PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**

with the collaboration of Paul T. Bateman, Mario Benedicty, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Dennis Eichhorn, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, Jerrold R. Griggs, Jerrold Grossman, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, John Henry Steelman, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before June 30, 2007. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11277. *Proposed by Prithwijit De, University College Cork, Republic of Ireland.* Find

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\frac{\log(2-\sin\theta\cos\phi)\sin\theta}{2-2\sin\theta\cos\phi+\sin^2\theta\cos^2\phi}\,d\theta\,d\phi.$$

11278. *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.* Let f be a nonconstant entire function with nonnegative Taylor series coefficients. Prove that $\lim_{r\to\infty} f(r)/rf'(r)$ exists and is rational.

11279. *Proposed by Vitaly Stakhovsky, Redwood City, CA.* Two test-mass beads are sliding along a vertical circular track under (Newtonian) constant gravity, without friction. The first bead M_1 starts from the highest point on the circle, at some nonzero velocity. Some time later the second bead M_2 starts from the same position at the top of the circle and with the same initial velocity as M_1 had. Prove that there is a circle to which the line through the current positions of M_1 and M_2 is always tangent, and find the center and radius of that circle in terms of the original circle.

11280. *Proposed by Harris Kwong, SUNY Fredonia, Fredonia, NY.* Let f be a positive nondecreasing function on the real line that is twice differentiable and concave down. For any list x of positive real numbers $x_1, \dots, x_n$, let $S=\sum_{k=1}^n x_k$. In terms of f and n, find

$$\max_x \prod_{k=1}^n (f(S-x_k))^{x_k/S}.$$

11281. *Proposed by Max Alekseyev, University of California-San Diego, La Jolla, CA, and Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Show that the number of permutations π of $\{1,\dots,n\}$ such that $\pi(k)-k$ takes exactly two distinct values is equal to $\sigma(n)-\tau(n)$, where $\sigma(n)$ is the sum of the divisors of n and $\tau(n)$ is the number of divisors.

11282. *Proposed by Dragomir Ž. Đoković, University of Waterloo, Waterloo, Canada, Konstanze Rietsch, King's College London, (U. K.), and Kaiming Zhao, Wilfred Laurier University, Waterloo, Canada and Chinese Academy of Sciences, Beijing, China.* Let n be an even positive integer, and let A be the $n \times n$ matrix with entries

$$a_{i,j} = \begin{cases} 1, & \text{if } j = i+1, \\ \frac{j(n-j)}{1-(n-2j)^2}, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

Prove by elementary means that A^2 is unipotent, that is, $A^2 - I$ is nilpotent.

11283. *Proposed by John Abbot, Genoa, Italy, and Umberto Zannier, Pisa, Italy.* Is there a quadratic polynomial g in $\mathbb{Q}[X]$ such that $g^4 + 1$ is reducible over $\mathbb{Q}$?

SOLUTIONS

Stepping to Regularity

10923 [2002, 201]. *Proposed by Stephen B. Gray, Santa Monica, CA.* Given a full-dimensional simplex S in $\mathbb{R}^n$, a *step* is an affine transformation that takes S into a new simplex S' by fixing all but one vertex and moving the remaining vertex parallel to the hyperplane determined by the others.
(**a**) Prove that every triangle in $\mathbb{R}^2$ can be made equilateral in at most two steps.
(**b**) Prove that for every positive integer n there exists a positive integer N_n such that every full-dimensional simplex in $\mathbb{R}^n$ can be made regular in at most N_n steps.

Solution by Thomas McMillan and Xiaoshen Wang, University of Arkansas at Little Rock, AR. Steps preserve both n-volume and dimension. Both (**a**) and (**b**) follow from a general claim that we prove by induction on n: Every full-dimensional simplex in $\mathbb{R}^n$ can be made regular in at most N_n steps, where $N_n = (n^2 + n - 2)/2$.

Since $N_1 = 0$ and all 1-simplexes are regular, the claim is true when $n = 1$. Since $N_n = N_{n-1} + n$, for the induction step it suffices to show that $N_{n-1} + n$ steps suffice to make a given simplex S in $\mathbb{R}^n$ regular.

Let $V_0, \dots, V_n$ be the vertices of S. Let h be the altitude of any regular n-dimensional simplex with the same n-volume as S. If every edge of S has length less than h, then the n-volume of S is smaller than that of a regular n-simplex with altitude h, so we may index the vertices of S so that the length a of $V_0 V_n$ is at least h.

Let A be the hyperplane through V_0 that is perpendicular to $V_0 V_n$. Let $\bar{S}$ be the simplex in $\mathbb{R}^n$ obtained from S by fixing V_n and projecting the remaining vertices $V_0, \dots, V_{n-1}$ orthogonally onto corresponding points $V_0', \dots, V_{n-1}'$ in A. Note that $V_0' = V_0$, and $\bar{S}$ has altitude a. Along the line through V_{n-1}' that is parallel to $V_0 V_n$, there is a point V_{n-1}'' such that the altitude at V_n of the simplex $\hat{S}$ with vertices $V_0', \dots, V_{n-2}', V_{n-1}'', V_n$ is h, since this altitude varies from 0 to a as the point moves in along this line from infinity to V_{n-1}'.

These orthogonal projections moving $V_1, \dots, V_{n-2}$ to $V_1', \dots, V_{n-2}'$ and V_{n-1} to V_{n-1}'' along lines parallel to $V_0 V_n$ are steps of the required form, since in each step the hyperplane determined by the points not moving contains $V_0 V_n$. That is, each desired motion is parallel to the opposite hyperplane, as required. Let S' be the $(n-1)$-dimensional simplex determined by the resulting points $V_0', \dots, V_{n-2}'$ and V_{n-1}''.

Since the altitude of $\hat{S}$ at V_n is h, and S' is the facet of $\hat{S}$ opposite V_n, and $\hat{S}$ has the same n-volume as S, the $(n-1)$-volume of S' is that of a regular $(n-1)$-simplex with altitude h. Using at most N_{n-1} steps within the hyperplane B containing S', we transform S' to a regular $(n-1)$-simplex S^* with the same $(n-1)$-volume as S'. With one final step, we move V_n parallel to B to reach a line perpendicular to B through the centroid of S^*. We have thus produced a regular n-simplex using at most $N_{n-1} + n$ steps.

Also solved by C. Anderson, J.-P. Grivaux (France), J. H. Lindsey II, R. Martin, S. Namil, R. Stong, A. Tissier (France), J. Tolosa, L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Heronian Triangles

11134 [2005, 180]. *Proposed by* Kent D. Boklan, Queens College, NY and the University of Iceland, Reykjavik, Iceland. Fix primes p and q. Prove that there are at most six integers x such that the area of the triangle with side-lengths p, q, and x is a positive integer.

Solution by Michael R. Avidon, Allston, MA. With θ denoting the angle between the sides with lengths p and q, the area A is given by

$$A = \frac{1}{2} pq \sin\theta. \tag{1}$$

Also,

$$x^2 = p^2 + q^2 - 2pq\cos\theta. \tag{2}$$

Assume, without loss of generality, that $p \geq q$.

If $q = 2$, then (1) and (2) imply $x^2 = p^2 + 4 \pm 4\sqrt{p^2 - A^2}$. Since the radical is rational, it must be an integer. Thus $x \equiv p \bmod 2$. The sides of the triangle satisfy $p - 2 < x < p + 2$. Therefore $x = p$, but then $p^2 - A^2 = 1$, which is impossible.

Hence we may assume that p and q are odd. Since (1) and (2) imply that $\sin\theta$ and $\cos\theta$ are rational, and (1) implies that the numerator of $\sin\theta$ is even, there exist positive integers a and b with $\gcd(a, b) = 1$ and $a \not\equiv b \bmod 2$ such that

$$\sin\theta = \frac{2ab}{a^2 + b^2} \quad \text{and} \quad \cos\theta = \frac{a^2 - b^2}{a^2 + b^2}. \tag{3}$$

Now (1) or (2) implies that $(a^2 + b^2) \mid pq$. Hence (i) $a^2 + b^2 = pq$, (ii) $a^2 + b^2 = p$, or (iii) $a^2 + b^2 = q$. The number of possible solutions (a, b) to any of these equals the number of possible values of x. We claim that there are respectively at most four, one, and zero possible solutions, yielding at most five values of x.

It is well known that the number of distinct ordered pairs (a, b) satisfying (i) is four, if $p \equiv q \equiv 1 \bmod 4$ and $p > q$. If $p = q \equiv 1 \bmod 4$, then there are two solutions. Otherwise there are no solutions.

Now (ii), together with (3) and (2), implies that $x^2 = p^2 + q^2 - 2q(a^2 - b^2)$, from which it follows that $x \equiv \pm p \bmod q$ and x is even. Since $p - q < x < p + q$, we cannot have $x \equiv p \bmod q$. For $x \equiv -p \bmod q$, there is exactly one value of x satisfying these constraints.

Finally, (iii), together with (3) and (2), implies that $x^2 = p^2 + q^2 - 2p(a^2 - b^2)$, from which it follows that $x \equiv \pm q \bmod p$. We still have that x is even and $p - q < x < p + q$, but this is impossible.

Editorial comment. Most solvers began by applying Heron's formula. The Microsoft Research Problems Group found six pairs of primes that produced three solutions for x, the smallest of which is $(4241, 2729)$, with $x = 1530,\ 6888,\ 1850$. It apparently remains open whether the maximum is 3, 4, or 5.

Also solved by T. Andebrhan (Eritrea), R. E. Prather, Q. Ruan, R. Stong, M. K. Uzun (Turkey), Z. Vörös (Hungary), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, National Security Agency Problems Group, and the proposers.

An Identity Involving Rooted Forests

11144 [2005, 274]. *Proposed by Ernst Schulte-Geers, BSI, Bonn, Germany.* Let $[n]$ denote $\{1, \dots, n\}$, and let K be a nonempty set of k elements in $[n]$. Say that K is *repelling* under a mapping $f\colon\ [n] \to [n]$ if $K \cap f^j(K) = \emptyset$ for all $j \geq 1$. Let $a_{n,k}$ be the number of mappings from $[n]$ to itself under which K is repelling.
(**a**) Prove that $a_{n,k} = k \sum_{i=1}^{n-k} \binom{n-k}{i} i^{i+k}(n-i)^{n-k-i-1}$.
(**b**) The ratio $a_{n,k}/n^n$ is the probability that K is repelling under a random mapping. For $k = \lfloor tn^{1/3} \rfloor$ with $t > 0$, find $\lim_{n\to\infty} a_{n,k}/n^n$.

Solution to part (a) by Karl David, Milwaukee School of Engineering, Milwaukee, WI. The functional digraph G of a function $f\colon\ [n] \to [n]$ consists of trees feeding into cycles, with one cycle in each component of the underlying undirected graph. The set K is repelling under f if and only if there is a partition (S, T) of $[n] - K$ such that G consists of a rooted forest of k components with root set K and non-root set S (with S consisting of the vertices that eventually pass through K under iteration of f), a functional digraph on T, and edges that map K into T. Furthermore, these three requirements can be satisfied independently once the partition (S, T) is chosen.

When $|T| = i$, there are $\binom{n-k}{i}$ ways to choose the partition, i^i maps from T to itself, and i^k ways to map K into T. Finally, we want a rooted forest on $n - i$ vertices in which the set of roots is precisely K. We prove by induction on m that the number $b_{m,r}$ of rooted forests on $[m]$ with specified root set R of size r is precisely rm^{m-r-1}. With this, we have $a_{n,k} = \sum_{i=1}^{n-k} \binom{n-k}{i} i^{i+k} b_{n-i,k}$, as desired.

For the basis, the formula reduces to 1 when $r = m$, and $b_{m,m} = 1$. Hence we may assume that $r < m$. Deleting the roots of such a forest F yields another rooted forest F', with $m - r$ vertices and some number j of roots. If we know the roots, then we can map them to the roots of F in r^j ways, and we can form F' in $b_{m-r,j}$ ways. Using the induction hypothesis for $b_{m-r,j}$, we obtain

$$b_{m,r} = \sum_{j=1}^{m-r} \binom{m-r}{j} r^j j(m-r)^{m-r-j-1}$$

$$= r \sum_{j=1}^{m-r} \binom{m-r-1}{j-1} r^{j-1}(m-r)^{m-r-1-(j-1)} = rm^{m-r-1}.$$

Solution to part (b) by the proposer. Let $p_{n,k} = a_{n,k}/n^n$. We show that if $k, n \to \infty$ so that $k/n^{1/3} \to t$ for fixed t, then $p_{n,k} \to e^{-\sqrt{2t^3}}$. First we set $j = n - k - i$ and define

$$q_{n,k,j} = \frac{1}{n^{n-k}} \binom{n-k}{j} k(j+k)^{j-1}(n-k-j)^{n-k-j}$$

to rewrite the result of part (a) as $p_{n,k} = \sum_{j=0}^{n-k} q_{n,k,j} \left(\frac{n-k-j}{n}\right)^k$. Clearly $q_{n,k,j} \geq 0$. We next show that $\sum_{j=0}^{n-k} q_{n,k,j} = 1$ to confirm that these numbers form a probability distribution.

Let $T(x) = \sum_{n\geq 1} n^{n-1} \frac{x^n}{n!}$; this gives the exponential generating function for rooted forests. Using the well known fact that $T(x) = xe^{T(x)}$, Lagrange Inversion yields power series expansions in x for various functions of T:

$$[x^n]e^{T(x)} = \frac{k(n+k)^{n-1}}{n!}, \quad [x^n]\frac{1}{1-T(x)} = \frac{n^n}{n!}, \quad [x^n]\frac{e^{kT(x)}}{1-T(x)} = \frac{(n+k)^n}{n!}.$$

Thus

$$\begin{aligned}\sum_{j=0}^{n-k} q_{n,k,j} &= \frac{(n-k)!}{n^{n-k}} \sum_{j=0}^{n-k} [x^j]e^{kT(x)}[x^{n-k-j}]\frac{1}{1-T(x)} \\ &= \frac{(n-k)!}{n^{n-k}}[x^{n-k}]\frac{e^{kT(x)}}{1-T(x)} = 1.\end{aligned}$$

Observe that $p_{n,k} = \mathrm{E}\left(1 - \frac{k+Y_{n,k}}{n}\right)^k$ when $Y_{n,k}$ is a random variable with distribution $q_{n,k}$. If now $k, n \to \infty$ with $k = o(\sqrt{n})$, then $Y_{n,k}/k^2$ converges in distribution to a nonnegative random variable S with Lesbesgue density

$$d(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-1/(2x)} & \text{for } x > 0 \\ 0 & \text{for } x \le 0. \end{cases}$$

To see this, set $j = k^2 x$. Stirling's Formula and the series expansion for $\log(1-x)$ yield $k^2 q_{n,k,k^2x} \to d(x)$ uniformly on each interval $[a, b]$, given $k = o(\sqrt{n})$.

For the final step, we need a lemma.

Lemma If $\langle X_k \rangle$ is a sequence of nonnegative random variables with values in the interval [0, 1] such that kX_k converges in distribution to X, then
(a) $(1 - X_k)^k \to e^{-X}$ in distribution, and
(b) $\mathrm{E}(1 - X_k)^k \to \mathrm{E}(e^{-X})$.

Proof (a) Let G and G_k denote the cumulative density functions for e^{-X} and for $(1 - X_k)^k$, respectively. Elementary considerations show that under the given conditions, $G_k(t) \to G(t)$ for each continuity point of G.

(b) The random variables $(1 - X_k)^k$ are bounded and hence uniformly integrable. Hence the assertion follows from part (a) and standard facts about integration to the limit. □

If $k/n^{1/3} \to t$, then $k\frac{k+Y_{n,k}}{n} = \frac{k+Y_{n,k}}{k^2}\frac{k^3}{n} \to t^3 S$ in distribution. Note that S is a positive stable distribution with index $1/2$. It coincides with the distribution of $1/N(0,1)^2$, where $N(0, 1)$ is a standard normal random variable. The Laplace transform of S is $\mathrm{E}(e^{-pS}) = e^{-\sqrt{2p}}$ (see W. Feller, *An Introduction to Probability Theorem and Its Applications, Vol. II*, 2nd ed., 436–437). By part (b) of the lemma,

$$p_{n,k} = \mathrm{E}\left(1 - \frac{k + Y_{n,k}}{n}\right)^k \to \mathrm{E}\left(e^{-t^3 S}\right) = e^{-\sqrt{2t^3}}.$$

Since the limit is continuous in t, it follows that $n^{1/3}$ is a threshold function for k in this problem. That is, $p_{n,k} \to 1$ if $k/n^{1/3} \to 0$, and $p_{n,k} \to 0$ if $k/n^{1/3} \to \infty$.

Editorial comment. The formula for the number of rooted forests with specified root set appears in Cayley's original paper of 1889. The proof in part (a) above is especially simple. Another recursive proof appeared in L. Lovász's problem book. J. W. Moon (in *Graph Theory and Theor. Physics* (Academic Press, 1967), 261–272) proved a more general statement. R. P. Stanley (*Enumerative Combinatorics, Volume II* (Cambridge Univ. Press, 1999), p. 25) presented a proof that extends the method of Prüfer codes.

This method was extended to simplicial complexes in C. Greene and G. Iba (*Discrete Math.* 13 (1975), 1–11).

The results about $q_{n,k}$ come from Y.D. Burtin, On a simple formula for random mappings and its applications, *J. Appl. Prob.* 17 (1980), 403–414. In this article, Burtin used a combinatorial lemma (which has been called "Burtin's Lemma" in this MONTHLY 106 (1999), 345–351) to show that $q_{n,k,j}$ describes the distribution of the number of ancestors of a given k-element set in the digraph of a random mapping, and he determined the asymptotic properties of $q_{n,k,j}$.

Part (a) also solved by GCHQ Problem Solving Group (U. K.) and the proposer.

Matrix Systems with the Same Solution Sets

11146 [2005]. *Proposed by Yongge Tian, University of Alberta, Edmonton, Canada.* Let A_1 and A_2 be matrices with real entries and the same shape. If the matrix equations $A_1X = B_1$ and $A_2X = B_2$ have a common real solution X_0, then there exists Y such that $(A_1 + A_2)Y = B_1 + B_2$. Let

$$S_1 = \{X\colon\ A_1X = B_1, A_2X = B_2\}, \quad S_2 = \{Y\colon\ (A_1 + A_2)Y = B_1 + B_2\}.$$

Let C^T denote the transpose of C. Show that $S_1 = S_2$ if and only if the ranges of A_1^T and A_2^T are subsets of the range of $(A_1 + A_2)^T$.

Solution by Eugene A. Herman, Grinnell College, Grinnell, Iowa. We read the problem as including the premise that such an X_0 exists; otherwise taking $A_1 = \binom{1\ 0}{0\ 0}$, $B_1 = \binom{0}{1}$, $A_2 = \binom{0\ 0}{0\ 1}$, and $B_2 = \binom{1}{0}$ leads to a counterexample. Let $A = \binom{A_1}{A_2}$ and $B = \binom{B_1}{B_2}$. Thus $S_1 = \{X\colon\ AX = B\}$. Let K_1 and K_2 denote the kernels of A and $A_1 + A_2$, respectively. Thus, since $X_0 \in S_1$ and $X_0 \in S_2$, we have

$$S_1 = X_0 + K_1, \quad S_2 = X_0 + K_2.$$

Since $K_1 \subseteq K_2$, we conclude that $S_1 = S_2$ if and only if $K_2 \subseteq K_1$. Finally, we use the fact that the orthogonal complement of the kernel of C is the range of C^T. Therefore, $S_1 = S_2$ if and only if the range of A^T is a subset of the range of $(A_1 + A_2)^T$. Since $A^T = [A_1^T\ A_2^T]$, we have $S_1 = S_2$ if and only if the ranges of A_1^T and A_2^T are subsets of the range of $(A_1 + A_2)^T$.

Also solved by S. Amghibech (Canada), F. S. Barger, R. Chapman (U. K.), J.-P. Grivaux (France), D. A. Huckaby, T. Jager, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Stong, Microsoft Research Problems Group, and the proposer.

Peaks in Dyck Paths

11150 [2005, 367]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* A *Dyck n-path* is a lattice path of n upsteps $(1, 1)$ and n downsteps $(1, -1)$ that starts at the origin and never goes below the x-axis. A two-step subpath in which the first step is an upstep and the second a downstep is called a *peak*. The *height* of a peak is the y-coordinate of the right endpoint of its upstep.

Let E_n and O_n denote the number of Dyck n-paths having an even and an odd number of peaks, respectively, at even height. Find E_n and O_n.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. If n is even, then $O_n = E_n = \frac{1}{2}C_n$, where C_n is the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$, which is the total number of Dyck n-paths. If n is odd, then $O_n = \frac{1}{2}(C_n - C_{\lfloor n/2\rfloor})$ and $E_n = \frac{1}{2}(C_n + C_{\lfloor n/2\rfloor})$.

A two-step subpath in which the first step is a downstep and the second an upstep is called a *valley*. The height of a valley is the y-coordinate of the left endpoint of its upstep. Let C_n^1 be the number of Dyck n-paths having at least one peak or valley at even height, and let C_n^0 be the number of paths having no peaks or valleys at even height. Changing the first peak or valley at even height into a valley or peak changes the parity of the number of peaks at even height, so C_n^1 contributes equally to E_n and O_n. That is,

$$O_n = \frac{1}{2}C_n^1 = \frac{1}{2}(C_n - C_n^0) \qquad \text{and} \qquad E_n = \frac{1}{2}C_n^1 + C_n^0 = \frac{1}{2}(C_n + C_n^0).$$

If a Dyck path has no peaks or valleys at even height, then the number of consecutive upsteps or downsteps is odd at the beginning or end and is even for all other runs. This requires an odd number of upsteps and an odd number of downsteps, which requires n to be odd. Thus $C_n^0 = 0$ when n is even, which completes the proof for even n.

When n is odd, the runs of even length allow us to delete the first and last step and then collapse the steps in pairs to obtain a Dyck $\lfloor n/2 \rfloor$-path. Each such path arises from a unique path counted by C_n^0. Thus $C_n^0 = C_{\lfloor n/2 \rfloor}$, which completes the proof.

Editorial comment. The sequences in this problem appear in the *On-Line Encyclopedia of Integer Sequences*. That defined by E_n is the highly annotated OEIS sequence A007595, while that defined by O_n is the ancient A000150.

Also solved by D. Beckwith, R. Chapman (U. K.), C. Delorme (France), R. A. Simon (Chile), R. Staum, R. Stong, L. Zhou, GCHQ Problem Solving Group, Microsoft Research Problems Group, and the proposer.

Partitions of a Circular Set

11151 [2005, 367]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Suppose n people are sitting at a circular table. Let $e_{m,n}$ denote the number of ways to partition them into m affinity groups with no two members of a group seated next to each other. (For example, $e_{3,4} = 2$, $e_{3,5} = 5$, and $e_{3,6} = 10$.) For $m \geq 2$, find the generating function $\sum_{n=0}^{\infty} e_{m,n} z^n$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. The desired generating function is given by

$$E_m(z) = \frac{z^m}{1+z} \prod_{k=1}^{m-1} \frac{1}{1-kz}.$$

Let us first consider n people sitting in a row. Let $f_{m,n}$ denote the number of ways to partition them into m affinity groups with no two members of a group seated next to each other. For $m = 2$, we have $f_{2,0} = f_{2,1} = 0$, and $f_{2,n} = 1$ when $n \geq 2$. For $m \geq 3$, we have $f_{m,n} = 0$ when $n < m$ and

$$f_{m,n} = f_{m-1,n-1} + (m-1) f_{m,n-1} \tag{1}$$

when $n \geq m$. Define $F_m(z) = \sum_{n=0}^{\infty} f_{m,n} z^n$. Multiplying (1) by z^n and summing over $n \geq 2$ gives $F_m(z) = zF_{m-1}(z) + z(m-1)F_m(z)$ for $m \geq 3$, or equivalently, $F_m(z) = \frac{z}{1+(m-1)z} F_{m-1}(z)$. Since $F_2(z) = \sum_{n=2}^{\infty} z^n = \frac{z^2}{1-z}$, we obtain

$$F_m(z) = z \prod_{k=1}^{m-1} \frac{z}{1-kz}.$$

To determine $e_{m,n}$, note that the row arrangements counted by $f_{m,n}$ may or may not use the same affinity group at both ends. Since the people are distinguishable, we obtain $f_{m,n} = e_{m,n} + e_{m,n-1}$, which yields $F_m(z) = E_m(z) + zE_m(z)$ and hence the expression claimed above for $E_m(z)$.

Editorial comment. The Microsoft Research Problems Group provided a closed form for the generating function,

$$E_m(z) = \frac{\Gamma\left(\frac{1}{z} - (m-1)\right)}{z\Gamma\left(\frac{1}{z} + 2\right)}.$$

Several solvers noted that $f_{m,n}$ equals the Stirling number $S(n-1, m-1)$ of the second kind, which counts the partitions of $n-1$ people into $m-1$ nonempty affinity groups without regard to adjacency. They all proved this by induction, noting that both sides satisfy the same recurrence. There is also a bijection. To obtain a partition from a row arrangement, delete the leftmost person and have anyone else in that group join instead the group of the person on her left. For the inverse, any segment of consecutive people in a single group decomposes by moving the even-indexed members of the segment into the new group with the new leftmost person.

Also solved by D. Beckwith, D. Callan, R. Chapman (U. K.), W. Chu (Italy), J. Dalbec, K. Dale (Norway), S. M. Gagola Jr., C. R. Pranesachar (India), M. A. Prasad (India), R. Pratt, R. Tauraso (Italy), GCHQ Problem Solving Group, Microsoft Research Problems Group, and the proposer.

Indivisibility of the Sum of Squares

11154 [2005, 467]. *Proposed by Kent Holing, Trondheim, Norway.* Let a and b be positive integers with $ab > 1$.
(**a**) Show that if a and b are relatively prime, then $(1 + ab) \nmid (a^2 + b^2)$.
(**b**)* Suppose now that a and b need not be relatively prime, but that $a^2 + b^2$ is a square. Can it happen that $(1 + ab) \mid (a^2 + b^2)$?

Solution to part (a) by Michael R. Avidon, Allston, MA. Suppose that (i) $a > b \geq 1$, (ii) $\gcd(a, b) = 1$, (iii) $(1 + ab) \mid (a^2 + b^2)$, and (iv) a is the smallest integer such that (i) through (iii) hold for some b. (This b cannot be 1.) There exist integers q and r such that $a = qb - r$ with $q \geq 2$ and $0 < r < b$. Note that $a \geq q$.

By (iii), $1 + ab$ divides $a^2 + b^2 - (q-1)(1 + ab)$. From $a^2 = a(qb - r)$, we get

$$\begin{aligned} a^2 + b^2 - (q-1)(1+ab) &= ab - ar + b^2 - q + 1 \\ &= a(b - r - 1) + b^2 + a - q + 1. \end{aligned}$$

Since $r < b$ and $a \geq q$, this quantity is positive. It is bounded by $(a + b)b$, which is less than $2 + 2ab$ since $a > b$. By (iii), it is divisible by $1 + ab$, and hence it must equal $1 + ab$.

From $1 + ab = ab - ar + b^2 - q + 1$, we obtain $q = b^2 - ar$, and adding qbr to both sides yields $q(1 + br) = b^2 + (qb - a)r = b^2 + r^2$. Hence $(1 + br) \mid (b^2 + r^2)$ with $a > b > r \geq 1$. Since $b < a$, (iv) requires $r \mid b$. Since $(1 + br) \mid (b^2 + r^2)$ prohibits $r = 1$, now a and b have a nontrivial common factor r.

Editorial comment. Several solvers, including the proposer, noted that part (a) follows from Problem 6 of the 1988 International Mathematical Olympiad. No solutions to part (b) were received.

Part (a) also solved by S. Amghibech (Canada), R. Chapman (U. K.), K. Foster, S. M. Gagola Jr., O. P. Lossers (The Netherlands), C. A. Meyer (Switzerland), M. A. Prasad (India), R. E. Prather, C. V. Riecke, T. P. Schonbek, A. Stadler (Switzerland), R. Stong, L. Zhou, GCHQ Problem Group (U. K.), Microsoft Problems Group, NSA Problems Group, and the proposer.