PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before February 28, 2013. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11663. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA.* The unit interval is broken at two randomly chosen points along its length. Show that the probability that the lengths of the resulting three intervals are the heights of a triangle is equal to

$$\frac{12\sqrt{5}\log((3+\sqrt{5})/2)}{25}-\frac{4}{5}.$$

11664. *Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ, and Darij Grinberg, Massachusetts Institute of Technology, Cambridge, MA.* Let a, b, and c be the side lengths of a triangle. Let s denote the semiperimeter, r the inradius, and R the circumradius of that triangle. Let $a'=s-a$, $b'=s-b$, and $c'=s-c$.
(**a**) Prove that $\frac{ar}{R}\le\sqrt{b'c'}$.
(**b**) Prove that

$$\frac{r(a+b+c)}{R}\left(1+\frac{R-2r}{4R+r}\right)\le 2\left(\frac{b'c'}{a}+\frac{c'a'}{b}+\frac{a'b'}{c}\right).$$

11665. *Proposed by Raitis Ozols, student, University of Latvia, Riga, Latvia.* Let $a=(a_1,\dots,a_n)$, where $n\ge 2$ and each a_j is a positive real number. Let $S(a)=a_1^{a_2}+\cdots+a_{n-1}^{a_n}+a_n^{a_1}$.
(**a**) Prove that $S(a)>1$.
(**b**) Prove that for all $\epsilon>0$ and $n\ge 2$ there exists a of length n with $S(a)<1+\epsilon$.

11666. *Proposed by Dmitry G. Fon-Der-Flaass (1962–2010), Institute of Mathematics, Novosibirsk, Russia, and Max. A. Alekseyev, University of South Carolina, Columbia,*

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SC. Let m be a positive integer, and let A and B be nonempty subsets of $\{0, 1\}^m$. Let n be the greatest integer such that $|A| + |B| > 2^n$. Prove that $|A + B| \geq 2^n$. (Here, $|X|$ denotes the number of elements in X, and $A + B$ denotes $\{a + b\colon\ a \in A, b \in B\}$, where addition of vectors is componentwise modulo 2.)

11667. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Dan Schwarz, Softwin Co., Bucharest, Romania.* Let f, g, and h be elements of an inner product space over $\mathbb{R}$, with $\langle f, g\rangle = 0$.
(**a**) Show that

$$\langle f, f\rangle\langle g, g\rangle\langle h, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

(**b**) Show that

$$(\langle f, f\rangle\langle h, h\rangle)\langle h, f\rangle^2 + (\langle g, g\rangle\langle h, h\rangle)\langle g, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

11668. *Proposed by Dimitris Stathopoulos, Marousi, Greece.* For positive integer n and $i \in \{0, 1\}$, let $D_i(n)$ be the number of derangements on n elements whose number of cycles has the same parity as i. Prove that $D_1(n) - D_0(n) = n - 1$.

11669. *Proposed by Herman Roelants, Catholic University of Leuven, Louvain, Belgium.* Prove that for all $n \geq 4$ there exist integers $x_1, \dots, x_n$ such that

$$\frac{x_{n-1}^2 + 1}{x_n^2}\prod_{k=1}^{n-2}\frac{x_k^2 + 1}{x_k^2} = 1$$

satisfying the following conditions: $x_1 = 1$, $x_{k-1} < x_k < 3x_{k-1}$ for $2 \leq k \leq n - 2$, $x_{n-2} < x_{n-1} < 2x_{n-2}$, and $x_{n-1} < x_n < 2x_{n-1}$.

SOLUTIONS

An Equation Satisfied only by the Identity Matrix

11510 [2010, 558]. *Proposed by Vlad Matei (student), University of Bucharest, Bucharest, Romania.* Prove that if I is the n-by-n identity matrix, A is an n-by-n matrix with rational entries, $A \neq I$, p is prime with $p \equiv 3 \pmod 4$, and $p > n + 1$, then $A^p + A \neq 2I$.

Solution by C. T. Stretch, University of Ulster at Coleraine, Coleraine, Londonderry, Northern Ireland. The primality and congruence conditions on p are not needed; we require only $p > n + 1$. We prove more generally that $A^p + (q - 1)A = qI$ cannot hold for any prime q.

If $A^p + (q - 1) - qI = 0$, then the minimal polynomial $m(x)$ of A divides $\phi(x)$, where $\phi(x) = x^p + (q - 1)x - q$. Note that $\phi(x) = (x - 1)\psi(x)$, where

$$\psi(x) = x^{p-1} + x^{p-2} + \cdots + x^2 + x + q.$$

Since $A \neq I$, we have $m(x) \neq x - 1$. Thus $m(x)$, which has degree at most n, is a factor of $\psi(x)$, which has degree greater than n. We obtain a contradiction and complete the proof by showing that $\psi(x)$ is irreducible over the rationals.

Since $\psi(1) \neq 0$ and $\psi(0) = q$, it suffices to show that $\psi(x)$ is irreducible over the integers. If α is a (complex root) of $\phi(x)$, then $\alpha^p = q - (q - 1)\alpha$. If $|\alpha| \leq 1$, then

$|q-(q-1)\alpha| = |\alpha|^p \le 1$. Since $|q-(q-1)\alpha| \ge q-(q-1)|\alpha| \ge 1$, we obtain $|q-(q-1)\alpha| = 1$, which occurs if and only if $\alpha = 1$. Thus every root α of $\psi(x)$ satisfies $|\alpha| > 1$.

Suppose that $\psi(x) = f(x)g(x)$, where both $f(x)$ and $g(x)$ have positive degree and integer coefficients. Consider the factorization $f(x) = \prod_{i=1}^{k}(x-\alpha_i)$ over the complex numbers. As shown above, $|\alpha_i| > 1$ for all i. Since $f(0) = (-1)^k \prod_{i=1}^{k} \alpha_i$, also $|f(0)| > 1$; similarly, $|g(0)| > 1$. Since $f(0)$ divides $\psi(0)$ and q is prime, we have $|f(0)| = q$; similarly, $|g(0)| = q$. But then $q = \psi(0) = f(0)g(0) = \pm q^2$, a contradiction. We conclude that $\psi(x)$ is irreducible.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), J. Simons (U. K.), N. C. Singer, R. Stong, M. Tetiva (Romania), Ellington Management Problem Solving Group, GCHQ Problem Solving Group, and the proposer.

An Expression for the kth Smallest Element of a Set

11520 [2010, 649]. *Proposed by Peter Ash, Cambridge Math Learning, Bedford, MA.* Let n and k be integers with $1 \le k \le n$, and let A be a set of n real numbers. For i with $1 \le i \le n$, let S_i be the set of all subsets of A with i elements, and let $\sigma_i = \sum_{s \in S_i} \max(s)$. Express the kth smallest element of A as a linear combination of $\sigma_0, \dots, \sigma_n$.

Solution by Mark Wildon, Royal Holloway, University of London, Egham, United Kingdom. Let $A = \{a_1, \dots, a_n\}$ with $a_1 < \cdots < a_n$. There are exactly $\binom{m-1}{k-1}$ k-subsets of A in which a_m is largest, so $\sigma_k = \sum_{m=1}^{n} \binom{m-1}{k-1} a_m$ for $1 \le k \le n$. The following computation expresses a_k as a linear combination of $\sigma_k, \dots, \sigma_n$, where the final step uses that $\sum_{r=1}^{n}(-1)^r\binom{m-k}{r-k} = 0$ except when $m = k$:

$$\begin{aligned}\sum_{r=1}^{n}(-1)^{k+r}\binom{r-1}{k-1}\sigma_r &= (-1)^k \sum_{m=1}^{n} a_m \sum_{r=1}^{n}(-1)^r \binom{m-1}{r-1}\binom{r-1}{k-1} \\ &= (-1)^k \sum_{m=1}^{n} a_m \sum_{r=1}^{n}(-1)^r \binom{m-1}{k-1}\binom{m-k}{r-k} \\ &= (-1)^k \sum_{m=1}^{n} a_m \binom{m-1}{k-1} \sum_{r=1}^{n}(-1)^r \binom{m-k}{r-k} \\ &= a_k.\end{aligned}$$

Editorial comment. The main step here can be viewed as inverting a matrix of binomial coefficients. Jayantha Senadheera cited Cramer's Rule. Li Zhou cited the book of Polya and Szegö, where the verification is a solved problem. Oliver Geupel cited a general binomial coefficient identity of which the verification is a special case (see R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete Mathematics: a Foundation for Computer Science*, (Addison–Wesley, 1989), page 70).

Also solved by T. Amdeberhan & V. De Angelis, R. Bagby, D. Beckwith, M. Benedicty, N. Caro (Brazil), R. Chapman (U. K.), W. J. Cowieson, P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, S. J. Herschkorn, Y. J. Ionin, J.-W. Kang (Korea), O. Kouba (Syria), J. H. Lindsey II, J. H. Nieto (Venezuela), W. Nuij (Netherlands), É. Pité (France), Á. Plaza & S. Falcón (Spain), R. Pratt, J. Schlosberg, J. Senadheera, J. Simons (U. K.), J. H. Smith, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), S. Xiao (Canada), L. Zhou, Barclays Capital Quantitative Analytics Group, GCHQ

Problem Solving Group (U. K.), Missouri State University Problem Solving Group, NSA Problems Group, Texas State University Problem Solving Group, and the proposer.

A Geometric Inequality

11552 [2011, 178]. *Proposed by Weidong Jiang, Weihai Vocational College, Weihai, China.* In triangle ABC, let A_1, B_1, C_1 be the points opposite A, B, C at which the angle bisectors of the triangle meet the opposite sides. Let R and r be the circumradius and inradius of ABC. Let a, b, c be the lengths of the sides opposite A, B, C, and let a_1, b_1, c_1 be the lengths of the line segments B_1C_1, C_1A_1, A_1B_1. Prove that

$$\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \geq 1 + \frac{r}{R}.$$

Solution by Prithwijit De, HBCSE, Mumbai, India. If X and Y are the feet of the perpendiculars on BC from C_1 and B_1, respectively, then $a_1 = |B_1C_1| \geq |XY|$. However,

$$|XY| = a - (|BC_1| \cos B + |B_1C| \cos C) = 1 - \left(\frac{ac \cos B}{a+b} + \frac{ab \cos C}{a+c} \right).$$

Therefore,

$$\frac{a_1}{a} \geq 1 - \left(\frac{c}{a+b} \cos C + \frac{b}{a+c} \cos C \right). \tag{1}$$

Similarly,

$$\frac{b_1}{b} \geq 1 - \left(\frac{a}{b+c} \cos C + \frac{c}{b+a} \cos A \right), \tag{2}$$

$$\frac{c_1}{c} \geq 1 - \left(\frac{b}{c+a} \cos A + \frac{a}{b+c} \cos B \right). \tag{3}$$

Adding, we get

$$\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \geq \tag{4}$$
$$3 - \left(\frac{a}{b+c} (\cos B + \cos C) + \frac{b}{c+a} (\cos C + \cos A) + \frac{c}{a+b} (\cos A + \cos B) \right).$$

Now,

$$\begin{aligned} \frac{a(\cos B + \cos C)}{b+c} &= \frac{\sin A(\cos B + \cos C)}{\sin B + \sin C} \\ &= \frac{2\sin(A/2)\cos(A/2) \cdot 2\cos((B+C)/2)\cos((B-C)/2)}{2\sin((B+C)/2)\cos((B-C)/2)} \\ &= \frac{4\sin(A/2)\cos(A/2)\sin(A/2)\cos((B-C)/2)}{2\cos(A/2)\cos((B-C)/2)} \\ &= 2\sin^2 \frac{A}{2} = 1 - \cos A. \end{aligned}$$

Similarly,

$$\frac{b}{c+a}(\cos C + \cos A) = 1 - \cos B,$$
$$\frac{c}{a+b}(\cos A + \cos B) = 1 - \cos C.$$

Putting these into (4), we have

$$\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \geq \cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Editorial comment. Peter Nüesch (Switzerland) notes that this problem may be viewed as a special case of Problem 1320 in *Mathematics Magazine*, proposedby V. Kovner in vol. 62 (1989), p. 137, solved by J. Heuver and Richard E. Pfiefer in vol. 63(1990) pp. 130–131.

Also solved by P. P. Dályay (Hungary), P. Nüesch (Switzerland), J. Posch, R. Stong, and the proposer.

Triangle Center $X(79)$

11554 [2011, 178]. *Proposed by Zhang Yun, Xi'an Jiao Tong University Sunshine High School, Xi'an, China.* In triangle ABC, let I be the incenter, and let A', B', C' be the reflections of I through sides BC, CA, AB, respectively. Prove that the lines AA', BB', and CC' are concurrent.

Solution by Alin Bostan, INRIA, Rocquencourt, France. First we identify this problem as a particular case of two different classical theorems in Euclidean geometry: Jacobi's Theorem and Kariya's Theorem (which is itself a particular case of an older theorem of Lemoine's, see below). We then give two proofs of Problem 11554.

Jacobi's Theorem (sometimes called "the Isogonal Theorem"): *If ABC is a triangle, and A', B', and C' are points in its plane such that $\angle B'AC = \angle BAC'$, $\angle C'BA = \angle CBA'$, and $\angle A'CB = \angle ACB'$, then the lines AA', BB', and CC' are concurrent.* This is a generalization of the famous "Napoleon's Theorem", available at `http://en.wikipedia.org/wiki/Napoleon's\_theorem`. It was seemingly discovered by Carl Friedrich Andreas Jacobi [not to be confused with Carl Gustav Jacob Jacobi], and published in 1825 in Latin: C. F. A. Jacobi, *De triangulorum rectilineorum proprietatibus quibusdam nondum satis cognitis,* Naumburg (1825).

Kariya's Theorem: *Let I be the incenter of a triangle ABC, and let X, Y, Z be the points where the incircle of $\triangle ABC$ touches the sides BC, CA, AB, respectively. If A', B', C' are three points on the half-lines IX, IY, IZ, respectively, such that $IA' = IB' = IC'$, then the lines AA', BB', and CC' are concurrent.* This theorem has a long history. It was discovered independently by Auguste Boutin and by V. Retali: A. Boutain, "Sur un groupe de quatre coniques remarquables,"*Journal de mathématiques spéciales* ser. 3, **4** (1890) 104–107, 124–127; A. Boutin, "Problèmes sur le triangle,"*Journal de mathématiques spéciales* ser. 3, **4** (1890) 265–269; V. Retali, *Periodico di Matematica* (Rome) **11** (1896) 71.

The result only became well known with Kariya's paper (which inspired many results appearing in *l'Enseignement* over the following years): J. Kariya, "Un probléme sur le triangle,"*L'Enseignement mathématique* **6** (1904) 130–132, 236, 406. Actually, a generalization of this result was obtained before Kariya by Emile Lemoine in Section 4 of: E. Lemoine, "Contributions à la géométrie du triangle,"*Congrès de l'AFAS,* Paris, 1889, p. 197–222.

Lemoine explicitly states and proves on page 202 the following: *Let ABC be a triangle, M a point in its plane, and X, Y, Z the projections of M on BC, CA, AB, respectively. If A', B', C' are points on the half-lines MX, MY, and MZ, respectively, such that $MX \cdot MA' = MY \cdot MB' = MZ \cdot MC'$, then AA', BB', CC' are concurrent.* Auric gave in 1915 another generalization of Kariya's Theorem: A. Auric, "Généralisation du théorème de Kariya,"*Nouvelles annales de mathématiques* 4e série**15** (1915) 222–225. The statement is the same as Lemoine's Theorem except that the assumption $MX \cdot MA' = MY \cdot MB' = MZ \cdot MC'$ is replaced by $MX/MA' = MY/MB' = MZ/MC'$.

Now we give the two solutions to Problem 11554, both based on Ceva's Theorem.

(1) This solution is possibly new (less elegant than the second one, but a bit shorter). Let P be the intersection of AA' and BC, and let Q be the intersection of AI and BC. Applying Menelaus' Theorem twice (once for $\triangle APQ$ and transversal IA', once for $\triangle AIA'$ and transversal BC), we find that $BP/PC = (a^2 + c^2 - b^2 + ca)/(b^2 + a^2 - c^2 + ab)$. Since the numerator is obtained from the denominator by the cyclic permutation $a \to b \to c \to a$, the conclusion follows from Ceva's Theorem.

(2) The second solution is much more elegant, and is possibly due to the Romanian geometer Gheorghe Titeica (it appears as Problem 1138 in his book *Problems of Geometry* (in Romanian)). Let the parallel to BC passing through A' intersect AB and AC in A_1 and A_2, respectively. Construct similarly the points B_1, B_2, C_1, and C_2. By symmetry, $A'A_1 = C'C_2$, $A'A_2 = B'B_1$, and $B'B_2 = C'C_1$. Let P be the intersection of AA' and BC, let Q be the intersection of BB' and AC, and let R be the intersection of CC' and AB. Thales' Theorem implies $BP/PC = A_1A'/A'A_2$, $CQ/QA = B_1B'/B'B_2$, and $AR/RB = C_1C'/C'C_2$. It follows that

$$\frac{BP}{PC}\,\frac{CQ}{QA}\,\frac{AR}{RB} = \frac{A_1A'}{A'A_2}\,\frac{B_1B'}{B'B_2}\,\frac{C_1C'}{C'C_2} = 1,$$

and the conclusion follows from Ceva's Theorem.

Final notes: (i) Nowadays the point J of concurrence in Problem 11554 is sometimes called "Gray's point" after Steve Gray who noted a seemingly new property, namely that the line IJ is parallel to the Euler line OH of $\triangle ABC$.

(ii) The point J is called $X(79)$ in Kimberling's *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

Also solved by Y. An (China), G. Apostolopoulos (Greece), M. Bataille (France), R. B. Campos (Spain), C. Curtis, P. P. Dályay (Hungary), P. De (India), C. Delorme (France), A. Ercan (Turkey), O. Faynshteyn (Germany), R. Frank & H. Riede (Germany), O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, S. Hitotumatu (Japan), Y. J. Ionin, M. E. Kidwell & M. D. Meyerson, O. Kouba (Syria), R. Mabry, R. Murgatroyd, C. R. Pranesachar (India), J. Schlosberg, T. Smith, R. Stong, M. Tetiva (Romania), R. S. Tiberio, Z. Vörös (Hungary), Z. Xintao (China), P. Yff, J. B. Zacharias, D. Zeilberger, GCHQ Problem Solving Group (U. K.), and the proposer.

Value Defined by an Integral

11555 [2011, 178]. *Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam.* Let f be a continuous real-valued function on $[0, 1]$ such that $\int_0^1 f(x)\,dx = 0$. Prove that there exists c in the interval $(0, 1)$ such that $c^2 f(c) = \int_0^c (x + x^2) f(x)\,dx$.

Solution I by Michael W. Botsko, Saint Vincent College, PA. First, let $F(x) = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt$ on $[0, 1]$. By its construction, $F'(x) = \int_0^x f(t)\,dt$ and

$F'(0) = 0$. Since $\int_0^1 f(x)\,dx = 0$, also $F'(1) = 0$. Now by Flett's Mean Value Theorem (T. M. Flett, "A mean value theorem", *Math. Gazette*, **42**(1958), 38-39), there exists $a \in (0, 1)$ such that

$$\frac{F(a) - F(0)}{a - 0} = F'(a).$$

Therefore,

$$\int_0^a x f(x)\,dx = 0, \quad a \in (0, 1). \tag{1}$$

Next, let $G(x) = e^{-x} \int_0^x t f(t)\,dt$. From (1), $G(0) = G(a) = 0$. By Rolle's Theorem, there exists $b \in (0, a)$ such that

$$0 = G'(b) = -e^{-b} \int_0^b x f(x)\,dx + e^{-b} b f(b).$$

Therefore,

$$\int_0^b x f(x)\,dx = b f(b), \quad b \in (0, a). \tag{2}$$

Finally, let $H(x) = x \int_0^x t f(t)\,dt - \int_0^x (t + t^2) f(t)\,dt$. Then

$$H'(x) = \int_0^x t f(t)\,dt - x f(x).$$

Using (2), we have that $H'(0) = H'(b) = 0$. Once again using Flett's Mean Value Theorem, there exists $c \in (0, b)$ such that

$$\frac{H(c) - H(0)}{c - 0} = H'(c).$$

Therefore,

$$c \int_0^c x f(x)\,dx - \int_0^c (x + x^2) f(x)\,dx = c \int_0^c x f(x)\,dx - c^2 f(c).$$

This implies $\int_0^c (x + x^2) f(x)\,dx = c^2 f(c)$.

Solution II by Hongwei Chen, Christopher Newport University, Newport News, VA. If f is identically zero, there is nothing to prove, so assume that $f(x)$ is not identically zero. Since $\int_0^1 f(x)\,dx = 0$, there exist a and $b \in [0, 1]$ such that $a \neq b$ and

$$f(a) = \max_{x \in [0,1]} f(x) > 0, \qquad f(b) = \min_{x \in [0,1]} f(x) < 0.$$

Define $F(x) = x^2 f(x) - \int_0^x (t + t^2) f(t)\,dt$. Note that $F(x)$ is continuous on $[0, 1]$. Since $(t + t^2) f(a) \geq (t + t^2) f(t)$ for all $t \in [0, 1]$,

$$F(a) \geq a^2 f(a) - \int_0^a (t + t^2) f(a)\,dt = a^2 \left(\frac{1}{2} - \frac{a}{3} \right) f(a) > 0.$$

Similarly, $(t + t^2) f(b) \leq (t + t^2) f(t)$ for all $t \in [0, 1]$, so

$$F(b) \leq b^2 f(b) - \int_0^b (t + t^2) f(b)\,dt = b^2 \left(\frac{1}{2} - \frac{b}{3} \right) f(b) < 0.$$

The Intermediate Value Theorem implies that there exists a number $c \in (a, b) \subset (0, 1)$ such that $F(c) = 0$, so that $c^2 f(c) = \int_0^c (x + x^2) f(x)\, dx$.

Also solved by K. F. Andersen (Canada), M. W. Botsko, P. Bracken, H. Chen, D. Constales (Belgium), P. P. Dályay (Hungary), N. Grivaux (France), L. Han, E. Ionascu, K.-W. Lau (China), J. H. Lindsey II, M. Omarjee (France), S. Pauley, N. Weir & A. Welter, P. Perfetti (Italy), Á. Plaza (Spain), A. Stenger, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), H. Wang & Y. Xia, GCHQ Problem Solving Group (U. K.), and the proposer.

A Four-Number Summetric Inequality

11556 [2011, 179]. *Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.* For positive real numbers a, b, c, d, show that

$$\frac{9}{a(b+c+d)} + \frac{9}{b(c+d+a)} + \frac{9}{c(d+a+b)} + \frac{9}{d(a+b+c)} \geq \frac{16}{(a+b)(c+d)} + \frac{16}{(a+c)(b+d)} + \frac{16}{(a+d)(b+c)}.$$

Solution by Marian Dincă, Bucharest, Romania. Suppose f is a convex function on the interval $I \subset \mathbb{R}$. Given numbers x, y, z, let $x' = (y+z)/2$, $y' = (z+x)/2$, and $z' = (x+y)/2$. Combining Popoviciu's Inequality

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 2\left[f(z') + f(y') + f(x')\right]$$

and Jensen's Inequality in the form

$$f(x) + f(y) + f(z) = \frac{f(x)+f(y)}{2} + \frac{f(x)+f(z)}{2} + \frac{f(y)+f(z)}{2} \geq f(z') + f(y') + f(x')$$

(specifically adding the first to twice the second) gives

$$3f(x) + 3f(y) + 3f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 4\left[f(z') + f(y') + f(x')\right].$$

Applying this to the convex function $f(t) = \frac{1}{t}$, $t > 0$, for $x, y, z > 0$ we have

$$\frac{3}{x} + \frac{3}{y} + \frac{3}{z} + \frac{9}{x+y+z} \geq \frac{8}{x+y} + \frac{8}{x+z} + \frac{8}{y+z}.$$

Summing the four inequalities we get by taking x, y, z to be any three of a, b, c, d we obtain

$$\frac{9}{a} + \frac{9}{b} + \frac{9}{c} + \frac{9}{d} + \frac{9}{a+b+c} + \frac{9}{a+b+d} + \frac{9}{a+c+d} + \frac{9}{b+c+d} \geq \frac{16}{a+b} + \frac{16}{a+c} + \frac{16}{a+d} + \frac{16}{b+c} + \frac{16}{b+d} + \frac{16}{c+d}.$$

Noting that

$$\frac{1}{a} + \frac{1}{b+c+d} = \frac{a+b+c+d}{a(b+c+d)} \quad \text{and} \quad \frac{1}{a+b} + \frac{1}{c+d} = \frac{a+b+c+d}{(a+b)(c+d)},$$

and symmetrically, we see that this is exactly the desired inequality multiplied by $a+b+c+d$.

Also solved by S. Hitotumatu (Japan), E. Hysnelaj & E. Bojaxhiu (Australia & Germany), O. Kouba (Syria), J. H. Lindsey II, P. H. O. Pantoja (Brazil), P. Perfetti (Italy), C. R. Pranesacher (India), A. Stenger, R. Stong, M. Tetiva (Romania), L. Zhou, Zhou X. (China), GCHQ Problem Solving Group (U. K.), and the proposer.