# PROBLEMS AND SOLUTIONS

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*Proposed problems and solutions should be sent in duplicate to the* MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before September 30, 2009. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

## PROBLEMS

**11432**. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let $P$ be a polynomial of degree $n$ with complex coefficients and with $P(0) = 0$. Show that for any complex $\alpha$ with $|\alpha| < 1$ there exist complex numbers $z_1, \dots, z_{n+2}$, all of norm 1, such that $P(\alpha) = P(z_1) + \cdots + P(z_{n+2})$.

**11433**. *Proposed by Marius Cavachi, University "Ovidius," Constanţa, Romania.* Let $n$ be a positive integer, and let $A_1, \dots, A_n$, $B_1, \dots, B_n$, and $C_1, \dots, C_n$ be points on the unit sphere $S^2$. Show that there exists $P$ on $S^2$ such that

$$\sum_{k=1}^{n} |P - A_k|^2 = \sum_{k=1}^{n} |P - B_k|^2 = \sum_{k=1}^{n} |P - C_k|^2.$$

**11434**. *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.* Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $x_1, \dots, x_n$ be distinct real numbers, and let $p_1, \dots, p_n$ be positive numbers summing to 1. Let

$$S = \frac{\sum_{k=1}^{n} p_k x_k^3 - \left(\sum_{k=1}^{n} p_k x_k\right)^3}{3\left(\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{k=1}^{n} p_k x_k\right)^2\right)}.$$

Show that $\min\{x_1, \dots, x_n\} \leq S \leq \max\{x_1, \dots, x_n\}$.

**11435**. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.* In a triangle $T$, let $a$, $b$, and $c$ be the lengths of the sides, $r$ the inradius, and $R$ the circumradius. Show that

$$\frac{a^2bc}{(a+b)(a+c)} + \frac{b^2ca}{(b+c)(b+a)} + \frac{c^2ab}{(c+a)(c+b)} \leq \frac{9}{4} rR.$$

**11436**. *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.* In a triangle $ABC$, let $B'$ and $C'$ be points on sides $AC$ and $AB$, respectively. Let $M$ be the intersection of $BB'$ and $CC'$. Let distinct lines $k$ and $l$ intersecting inside triangle $MBC$

meet segments $C'B$, $MB$, $MC$, and $B'C$ at $K_1$, $K_2$, $K_3$, $K_4$ and $L_1$, $L_2$, $L_3$, $L_4$, respectively. Show that the intersection points of the diagonals of $K_1K_2L_2L_1$, $K_2L_2K_3L_3$, and $K_3L_3L_4K_4$ are not collinear.

**11437**. *Proposed by Tamás Erdélyi, Texas A&M University, College Station, TX.* Let $\mathcal{L}_k$ denote the set of all polynomials of degree $k$ in $x$ with each of their $k+1$ coefficients in $\{-1, 1\}$. Let $M_k$ denote the largest multiplicity that a zero of a $P$ in $\mathcal{L}_k$ can have at 1. Let $\langle C_k \rangle$ be a sequence of positive integers tending to infinity. Show that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k\colon 1 \le k \le n \text{ and } M_k \ge C_k\}\right| = 0.$$

**11438**. *Proposed by David H. Bailey, Lawrence Berkeley National Laboratory, Berkeley, CA, Jonathan M. Borwein, University of Newcastle, Newcastle, Australia and Dalhousie University, Halifax, Canada, and Jörg Waldvogel, Swiss Federal Institute of Technology ETH, Zurich, Switzerland.* Let

$$P(x) = \sum_{k=1}^{\infty} \arctan\left(\frac{x-1}{(k+x+1)\sqrt{k+1} + (k+2)\sqrt{k+x}}\right).$$

(**a**) Find a closed-form expression for $P(n)$ when $n$ is a nonnegative integer.
(**b**) Show that $\lim_{x\to -1^+} P(x)$ exists, and find a closed-form expression for it.

# SOLUTIONS

### A Sequence of Partitions into Sets with Equal Sum

**11273** [2007, 164]. *Proposed by Marian Tetiva, Bîrlad, Romania.* Let $\langle a_n \rangle$ be the sequence such that $a_n = n$ for $n \le 6$ and $a_n = \lfloor (a_1 + \cdots + a_{n-1})/2 \rfloor$ for $n > 6$. Let $r_n$ be the number in $\{0, 1, 2\}$ congruent to $\sum_{k=1}^{n} a_k$ modulo 3. Show that for $n \ge 6$, the set $\{a_1, \dots, a_n\} - \{r_n\}$ can be partitioned into three subsets with equal sum. (For example, if $n = 7$, then $\{2, 3, 4, 5, 6, 10\} = \{2, 3, 5\} \cup \{4, 6\} \cup \{10\}$.)

*Solution by Byron Schmuland, University of Alberta, Edmonton, Alberta, Canada.* For $n = 6$, we have $\{1, 2, 3, 4, 5, 6\} = \{1, 6\} \cup \{2, 5\} \cup \{3, 4\}$, so we henceforth assume that $n > 6$. Let $p$ be 0 or 1 depending on whether $\sum_{k=1}^{n-1} a_k$ is even or odd, so $a_n = \frac{1}{2}(\sum_{k=1}^{n-1} a_k - p)$. Thus $2a_n + p = \sum_{k=1}^{n-1} a_k$, and $3a_n + p = \sum_{k=1}^{n} a_k$. We conclude that $r_n = p$ and that each set in the partition should sum to $a_n$.

It suffices to show that there is a subset $S$ of $\{a_2, \dots, a_{n-1}\}$ with sum $a_n$. If so, then the sets in the desired partition are $\{a_n\}$, $S$, and $T - \{r_n\}$, where $T = \{a_1, \dots, a_{n-1}\} - S$, since $T$ contains 1 and has sum $a_n + r_n$.

Say that a set of integers with sum $s$ is *good* if it has subsets with sums 2 through $s - 2$. It suffices to show that $\{a_2, \dots, a_{n-1}\}$ is good whenever $n > 6$. We use induction on $n$. For the basis step, check explicitly that $\{2, 3, 4, 5, 6\}$ has subsets with all sums from 2 through 18.

For larger $n$, suppose that $\{a_2, \dots, a_{n-2}\}$ has sum $s$ and is good. The new set $\{a_2, \dots, a_{n-1}\}$ inherits the subsets of $\{a_2, \dots, a_{n-2}\}$ with sums $2, 3, \dots, s-2$. By taking the complements of such sets in $\{a_2, \dots, a_{n-1}\}$, we also find subsets with sums $s + a_{n-1} - j$ for $2 \le j \le s - 2$. If $a_{n-1} \le s - 3$, then there is no gap and the new set $\{a_2, \dots, a_{n-1}\}$ is also good. Since $s \ge 20 \ge 7$ for $n > 6$, the claim follows from

$$a_{n-1} \le \frac{1}{2} \sum_{k=1}^{n-2} a_k = \frac{1+s}{2} \le s - 3.$$

Finally, since $\{a_2, \ldots, a_{n-1}\}$ with sum $s$ is good, it contains the desired subset $S$ with sum $a_n$, since $2 \le a_n = \lfloor (1+s)/2 \rfloor \le s-2$.

Also solved by R. Chapman (U. K.), G. Keselman, J. H. Lindsey II, O. P. Lossers (Netherlands), N. C. Singer, A. Stadler (Switzerland), CMC 328 (Carleton College), Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Hofstra University Problem Solvers, Microsoft Research Problems Group, and the proposer.

## Counting Permutations With Two Displacement Values

**11281** [2007, 259]. *Proposed by Max Alekseyev, University of California-San Diego, La Jolla, CA and Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Show that the number of permutations $\pi$ of $\{1, \ldots, n\}$ such that $\pi(k) - k$ takes exactly two distinct values is equal to $\sigma(n) - \tau(n)$, where $\sigma(n)$ is the sum of the divisors of $n$ and $\tau(n)$ is the number of divisors.

*Solution by John Henry Steelman, Indiana University of Pennsylvania, Indiana, PA.* Clearly $\sum_{k=1}^{n}(\pi(k) - k) = \sum_{k=1}^{n}\pi(k) - \sum_{k=1}^{n} k = 0$. Thus if $\pi(k) - k$ takes exactly two distinct values, one must be positive and the other negative, say $a$ and $-b$.

Since $\pi(k) > 0$, it follows that $\pi(k) - k$ cannot be as small as $-b$ for $1 \le k \le b$, and hence $\pi(k) = k + a$ for such $k$. Similarly, $\pi(k) - k$ cannot be as large as $a$ if $1 \le \pi(k) \le a$, and hence $\pi(k) - k = -b$ for such $k$. Thus we have $\pi(k) = k - b$ for $b + 1 \le k \le a + b$. We conclude that on the set $\{1, \ldots, a + b\}$ the permutation $\pi$ restricts to the permutation of $\{1, \ldots, a + b\}$ that takes $\{1, \ldots, b\}$ to $\{a + 1, \ldots, a + b\}$ and takes $\{b + 1, \ldots, b + a\}$ to $\{1, \ldots, a\}$.

The same argument applies to the remaining set $\{a + b + 1, \ldots, n\}$. We conclude that $\pi(k) = k + a$ for $a + b + 1 \le k \le a + 2b$ and $\pi(k) = k - b$ for $a + 2b + 1 \le k \le 2(a + b)$. Continuing in this way, we find that $a + b$ divides $n$. Conversely, given any divisor $d$ of $n$ and any $a \in \{1, \ldots, d - 1\}$, with $b = d - a$ one can construct a permutation of $\{1, \ldots, n\}$ in this manner. Thus the number of such permutations is the number of ordered pairs $(a, b)$ such that $a + b$ divides $n$. We compute

$$\sum_{d|n}(d-1) = \sum_{d|n} d - \sum_{d|n} 1 = \sigma(n) - \tau(n).$$

Also solved by O. Antolin-Camarena (Canada), D. A. Beckwith, D. R. Bridges, R. J. Chapman (U. K.), K. David, T. J. Evans & K. P. Yanosko, J. Ferdinands, D. G. Fleishman, J.-P. Grivaux (France), C. P. Lanski, J. H. Lindsey II, G. Lord, O. P. Lossers (Netherlands), J. M. Metzger, A. Nakhash, D. J. Opitz, Á. Plaza (Spain), T. Rucker, V. Stakhovsky, R. Stong, M. Tetiva (Romania), S. V. Witt, CMC 328 (Carleton College), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposers.

## Irreducible Polynomials from a Theorem of Fermat

**11283** [2007, 260]. *Proposed by John Abbot, Genoa, Italy and Umberto Zannier, Pisa, Italy.* Is there a quadratic polynomial $g \in \mathbb{Q}[X]$ such that $g^4 + 1$ is reducible over $\mathbb{Q}$?

*Solution by John Wilkinson, San Jose, CA.* There is no such polynomial $g$. Let $g(X) = aX^2 + bX + c$, and suppose that $g^4 + 1$ is reducible over $\mathbb{Q}$. Let $\omega$ be a primitive eighth root of unity. Note that $g^4 + 1$ factors over $\mathbb{Q}[\omega]$:

$$g^4 + 1 = (g - \omega)(g - \omega^3)(g - \omega^5)(g - \omega^7)$$

Since $X^4 + 1$ is irreducible over $\mathbb{Q}$, one of these quadratic factors of $g^4 + 1$ must be reducible over $\mathbb{Q}[\omega]$. It follows that they all are, since they are conjugates. However,

$g - \omega^i$ is reducible over $\mathbb{Q}[\omega]$ only if its discriminant $b^2 - 4a(c - \omega^i)$ is a square in $\mathbb{Q}[\omega]$. Let $D = b^2 - 4ac$. Since $b^2 - 4a(c - \omega^i) = D + 4a\omega^i$, it follows that

$$(D + 4a\omega)(D + 4a\omega^3)(D + 4a\omega^5)(D + 4a\omega^7) = D^4 + (4a)^4$$

is a square in $\mathbb{Q}$. Clearing fractions gives us two nonzero integers whose fourth powers add to a square. However, Fermat proved that the equation $x^4 + y^4 = z^2$ has no nonzero integer solutions, so the left side of the preceding equation cannot be a square. We conclude that no such quadratic polynomial $g$ exists.

*Editorial comment.* Stephen Gagola and the team of Gabriel Dospinescu and Marian Tetiva both extended the result by proving that if $n$ is an integer greater than 1, then there is no quadratic polynomial $g \in \mathbb{Q}[X]$ such that $g^{2^n} + 1$ is reducible over $\mathbb{Q}$.

Also solved by C. P. Anilkumar (India), A. J. Bevelacqua, R. Chapman (U. K.), G. Dospinescu (France) & M. Tetiva (Romania), S. M. Gagola, J. Grivaux (France), F. Luca, V. Miller, A. Nakhash, C. R. Pranesachar (India), R. Stong, J. Sun, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposers.

## Rings that are Fields

**11284** [2007, 358]. *Proposed by Greg Oman, Ohio State University, Columbus, OH.* Let $R$ be an infinite commutative ring with identity. Suppose that every proper ideal of $R$ has smaller cardinality than $R$. Prove that $R$ is a field.

*Solution by Manuel Reyes, University of California, Berkeley, CA.* We prove a stronger statement: if $R$ is an infinite ring with identity (not necessarily commutative) and every proper right ideal of $R$ has smaller cardinality than $R$, then $R$ is a division ring.

It suffices to show that every nonzero element $a \in R$ is right-invertible. The mapping from $R$ to $aR$ given by $x \mapsto ax$ is a surjective right $R$-module homomorphism. If $I$ is its kernel, then $R/I$ and $aR$ are isomorphic as right $R$-modules. Considering these modules as abelian groups, a basic result of group theory gives the cardinal equation $|R| = |aR| \cdot |I|$.

Since $|R|$ is infinite, at least one of $|aR|$ and $|I|$ must also be infinite, and by cardinal arithmetic $|R| = |aR| \cdot |I| = \max\{|aR|, |I|\}$. Since $a \neq 0$, the kernel $I$ is a proper right ideal of $R$, so $|I| < |R|$, and we conclude that $|R| = |aR|$. The hypothesis then implies that $aR = R$, proving that $a$ is right-invertible.

*Editorial comment.* Alun Wyn-jones proved also that every finite commutive ring $R$ with identity in which every proper ideal has cardinality less than $\sqrt{|R|}$ is a field.

Also solved by D. Anderson, A. J. Bevelacqua, J. Bloom & D. Saracino, D. R. Bridges, P. Budney, S. Butcher, R. Chapman (U. K.), W. E. Duckworth, Z. Engberg, D. Fleischman, A. Fok (China), M. Goldenberg & M. Kaplan, J. Grivaux (France), J. W. Hagood, T. Kezlan, J. Konieczny, C. Lanski, J. H. Lindsey II, O. P. Lossers (Netherlands), C. Malkiewich, A. Nakhash, D. Opitz, P. S. Peck, K. Schilling, S. Siciliano (Italy), V. Stakhovsky, S. Vagi, N. Vonessen, J. Wilkinson, A. Wyn-jones, K. Yanoski, J. Young, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposers.

## Descending Dungeons

**11286** [2007, 358]. *Proposed by Marc LeBrun, Fixpoint Inc., Larkspur, CA and David Applegate & N.J.A. Sloane, AT&T Shannon Labs, Florham Park, NJ.* When $a$ and $b$ are positive integers with $b \geq 10$, write $a_b$ (or $a : b$ inline) for the integer whose base $b$ expansion is the decimal expansion of $a$. That is, if $a = \sum_{i=0}^{k} a_i 10^i$ with each $a_i$ in $\{0, 1, \ldots, 9\}$, then $a_b = a : b = \sum_{i=0}^{k} a_i b^i$. Thus,

$$10_{11_{12_{13}}} = 10 : (11 : (12 : 13)) = 16.$$

Consider the "dungeon sequences"

$$10,\ 10:11,\ (10:11):12,\ ((10:11):12):13\dots,$$
$$10,\ 10:11,\ 10:(11:12),\ 10:(11:(12:13))\dots,$$
$$10,\ 11:10,\ (12:11):10,\ ((13:12):11):10\dots,$$
$$10,\ 11:10,\ 12:(11:10),\ 13:(12:(11:10))\dots.$$

Let $s_n$ be the $n$th term in any of these sequences. Show that $\log\log s_n/(n\log\log n)$ approaches 1 as $n$ goes to infinity.

*Solution by the proposers.* The same argument applies to all four sequences. With all logarithms being base-10 logarithms, we prove that if $a, b \geq 10$, then $a : b$ is roughly $10^{\log a \log b}$. More precisely, we prove the following inequalities.

**Lemma:** If $a, b \geq 10$, then $10^{\lfloor\log a\rfloor\lfloor\log b\rfloor} \leq 10^{\lfloor\log a\rfloor\log b} \leq a : b \leq 10^{\log a\log b}$.

To prove the lemma, let $\sum_{i=0}^{k} a_i 10^i$ be the decimal expansion of $a$, where $k = \lfloor\log a\rfloor$ and $a_k \neq 0$. Thus $a : b = \sum_{i=0}^{k} a_i b^i$. For the lower bound, note that $a : b \geq a_k b^k \geq 10^{k\log b} = 10^{\lfloor\log a\rfloor\log b}$. We prove the upper bound by comparing logarithms; note first that $b \geq 10$ implies

$$\log(a_k b^k) = \log a_k + k\log b \leq (k + \log a_k)\log b = (\log a_k 10^k)(\log b).$$

Using this inequality and using $b \geq 10$ twice more, we compute

$$\log\sum_{i=0}^{k} a_i b^i = \log\left\{a_k b^k\left(1+\sum_{i=0}^{k-1}\frac{a_i}{a_k b^{k-i}}\right)\right\} = \log(a_k b^k) + \log\left(1+\sum_{i=0}^{k-1}\frac{a_i}{a_k b^{k-i}}\right)$$
$$\leq (\log b)(\log a_k 10^k) + (\log b)\log\left(1+\sum_{i=0}^{k-1}\frac{a_i}{a_k 10^{k-i}}\right) = (\log b)(\log a).$$

The bounds proved in the lemma are symmetric in $a$ and $b$. Hence they do not depend on the order of grouping the base values from which $s_n$ is computed. For clarity, let the desired sequences start with $s_0$, and let $t_n = s_{n-10}$ for $n \geq 10$, so the base values in computing $t_n$ are 10 through $n$. By the lemma, $\prod_{i=10}^{n}\lfloor\log i\rfloor \leq \log t_n \leq \prod_{i=10}^{n}\log i$. Thus

$$\log\log t_n \leq \sum_{i=10}^{n}\log\log i \leq n\log\log n.$$

In the other direction, $\log t_n \geq \prod_{i=11}^{n}\lfloor\log i\rfloor$, so $\log\log t_n \geq \sum_{i=11}^{n}\log\lfloor\log i\rfloor$. Now

$$\sum_{i=11}^{n}\log\lfloor\log i\rfloor = \frac{1}{\ln(10)}\int_{x=10}^{n}(\ln\ln x + O(1))\,dx = n\log\log n + O(n),$$

so $\log\log t_n \geq n\log\log n + O(n)$. Combining bounds gives

$$\log\log s_n = n\log\log n + O(n).$$

Also solved by D. R. Bridges, J. H. Lindsey II, and GCHQ Problem Solving Group (U. K.).

## A Polynomial Identity

**11288** [2007, 359]. *Proposed by Christopher Hillar, Texas A&M University, College Station, TX and Troels Windfeldt, University of Copenhagen, Copenhagen, Denmark.* Let $n$ be a positive integer, and let $U = \{1, \dots, 2n\}$. For a set $S \subseteq U$ and a positive integer $d$, let $h_S^d$ be the sum of all monomials of degree $d$ in the indeterminates $\{X_i : i \in S\}$. Let $\mathcal{T}$ be the set of all $n$-element subsets of $U$ with the property that for any odd element $k$ of the set, $k+1$ is not a member. For $S$ in $\mathcal{T}$, let $o(S)$ denote the number of odd elements of $S$. Show that for every positive integer $d$,

$$h_U^d \prod_{i=1}^{n} (X_{2i-1} - X_{2i}) = \sum_{S \in \mathcal{T}} (-1)^{o(S)} h_{U \setminus S}^{d+n}.$$

*Solution by Said Amghibech, Quebec, Canada.* First note that a subset $S$ of $U$ is in $\mathcal{T}$ if and only if for each $i$ from 1 to $n$, the set $S$ contains $2i-1$ or $2i$ but not both. Thus $S \in \mathcal{T}$ if and only if $U - S \in \mathcal{T}$. Hence it suffices to prove that

$$h_U^d \prod_{i=1}^{n} (X_{2i-1} - X_{2i}) = \sum_{S \in \mathcal{T}} (-1)^{e(S)} h_S^{d+n}, \tag{1}$$

where $e(S)$ is the number of even elements of $S$.

Using that $\sum_{d=0}^{\infty} t^d h_U^d = \prod_{j \in U} \frac{1}{1-tX_j}$, we have

$$\sum_{d=0}^{\infty} t^{d+n} h_U^d \prod_{i=1}^{n} (X_{2i-1} - X_{2i}) = \prod_{i=1}^{n} t \frac{X_{2i-1} - X_{2i}}{(1-tX_{2i-1})(1-tX_{2i})}$$
$$= \prod_{i=1}^{n} \left( \frac{1}{1-tX_{2i-1}} - \frac{1}{1-tX_{2i}} \right).$$

The coefficient of $t^{d+n}$ in this expression is $\sum_{S \in \mathcal{T}} (-1)^{e(S)} h_S^{d+n}$, as desired.

*Editorial comment.* In the original statement of the problem, $\{X_i : i \in S\}$ was written incorrectly as $\{X_i : X_i \in S\}$.

Also solved by D. Beckwith, R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, J. H. Lindsey II, O. P. Lossers (Netherlands), K. McInturff, Á. Plaza (Spain), T. Rucker, M. Tetiva (Romania), GCHQ Problem Solving Group (U. K.), and the proposer.

## Divisibility of a Central Binomial Sum

**11292&11307** [2007, 451&640]. *Proposed by David Callan, University of Wisconsin, Madison, WI.* Show that if $p$ is a prime and $p \geq 5$, then $p^2$ divides $\sum_{k=1}^{p^2-1} \binom{2k}{k}$.

*Solution by Robin Chapman, University of Bristol, Bristol, U. K.* Let $S = \sum_{k=0}^{p^2-1} \binom{2k}{k}$; we show instead that $S \equiv 1 \pmod{p^2}$. The "coefficient operator" $[x^n]$ extracts the coefficient of $x^n$ from a Laurent series in $x$. Note that $\binom{2k}{k} = [x^0](x + x^{-1})^{2k}$. Also,

$[x^0](x+x^{-1})^m = 0$ whenever $m$ is odd. Hence

$$S = \left[x^0\right]\sum_{m=0}^{2p^2-1}(x+x^{-1})^m = \left[x^0\right]\frac{(x+x^{-1})^{2p^2}-1}{x+x^{-1}-1}$$
$$= \left[x^0\right]x^{1-2p^2}\frac{(1+x^2)^{2p^2}-x^{2p^2}}{1-x+x^2} = \left[x^0\right]x^{1-2p^2}\frac{(1+x)[(1+x^2)^{2p^2}-x^{2p^2}]}{1+x^3}$$
$$= \left[x^{2p^2-1}\right]\frac{(1+x)(1+x^2)^{2p^2}}{1+x^3} = \left[x^{2p^2-1}\right]\sum_{m=0}^{2p^2}\binom{2p^2}{m}\frac{x^{2m}+x^{2m+1}}{1+x^3}.$$

For $r \geq 0$, we have $\frac{x^r}{1+x^3} = \sum_{j=0}^{\infty}(-1)^j x^{r+3j}$. The coefficient of $x^{2p^2-1}$ in this series is nonzero if and only if $r < 2p^2$ and $r \equiv 2p^2 - 1 \equiv 1 \pmod 3$. The coefficient is $(-1)^j$ when $3j = 2p^2 - 1 - r$, and then $(-1)^j = (-1)^{3j} = (-1)^{r-1}$. Letting $\chi(m)$ be 1, 0, or $-1$ according as $m$ is congruent to 0, 1, or 2 modulo 3, we conclude that

$$S = \sum_{m=0}^{p^2-1}\chi(m)\binom{2p^2}{m}.$$

Next we study the value of $\binom{2p^2}{m}$ modulo $p^2$. For $0 < k < p$, the binomial coefficient $\binom{p}{k}$ is divisible by $p$. Hence $(1+x)^p \equiv 1 + x^p \pmod p$. Iterating gives $(1+x)^{p^2} \equiv (1+x^p)^p \equiv 1 + x^{p^2} \pmod p$. Define $v(x)$ by $pv(x) = (1+x)^{p^2} - (1+x^{p^2})$, so that

$$(1+x)^{2p^2} = 1 + 2x^{p^2} + x^{2p^2} + 2p(1+x^{p^2})v(x) + p^2v(x)^2.$$

Since $[x^k]pv(x) = \binom{p^2}{k}$ for $0 < k < p^2$, comparing coefficients of $x^k$ gives $\binom{2p^2}{k} \equiv 2\binom{p^2}{k} \pmod{p^2}$. It follows that $S - 1 \equiv 2\sum_{m=1}^{p^2-1}\chi(m)\binom{p^2}{m} \pmod{p^2}$.

Let $\omega = e^{2\pi i/3} = \frac{1}{2}(-1+i\sqrt{3})$. For $n \geq 0$,

$$\omega^{-1}(1+\omega x)^n - \omega(1+\omega^{-1}x)^n = \sum_{m=0}^{n}\binom{n}{m}(\omega^{m-1}-\omega^{1-m})x^m$$
$$= -i\sqrt{3}\sum_{m=0}^{n}\chi(m)\binom{n}{m}x^m.$$

Setting $n = p^2$ and $x = 1$ gives

$$-i\sqrt{3}\sum_{m=0}^{p^2}\chi(m)\binom{p^2}{m} = \omega^{-1}(1+\omega)^{p^2} - \omega(1+\omega^{-1})^{p^2}.$$

Since $1+\omega$ equals $e^{2\pi i/6}$, it is a sixth root of unity. Also $p^2 \equiv 1 \pmod 6$, so

$$\omega^{-1}(1+\omega)^{p^2} = \omega^{-1}(1+\omega) = 1 + \omega^{-1}.$$

Similarly, $\omega(1+\omega^{-1})^{p^2} = 1 + \omega$, so

$$-i\sqrt{3}\sum_{m=0}^{p^2}\chi(m)\binom{p^2}{m} = \omega^{-1} - \omega = -i\sqrt{3}.$$

Hence

$$\sum_{m=1}^{p^2-1} \chi(m)\binom{p^2}{m} = -1 + \sum_{m=0}^{p^2} \chi(m)\binom{p^2}{m} = 0.$$

We conclude that $S - 1 \equiv 0 \pmod{p^2}$, as desired.

*Editorial comment.* Problem 11292 was inadvertently republished as problem 11307. A shorter proof is possible by using a result from Hao Pan and Zhi-Wei Sun, A combinatorial identity with applications to Catalan numbers, *Discrete Math.* **306** (2006) 1921–1940. More general results than proved here appear in Zhi-Wei Sun, Congruences involving Catalan numbers, arXiv:0709.1665 and in Robert Tauraso, On congruences involving central binomial coefficients, arXiv:0805.0563.

Problem 11292 was also solved by O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Tauraso (Italy), and the proposer. Problem 11307 was solved in addition by P. P. Dályay (Hungary), J. Grivaux (France), K. McInturff, Á. Plaza & S. Falcón (Spain), Fosheng Wang (China), A. Wyn-jones, BSI Problems Group (Germany).

## The Prime Exponentiation of an Integer

**11315** [2007, 744]. *Proposed by M. Farrokhi D. G., Ferdowsi University of Mashad, Mashad, Iran.* Define $f$ on the positive integers by letting $f(n) = \prod_{k=1}^{r} {a_k}^{p_k}$ when $n$ has prime factorization $\prod_{k=1}^{r} {p_k}^{a_k}$, with the empty product yielding $f(1) = 1$. Prove that for all $n$ the sequence $n, f(n), f(f(n)), \ldots$ is eventually periodic, with period at most 2.

*Solution by Tuomas Hytönen, University of Helsinki, Finland.* We show equivalently that the sequence $n, f^2(n), f^4(n), \ldots$ is eventually constant. Since its terms are positive integers, it suffices to show that $f(f(n)) \le n$ for every positive integer $n$.

Let $P$ be the set of prime factors of $n$, with $n = \prod_{p\in P} p^{a_p}$. For $p \in P$, let $Q_p$ be the set of prime factors of $a_p$, with $a_p = \prod_{q\in Q_p} q^{u_{pq}}$. Let $Q = \cup_{p\in P} Q_p$. With $u_{pq} = 0$ for $q \notin Q_p$, we have

$$f(n) = \prod_{p\in P}\left(\prod_{q\in Q} q^{u_{pq}}\right)^{p} = \prod_{q\in Q} q^{\sum_{p\in p} u_{pq} p}, \quad f(f(n)) = \prod_{q\in Q}\left(\sum_{p\in P} u_{pq} p\right)^{q}.$$

If $x_2 \ge x_1 \ge 2$, then $x_1 + x_2 \le 2x_2 \le x_1 x_2$. By induction on $k$, the sum of $k$ natural numbers all at least 2 is at most their product. Two applications of this inequality yield

$$\begin{aligned} f(f(n)) = \prod_{q\in Q}\left(\sum_{p\in P} u_{pq} p\right)^{q} &\le \prod_{q\in Q}\left(\prod_{p\in P} p^{u_{pq}}\right)^{q} \\ = \prod_{p\in P} p^{\sum_{q\in Q} u_{pq} q} &\le \prod_{p\in P} p^{\prod_{q\in Q} q^{u_{pq}}} = n. \end{aligned}$$

Also solved by A. Bandeira & J. Moreira (Portugal), P. Budney, J. Guerreiro (Portugal), A. Hesterberg, T. Kezlan, J. H. Lindsey II, O. P. Lossers (Netherlands), J. H. Nieto (Venezuela), V. Pambuccian, V. Stakhovsky, R. Stong, M. Tetiva (Romania), Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.