,\ \text{for } i = 1, ..., r\}.$$

[1] Actually, each finite-dimensional commutative semisimple Hopf algebra is the group algebra of its character group.

Since A is dense in $C(G)$, the $*$-weak topology on $\mathcal{X}$ is defined by the family of neighborhoods

$$\{U_{\chi_0}(a_1, a_2, ..., a_r, \epsilon) \ : \ \chi_0 \in \mathcal{X}, \ r \in \mathbb{N}, \ a_1, a_2, ..., a_r \in A, \ \epsilon \in \mathbb{R}^+\}.$$

We need the following lemma.

Lemma 5.5. *For all $\{a_1, a_2, ..., a_r\} \subset A$ and $\epsilon \in \mathbb{R}^+$ there exist $\{b_1, b_2, ..., b_k\} \subset A$ and $\epsilon' \in \mathbb{R}^+$ such that for all $\chi_1, \chi_2 \in \mathcal{X}$ the following inclusion holds:*

$$\chi_1 U_{\chi_2}(b_1, ..., b_k, \epsilon') \subseteq U_{\chi_1\chi_2}(a_1, ..., a_r, \epsilon).$$

Proof. By the definition of the Hopf algebra A,

$$\Delta a_i = \sum_{j=1}^{k_i} a_i^j \otimes b_{i,j}.$$

We can take the set of all $b_{i,j}$ in the role of the set $\{b_1, ..., b_k\}$ (this set is finite), and any positive number ϵ' satisfying the inequality

$$\epsilon' \leq \frac{\epsilon}{\sup_i \sum_j \|a_i^j\|}.$$

Indeed, let $\chi \in U_{\chi_2}(b_1, ..., b_k, \epsilon')$. Then, since $\|\chi_1\| = 1$, we have

$$|\chi_1\chi(a_i) - \chi_1\chi_2(a_i)| = |(\chi_1 \otimes \chi - \chi_1 \otimes \chi_2)(\sum a_i^j \otimes b_{i,j})|$$

$$\leq \sum |\chi_1(a_i^j)| \cdot |\chi(b_{i,j}) - \chi_2(b_{i,j})| < \epsilon' \sum \|a_i^j\| \leq \epsilon,$$

but this means that $\chi_1\chi \in U_{\chi_1\chi_2}$. □

Putting $\chi_2 = \chi, \ \chi_1 = \chi^{-1}$ in this lemma, we come to the following proposition.

Corollary 5.6. *For all $\{a_1, a_2, ..., a_r\} \subset A$ and $\epsilon \in \mathbb{R}^+$ there exist $\{b_1, b_2, ..., b_k\} \subset A$ and $\epsilon' \in \mathbb{R}^+$ such that for all $\chi \in \mathcal{X}$ the following inclusion holds:*

$$U_\chi(b_1, ..., b_k, \epsilon') \subseteq \chi U_e(a_1, ..., a_r, \epsilon).$$

Turning to the proof of the theorem, we can see that all that remains to be proved is that

$$\forall U_e \ \exists m \in \mathbb{N} \ \forall n > m \ \forall \chi_1, \chi_2 \in \mathcal{X}_n \ \ \phi_n^{-1}(\chi_1 *_n \chi_2)\phi_n(\chi_1)\phi_n(\chi_2) \in U_e,$$

where one can assume that $U_e = U_e(a_1, a_2, ..., a_r, \epsilon)$. By Corollary 5.6, the condition

$$\phi_n(\chi_1)\phi_n(\chi_2) \in \phi_n(\chi_1 *_n \chi_2)U_e(a_1, a_2, ..., a_r, \epsilon)$$

may be replaced by the condition

$$\phi_n(\chi_1)\phi_n(\chi_2) \in U_{\phi_n(\chi_1 *_n \chi_2)}(b_1, ..., b_k, \epsilon').$$

We show that the last condition is fulfilled for n large enough:

$$|\phi_n(\chi_1)\phi_n(\chi_2)(b) - \phi_n(\chi_1 *_n \chi_2)(b)| = |\chi_1 \otimes \chi_2(\psi_n \otimes \psi_n \Delta b) - \chi_1 \otimes \chi_2(\Delta_n \psi_n(b))| =$$

$$= |\chi_1 \otimes \chi_2(\psi_n \otimes \psi_n \Delta b - \Delta_n \psi_n(b))| \leq \|\psi_n \otimes \psi_n \Delta b - \Delta_n \psi_n(b)\|_{n \times n}.$$

Now by condition (3) of Definition 5.1 we get

$$\|\psi_n \otimes \psi_n \Delta b - \Delta_n \psi_n(b)\|_{n \times n} \to 0 \text{ for } n \to \infty.$$

□

Remarks. We show in the next section that many compact Lie groups, for example, $SO(3)$, are not approximable by finite groups in the sense of Definition 5.2. Thus a question arises on how to modify this definition so that the class of groups approximable by finite ones would be significantly wider. Following the ideas of A. M. Vershik [16, 15], one can modify Definition 5.1 of the approximability of Hopf algebras in several ways. We give some possibilities.

1. If, as in the previous section, we keep considering the approximating algebras to be only bialgebras and not necessarily Hopf algebras, we shall come to a definition of approximability of a group by finite ones that is similar to Definition 5.2. Moreover, one can prove the following statement exactly in the same way as Theorem 5.6.

Theorem 5.7. *The commutative Hopf algebra of a compact group G is approximable (as a normed algebra) by finite-dimensional commutative semisimple normed algebras if and only if G is approximable by finite groups.*

2. One can weaken Definition 5.1 by assuming that the linear operators ψ_n are not homomorphisms in general but they satisfy the condition

$$\forall a, b \in A \quad \lim_{n\to\infty} \|\psi_n(ab) - \psi_n(a)\psi_n(b)\|_n = 0.$$

This modification of Definition 5.1 has not yet been studied.

3. In the above modification of Definition 5.1, one can abandon the commutativity of approximating finite-dimensional algebras. We note that noncommutative finite-dimensional deformations of function algebras on some simplectic manifolds have been considered in [3] in connection with the quantization problem. I. A. Shereshevskii pointed out that these deformations can also be considered as approximations in our sense.

5.1. Approximability of compact Lie groups.

For later use, it is convenient to reformulate the definition of approximability of a compact group by finite ones. As is known, the topology of a separable compact group can be defined by means of an invariant metric. In the sequel, we shall assume that such a metric is given and is denoted by d.

Definition 5.8. *We say that $A \subset G$ is an ϵ-net in G if*

$$\forall g \in G \quad \exists a \in A \quad (d(g, a) < \epsilon).$$

Definition 5.9. *Let H be a group and G be a compact topological group with invariant metric d. A map $\rho : H \to G$ is called an ϵ-homomorphism if*

$$\forall x, y \in H \;\; (d(\rho(xy), \rho(x)\rho(y)) < \epsilon).$$

The following corollary is a straightforward consequence of these definitions and the definition of approximability of a compact group by finite ones.

Corollary 5.10. *A compact group G is approximable by finite groups if and only if for every $\epsilon > 0$ there exists a finite group H and an ϵ-homomorphism ρ such that $\rho(H)$ is an ϵ-net in G.*

The next theorem gives a necessary and sufficient condition for approximability of compact Lie groups.

Theorem 5.11. *A compact finite-dimensional Lie group G is approximable by finite groups if and only if for any $\epsilon > 0$ there exists a finite subgroup $H \subset G$ that is an ϵ-net in G.*

The next statement is a consequence of this theorem and the fact that there are only a finite number of subgroups of $SO(3)$ that are not reduced to rotations around the same axis.

Corollary 5.12. *The group $SO(3)$ is not approximable by finite ones.*

Now we turn back to Theorem 5.11. The approximability of the groups satisfying the hypothesis of the theorem is obvious. The difficult part follows from the next theorem.

Theorem 5.13. *For any finite-dimensional compact Lie group G there exists an $\epsilon' > 0$ such that for any $0 < \epsilon < \epsilon'$ and any ϵ-homomorphism $\rho : H \to G$ of a compact (in particular, finite) group H there exists a homomorphism $\phi : H \to G$ such that $d(\phi(h), \rho(h)) < 2\epsilon$ for all $h \in H$.*

Actually, Theorem 5.13 is a modification of Kazhdan's theorem on ϵ-representations proved for amenable groups [7]. For the sake of completeness, we give here a somewhat more elementary proof of Theorem 5.13 than that given in [7]. This proof is also valid for amenable groups.

We may regard G as a Lie subgroup of a finite-dimensional unitary group $U(n)$ because any compact Lie group has a faithful unitary representation. Then we can define the metric by $d(g_1, g_2) = \|g_1 - g_2\|$, where $\|.\|$ is a standard norm in $L(n) \supset U(n)$, the space of all linear operators in the n-dimensional unit space. We denote by $\int\limits_{x\in H} ...dx$ the integration over an invariant measure. In the case of a finite group,

$$\int\limits_{x\in H} f(x)\delta x = \frac{1}{|H|}\sum_{x\in H} f(x).$$

The following statement is the main point in the proof of the theorem.

Proposition 5.14. *Let $\epsilon_n > 0$ be sufficiently small. Then for any ϵ_n-homomorphism ρ_n : $H \to G$ one can construct a $6\epsilon_n^2$-homomorphism $\rho_{n+1} : H \to G$ such that $\|\rho_n(x) - \rho_{n+1}(x)\| \leq \epsilon_n + \frac{\epsilon_n^2}{1-\epsilon_n}$.*

Then we take $\epsilon_0 = \epsilon$, $\rho_0 = \rho$, and construct a sequence ρ_k as in Proposition 5.14. It is easy to check that $\phi = \lim\limits_{k\to\infty} \rho_k$ satisfies the claim of the theorem when $\epsilon > 0$ is small enough.

Now we prove Proposition 5.14.

1. Let $\tilde{\rho}_1 : H \to L(n)$ be given by

$$\tilde{\rho}_1(h) = \int\limits_{x\in H} \rho^{-1}(x)\rho(xh)\mathrm{d}x;$$

then for all $h, g \in H$ we have

$$\|\tilde{\rho}_1(h)\tilde{\rho}_1(g) - \tilde{\rho}_1(hg)\| \leq 2\epsilon^2 \text{ and } \|\tilde{\rho}_1(h) - \rho(h)\| \leq \epsilon. \tag{18}$$

Indeed,

$$\tilde{\rho}_1(h)\tilde{\rho}_1(g) - \tilde{\rho}_1(hg) = \int\limits_{x\in H}\int\limits_{y\in H} (\rho^{-1}(x)\rho(xh)\rho^{-1}(y)\rho(yg) - \rho^{-1}(x)\rho(xhg))\mathrm{d}x\mathrm{d}y = I_1 + I_2,$$

where

$$I_1 = \int\int [\rho^{-1}(x)\rho(xh) - \rho(h)][\rho^{-1}(y)\rho(yg) - \rho(g)]\mathrm{d}x\mathrm{d}y$$

and

$$I_2 = \int\int [\rho^{-1}(x)\rho(xh)\rho(g) + \rho(h)\rho^{-1}(y)\rho(yg) - \rho(h)\rho(g) - \rho^{-1}(x)\rho(xhg)]\mathrm{d}x\mathrm{d}y.$$

Then

$$\|I_1\| \le \|\int \rho^{-1}(x)[\rho(xh) - \rho(x)\rho(h)]\mathrm{d}x\| \cdot \|\int \rho^{-1}(y)[\rho(yg) - \rho(y)\rho(g)]\mathrm{d}x\| \le \epsilon^2,$$

$$I_2 = \int [\rho^{-1}(x)\rho(xh)\rho(g) - \rho^{-1}(x)\rho(xhg)]\mathrm{d}x - \int [\rho(h)\rho^{-1}(x)\rho(x)\rho(g) - \rho(h)\rho^{-1}(x)\rho(xg)]\mathrm{d}x$$
$$= \int \rho^{-1}(x)[\rho(xh)\rho(g) - \rho(xhg)]\mathrm{d}x - \int \rho(h)\rho^{-1}(x)[\rho(x)\rho(g) - \rho(xg)]\mathrm{d}x.$$

Replacing in the last integral xh by x, we get

$$\|I_2\| = \|\int [\rho^{-1}(x) - \rho(h)\rho^{-1}(xh)][\rho(xh)\rho(g) - \rho(xhg)]\mathrm{d}x\| \le \epsilon^2$$

because

$$\|\rho^{-1}(x) - \rho(h)\rho^{-1}(xh)\| = \|\rho^{-1}(x)(\rho(xh) - \rho(x)\rho(h))\rho^{-1}(xh)\|.$$

Thus the first inequality (18) is proved, whereas the second is obvious.

2. In general, $\tilde{\rho}_1(H) \not\subset G$, so we construct, starting from $\tilde{\rho}_1$, a map ρ_1 satisfying the hypothesis of the proposition. Let $U_\epsilon = \{g \in U(n) \mid \|g - e\| < \epsilon\}$. For $\epsilon < 1$, there exists a map $\ln : U_\epsilon \to su(n)$ that can be represented as a sum of convergent series:

$$\ln(u) = \sum_{i=1}^{\infty} -(-1)^i \frac{(u-e)^i}{i}.$$

We need some estimation of ln and exp resulting from their expansions into power series.

Lemma 5.15. *If* $\|u - e\| < \epsilon < 1$, *then the inequality*

$$\|\ln(u) - (u - e)\| < \frac{\epsilon^2}{2(1-\epsilon)}$$

holds. If $x \in L(n)$ *and* $\|x\| < \epsilon$, *then*

$$\|\exp(x) - (x + e)\| < \frac{\epsilon^2}{2(1-\epsilon)}.$$

Proof. Estimate from above by geometric progressions. □

According to [12], one can choose $\epsilon > 0$ small enough for $\ln(U_\epsilon \cap G)$ to be in a Lie algebra of the group G and, therefore,

$$\forall x \ \ (x \in \mathrm{span}(\ln(U_\epsilon \cap G)) \Longrightarrow \exp(x) \in G)\,. \tag{19}$$

Let

$$b(h) = \int \rho^{-1}(x)[\ln(\rho(xh)\rho^{-1}(h)\rho^{-1}(x))]\rho(x)\mathrm{d}x.$$

By (19), $\exp[b(h)] \in G$. Set $\rho_1(h) = \exp[b(h)]\rho(h)$. Then it follows from Lemma 5.15 that

$$\|\rho_1(h) - \tilde{\rho}_1(h)\| \le \frac{\epsilon^2}{1-\epsilon}.$$

Hence

$$\|\rho_1(gh) - \rho_1(g)\rho_1(h)\| = \|\rho_1(gh) - \tilde{\rho}_1(gh)\| + \|\tilde{\rho}_1(gh) - \tilde{\rho}_1(g)\tilde{\rho}_1(h)\|$$
$$+\|\tilde{\rho}_1(g)\tilde{\rho}_1(h) - \rho_1(g)\tilde{\rho}_1(h)\| + +\|\rho_1(g)\tilde{\rho}_1(h) - \rho_1(g)\rho_1(h))\|$$
$$\le \frac{\epsilon^2}{1-\epsilon} + 2\epsilon^2 + \epsilon^2\frac{1+\epsilon}{1-\epsilon} + \frac{\epsilon^2}{1-\epsilon} = \epsilon^2(2 + \frac{3+\epsilon}{1-\epsilon}).$$

The lemma is proved.

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