CONNECTED PSEUDOACHROMATIC INDEX OF COMPLETE GRAPHS

LOWELL ABRAMS AND YOSEF BERMAN

Abstract. A connected pseudocomplete n-coloring of a graph G is a (non-proper) n-coloring of the vertices of G such that each color class induces a connected subgraph and for each pair of color classes there is an edge with one end of each color; this can be viewed as a kind of “inverse image” of a clique minor. The connected pseudoachromatic index of a graph G is the largest n for which the line graph of G has a connected pseudocomplete n-coloring. For all j, k we show that the connected pseudoachromatic index of the complete graph on 5k + j + 1 vertices is at least 9k + j. We also provide several results on connections between connected pseudoachromatic index of complete graphs and the Erdős-Faber-Lovász conjecture.

Keywords: pseudoachromatic number, pseudoachromatic index, Hadwiger number, Erdős-Faber-Lovász

1. Introduction

Let G be a finite simple graph with vertex set V(G) and edge set E(G). A pseudocomplete k-coloring of G is an assignment α: V(G) → [k], where [k] denotes the set {1, . . . , k}, such that for each i, j ∈ [k] there is an edge in E(G) having one end in α−1(i) and the other in α−1(j). The pseudochromatic number ψ(G) of G is the maximum k for which there is a pseudocomplete k-coloring of G [9]. Basic results on pseudochromatic number and the related notion of achromatic number (which presumes that no two adjacent vertices have the same color) were presented in [3, 5, 4, 11]. Along these lines also is [12], which discusses achromatic number of the line graph of Kn. Of more recent note is the calculation of ψ for complete multipartite graphs [14, 16] and for the line graph of Kn for special values of n [1, 2].

Suppose H is a minor of G obtained from a subgraph G′ of G by contracting some edges, and that V(H) = [k]. Then there is a naturally corresponding pseudocomplete k-coloring α: G′ → [k] for which α−1(i) is exactly the set of vertices of G′ which contract to vertex i in H. In this case, the classes α−1(i) have the additional property that
for each $i$ the induced subgraph $G[\alpha^{-1}(i)]$ is connected. Since this is so, without loss of generality we may presume that $G' = G$. Define the connected pseudoachromatic number $\psi_c(G)$ to be the maximum $k$ for which there is a connected pseudocomplete $k$-coloring of $G$, i.e., a pseudocomplete coloring in which each color-class induces a connected subgraph. With this definition, we see that $\psi_c(G)$ is the size of the largest complete-graph minor of $G$; this value is also called the Hadwiger number of $G$. Since for any graph $G$ we have $\psi(G) \geq \psi_c(G)$, study of the pseudoachromatic number has been useful for bounding the Hadwiger number, as in [13] and, from a probabilistic perspective, [6].

The pseudoachromatic number of the line graph $LG$ for any graph $G$ is also referred to as the pseudoachromatic index of $G$ [4]. We focus in this work on the line graph $LK_n$ of the complete graph $K_n$. Note that any connected pseudocomplete $k$-coloring of $LK_n$ may be viewed as an edge coloring of $K_n$ in which each edge color class induces a connected subgraph, and each pair of edge color classes share at least one vertex. We will make use of this point of view below when it is convenient.

There are a few existing results on $\psi(LK_n)$. Bosák and Nešetřil provide the following values.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(LK_n)$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>8</td>
<td>11</td>
</tr>
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</table>

Araujo-Pardo et al. prove that if $n = 2^{2^\beta} + 2^\beta + 1$ then $\psi(LK_n) \geq 2^{3\beta} + 2^\beta$, and if $n = 2^{2^\beta} + 2^{3+1} + 2$ then $\psi(LK_n) = 2^{3\beta} + 2^{2\beta} + 3 \cdot 2^\beta$.

In Section 2 we provide several results on lower bounds for $\psi_c(LK_n)$ for various values of $n$. Together they imply Theorem 2.4: For $j, k \geq 1$ we have $\psi_c(LK_{5k+j+1}) \geq 9k + j$.

Our results on $LK_n$ have an interesting implication for the relationship between two famous conjectures.

**Conjecture 1.1** (Hadwiger [8, 10]). *If graph $G$ does not contain $K_{n+1}$ as a minor, then $G$ is $n$-colorable.*

Of course, Hadwiger’s Conjecture is the reason for the term “Hadwiger number.”

**Conjecture 1.2** (Erdős-Faber-Lovász [7]). *If graph $G$ can be constructed by joining $n$ copies of $K_n$ so that no two copies of $K_n$ share more than one vertex, then $G$ is $n$-colorable.*

We refer to a graph which meets the hypothesis of the Erdős-Faber-Lovász Conjecture as an EFL graph. In Section 4 we exhibit an infinite family of EFL graphs which contain $K_{n+1}$ as a minor, demonstrating
that Hadwiger’s Conjecture does not imply the Erdős-Faber-Lovász Conjecture.

2. Connected Pseudoachromatic Number

Throughout this section we let \( V(K_n) = \{v_1, v_2, \ldots, v_n\} \) and write \( v_{i,j} \) for the vertex of \( LK_n \) corresponding to the undirected edge \( v_i v_j \) of \( K_n \). We begin with two basic cases.

Proposition 2.1. We have \( \psi_c(LK_4) = 4 \) and \( \psi_c(LK_5) = 6 \).

Proof. Note first that \( LK_4 \), which is the 1-skeleton of the octahedron, is planar and hence does not contain \( K_5 \) as a minor. Given the connection between \( \psi_c \) and clique-minors, we see that \( \psi_c(LK_4) \leq 4 \). On the other hand, it is not difficult to construct a connected pseudocomplete 4-coloring \( \alpha_4 \) of \( LK_4 \); here is one:

\[
\alpha_4^{-1}(1) = \{v_{1,2}\}, \quad \alpha_4^{-1}(3) = \{v_{2,3}\}, \\
\alpha_4^{-1}(2) = \{v_{1,3}\}, \quad \alpha_4^{-1}(4) = \{v_{1,4}, v_{2,4}, v_{3,4}\}.
\]

It is easy to check that each of these vertex classes is connected and between each pair of vertex classes there is an edge connecting a vertex in one of the classes to a vertex in the other class. This completes the proof of the first assertion in the proposition.

For the second assertion, here is one way to construct a connected pseudocomplete 6-coloring \( \alpha_6 \) of \( LK_5 \). Define \( \alpha_6 \) by

\[
\alpha_6^{-1}(1) = \{v_{1,2}, v_{2,3}\}, \quad \alpha_6^{-1}(4) = \{v_{1,5}, v_{4,5}\}, \\
\alpha_6^{-1}(2) = \{v_{1,3}, v_{3,5}\}, \quad \alpha_6^{-1}(5) = \{v_{2,4}, v_{2,5}\}, \\
\alpha_6^{-1}(3) = \{v_{4,1}\}, \quad \alpha_6^{-1}(6) = \{v_{3,4}\}.
\]

Again, it is not difficult to verify the validity of this connected pseudocomplete 6-coloring.

Now suppose for the sake of contradiction that \( \alpha : V(LK_5) \to [7] \) is a connected pseudocomplete 7-coloring. Since \( |V(LK_5)| = 10 \), at least 4 classes \( \alpha^{-1}(i) \) contain a single vertex. It is not difficult to see that the corresponding singleton edge classes in \( K_5 \) must share a single vertex \( x \) of \( K_5 \) in order to satisfy the mutual adjacency requirement. But now, since all edges incident to \( x \) in \( K_5 \) have been used, each additional edge class in \( K_5 \) must have at least three edges in order to simultaneously satisfy the connectedness and mutual adjacency requirements; see Figure 1. This implies that there are at most two non-singleton classes, contradicting the assumption that we have a 7-coloring. \( \square \)

The next result shows that any lower bound on \( \psi_c(LK_n) \) for some \( n \) gives lower bounds for all larger values of \( n \) as well.
Figure 1. Edge classes in $K_5$. Each of the four edges incident to $x$ comprises its own class, the bold edges comprise a single class, and the dashed edges comprise a single class.

**Proposition 2.2.** If for some $n$ we have $\psi_c(LK_n) \geq m$ for some $m$, then $\psi_c(LK_{n+j}) \geq m + j$ for all $j \geq 0$.

**Proof.** Proceeding by induction we assume the existence of a connected pseudocomplete $m$-coloring $\alpha_m$ of $LK_{n+j-1}$ for some $j > 0$ and extend it to construct a connected pseudocomplete $(m + 1)$-coloring $\alpha_{m+1}$ of $LK_{n+j}$. Explicitly, for $1 \leq i \leq m$ let $\alpha_{m+1}^{-1}(i) = \alpha_m^{-1}(i)$ and let $\alpha_{m+1}^{-1}(m+1) = \{v_1, n+j, \ldots, v_{n+j}, n+j\}$. Clearly, the subgraph induced on $\alpha_{m+1}^{-1}(m+1)$ is connected. To see that $\alpha_{m+1}$ is pseudocomplete we can interpret $\{v_1, n+j, \ldots, v_{n+j-1}, n+j\}$ as the edges $v_1v_{n+j}, \ldots, v_{n+j-1}v_{n+j}$ and note that each vertex of $K_{n+j-1}$ is incident to one of these. It then follows that each vertex in $LK_{n+j}$ is adjacent to some vertex in $\alpha_{m+1}^{-1}(m+1)$, so we are done. \hfill \Box

Propositions 2.1 and 2.2 combine to yield a proof by induction that $\psi_c(LK_n) \geq n + 1$ for $n \geq 5$. Nevertheless, Proposition 2.3 provides a far better result, formulated below as Theorem 2.4.

**Proposition 2.3.** For $k \geq 1$ we have $\psi_c(LK_{5k+2}) \geq 9k + 1$.

**Proof.** We first verify the result for $k = 1$. Construct a connected pseudocomplete 10-coloring $\alpha_{10}$ of $LK_7$ as follows.

\begin{align*}
\alpha_{10}^{-1}(1) &= \{v_{3,7}\} & \alpha_{10}^{-1}(6) &= \{v_{5,6}, v_{6,7}\} \\
\alpha_{10}^{-1}(2) &= \{v_{1,7}, v_{1,3}\} & \alpha_{10}^{-1}(7) &= \{v_{3,5}, v_{2,5}\} \\
\alpha_{10}^{-1}(3) &= \{v_{2,6}, v_{2,7}\} & \alpha_{10}^{-1}(8) &= \{v_{3,6}, v_{4,6}\} \\
\alpha_{10}^{-1}(4) &= \{v_{2,4}, v_{4,7}\} & \alpha_{10}^{-1}(9) &= \{v_{2,3}, v_{1,2}, v_{1,5}\} \\
\alpha_{10}^{-1}(5) &= \{v_{4,5}, v_{5,7}\} & \alpha_{10}^{-1}(10) &= \{v_{1,6}, v_{1,4}, v_{3,4}\}
\end{align*}
It is easy to check that each of these vertex classes is connected and between each pair of vertex classes there is an edge connecting a vertex in one of the classes to a vertex in the other class.

We now provide a construction, working in terms of edges colorings of complete graphs, that verifies the result for general $k$. Take $k$ disjoint copies $P_1, \ldots, P_k$ of the edge-colored $K_7$ specified above, and identify all vertices $v_3$ and all vertices $v_7$, respectively, as well as all edges $v_3v_7$ and their color classes. We maintain the distinct identities of the colors, other than color 1, for each $P_i$. The resulting graph has $5k + 2$ vertices (a schematic is shown in Figure 3); add in edges to obtain the complete graph $K_{5k+2}$.

Now consider a pair $P_r, P_s$. Refer to the vertices in $P_r$ as $v_1, v_2, \ldots, v_7$, to the edge colors in $P_r$ as 1, 2, $\ldots$, 10, and to the corresponding vertices and colors in $P_s$ as $v_1', v_2', v_3, v_4', v_5', v_6', v_7$, and 1, 2', $\ldots$, 10', respectively. Let $\alpha_{9k+1}$ be partially defined by

$$
\begin{align*}
    v_{1,2'} &\mapsto 9 & v_{2,2'} &\mapsto 3 & v_{4,4'} &\mapsto 4 & v_{5,5'} &\mapsto 5 \\
    v_{1,4'} &\mapsto 10 & v_{2,4'} &\mapsto 3 & v_{4,5'} &\mapsto 8 & v_{6,2'} &\mapsto 8 \\
    v_{1,5'} &\mapsto 9 & v_{2,5'} &\mapsto 7 & v_{5,2'} &\mapsto 7 & v_{6,5'} &\mapsto 6 \\
    v_{1,6'} &\mapsto 10 & v_{4,2'} &\mapsto 4 & v_{5,4'} &\mapsto 5 & v_{6,6'} &\mapsto 6
\end{align*}
$$

It is not difficult to check that each color class in $P_r \cup P_s$ is now incident to each other color class. To assist in this check, here is a table showing the classes incident to each vertex:
vertex : incident classes
\[ v_3 : 1, 2, 7, 8, 9, 10, 1', 2', 7', 8', 9', 10' \]
\[ v_7 : 1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6' \]
\[ v_1 : 2, 9, 10 \]
\[ v_2 : 3, 4, 7, 9 \]
\[ v_4 : 4, 5, 8, 10 \]
\[ v_5 : 5, 6, 7, 9 \]
\[ v_6 : 3, 6, 8, 10 \]
\[ v'_1 : 2', 9', 10' \]
\[ v'_2 : 3', 4', 7', 9', 3, 4, 7, 8, 9 \]
\[ v'_4 : 4', 5', 8', 10', 3, 4, 5, 10 \]
\[ v'_5 : 5', 6', 7', 9', 5, 6, 7, 8, 9 \]
\[ v'_6 : 3', 6', 8', 10', 6, 10 \]

Follow the analogous procedure for all other pairs \( P_r, P_s \). Finally, define \( \alpha_{9k+1} \) on the remaining edges in any way that preserves the connectivity of the color classes. Since each \( P_i \) contributes nine color classes in addition to the class 1, we indeed have a connected pseudocomplete \((9k + 1)\)-coloring of \( LK_{5k+2} \). \( \square \)

Combining Propositions 2.3 and 2.2 we obtain Theorem 2.4.

**Theorem 2.4.** For \( j, k \geq 1 \) we have \( \psi_c(LK_{5k+j+1}) \geq 9k + j \).

### 3. Computer calculation and what is known

Proposition 2.1 tells us that \( \psi_c(LK_4) = 4 \) and \( \psi_c(LK_5) = 6 \), and exhaustive computer calculation confirms that \( \psi_c(LK_6) = 7 \) and \( \psi_c(LK_7) = 10 \) (the computer calculations are described below). Beyond this, we know only the lower bounds derived from Theorem 2.4. This information is depicted in Figure 3.

We now describe how the computer calculations were done. Without loss of generality we can assume that the vertex classes in \( LK_n \) that constitute a connected pseudocomplete coloring are trees in \( K_n \), since all other edges may be colored in any way that preserves the connectedness of the individual classes. Using Prüfer sequences it is straightforward to iterate over all spanning trees in a clique, and by considering all possible sizes of subsets of the vertices of \( K_n \) we can find all subgraphs of \( K_n \) which are trees. Let \( T_n \) denote the set of all such trees and define a graph structure on \( T_n \) by declaring two trees to be adjacent if they share at least one vertex but share no edges. A clique in \( T_n \) represents a partial covering of the edges of \( K_n \) with a family of edge-disjoint trees such that each pair of trees shares at least one vertex. By extending the trees as described above, we obtain a connected pseudocomplete coloring of \( LK_n \).

Thus we have reduced the problem of finding connected pseudocomplete colorings of \( LK_n \) to the problem of finding cliques in \( T_n \). Of course, \( T_n \) grows in \( n \) faster than any exponential. Nevertheless, for sufficiently small \( n \), this approach yielded new results, specifically that
Figure 3. A schematic indicating what is known. A solid point at \((x, y)\) indicates that \(\psi_c(LK_y) = x\), whereas a hollow point indicates that \(\psi_c(LK_y) \geq x\). The shading emphasizes that the actual values of the various pseudoachromatic indices may lie to the right.

\(\psi_c(LK_6) = 7\) and \(\psi_c(LK_7) = 10\). Note that finding maximum cliques was done using the publicly available software package Cliquer [15].

4. Implications for EFL Graphs

An \(n\)-EFL graph is a connected graph \(G\) produced by joining \(n\) copies of \(K_n\), which we call panels, so that no two panels share more than one vertex. We refer to vertices contained in more than one panel as vertices of attachment. For each \(n\), define the standard \(n\)-EFL graph \(E_n\) to be the graph obtained from \(LK_n\) by adding new vertices \(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n\) and an edge connecting \(\hat{v}_i\) and \(v_{i,j}\) for each \(i, j \in [n]\). Note that, for each \(i\), the subgraph of \(E_n\) induced on the vertices \(\{\hat{v}_i\} \cup \{v_{i,j} \mid j \neq i\}\) is an \(n\)-clique, so \(E_n\) is indeed an \(n\)-EFL-graph. Figure 4 shows a drawing of \(E_4\).

The following result indicates a sense in which the graphs \(E_n\) are universal.

**Theorem 4.1.** If \(G\) is an \(n\)-EFL graph and \(K_m\) is a minor of \(G\), then \(K_m\) is a minor of \(E_n\) as well.
Figure 4. A plane drawing of $E_4$; the shading indicates copies of $K_4$.

Proof. Certainly, if $m \leq n$ then $K_m$ is a subgraph of any panel in $E_n$, so is a minor of $E_n$.

Suppose, then, that $m > n$ and that $K_m$ is a minor of $G$. We may realize this minor with a pair $(T, \beta)$ where $T = \{t_1, \ldots, t_m\}$ is a family of vertex-disjoint trees in $G$ and $\beta: \{\{i, j\}|i, j \in [m], i \neq j\} \rightarrow E(G)$ is such that for $i \neq j$ the edge $\beta(i, j)$ has one vertex in $t_i$ and the other in $t_j$. The trees can be viewed as sitting in distinct color classes of a connected pseudocomplete $m$-coloring of $G$; contracting them yields the vertices of $K_m$ and the edges in $\text{Im} \beta$ are the edges of $K_m$.

Suppose $t \in T$ contains no vertices of attachment. In that case, $t$ must be contained entirely in a single panel $P$. Moreover, all edges in $\text{Im} \beta$ which have one vertex in $t$ must have their other vertex in $P$ as well, and thus there can be at most $n - 1$ such edges. Since $m > n$, this contradicts the assumption that $(T, \beta)$ represents a $K_m$ minor. Thus every tree $t$ in $T$ contains vertices of attachment, and indeed must have a vertex of attachment in each panel in which it has any vertex at all.

We now produce a new pair $(T', \beta')$ which realizes $K_m$ as a minor of $G$ but which has the additional property that for each $t \in T'$ all vertices of $t$ are vertices of attachment. Suppose $v$ is a vertex in panel $P$ which is not a vertex of attachment but is contained in tree $t_i \in T$. As shown above, there is some vertex $v_i$ in $P$ which is contained in $t_i$ and is a vertex of attachment. Modify $t_i$ by deleting $v$ and, for each neighbor $w$ of $v$ which is contained in $t_i$ but is not connected to $v_i$ in $t_i - v_i$, adding in the edge $v_i w$. Because $v$ is not a vertex of attachment, $w$ must be contained in panel $P$, and therefore the edge $v_i w$ exists. Because the only path in $t_i$ between neighbors of $v$ is a path of length two through $v$, the result of this deletion and addition
is itself a tree. Corresponding to this new tree, we also modify \( \beta \). For every \( j \in [m] \) such that \( \beta(\{i, j\}) = vw_j \) for some \( w_j \) in \( t_j \), redefine \( \beta \) to map \( \{i, j\} \mapsto v_iw_j \). As before, because \( v \) is not a vertex of attachment, \( w_j \) must be contained in panel \( P \), and therefore the edge \( v_iw_j \) exists. See Figure 5 for an illustration of this two step process. Repeating the process for every such vertex \( v \) and panel \( P \) yields the desired pair \( (T', \beta') \).

We now modify \( G \) so as to convert it to \( E_n \); adjusting the pair \( (T', \beta') \) appropriately through this process yields the desired \( K_m \) minor of \( E_n \). Define the attachment weight of \( G \) to be the sum \( W(G) := \sum_v (d_v - 2) \) where the summation is over vertices of attachment \( v \) and \( d_v \) is the number of panels containing \( v \). If \( W(G) = 0 \), then we already have \( G = E_n \) and we are done. Otherwise, suppose that \( W(G) > 0 \), that \( v \) is a vertex of attachment with \( d_v > 2 \), and that \( P_1, P_2 \) are two of the panels containing \( v \). Since there are a total of \( n \) panels, and \( P_2 \) has \( n \) vertices, and \( d_v > 2 \), there is some vertex \( w \) of \( P_2 \) which is not a vertex of attachment and which therefore is not contained in any tree of \( T' \). Modify \( G \) by detaching panel \( P_1 \) at \( v \) from the other panels at \( v \), splitting off a new copy \( v_1 \) of vertex \( v \) in panel \( P_1 \) and renaming the “original” copy of \( v \) to \( v_2 \), then identify \( v_1 \) and \( w \). See Figure 6 for an illustration of this.

If \( v \) was contained in a tree \( t_i \) of \( T' \) (as illustrated in Figure 6), now adjust \( t_i \) to include the edge \( e = v_1v_2 \); refer to this newly adjusted \( t_i \) as \( t' \). To see that \( t' \) is connected, consider any vertices \( x \) and \( y \) in \( t' \).
Figure 6. An illustration of the process of detaching panel $P_1$, reattaching it, and then relabeling vertices. The diagram on the left depicts a tree $t_i$ containing $v$, and on the right is the tree $t'$ replacing it.

Suppose first that both $x$ and $y$ are also vertices of $t_i$, so that there is an $x$-$y$ path $P_{x,y}$ in $t_i$. If $P_{x,y}$ does not contain $v$, then $P_{x,y}$ is also an $x$-$y$ path in $t'$, so $x$ and $y$ are connected in $t'$. If $P_{x,y}$ does contain $v$, express $P_{x,y}$ as the concatenation $P_{x,v}P_{v,y}$ where $P_{x,v}$ is the $x$-$v$ path in $t_i$ and $P_{v,y}$ is the $v$-$y$ path. For some $v', v'' \in \{v_1, v_2\}$, $P_{x,v}$ and $P_{v,y}$ correspond to an $x$-$v'$ path $P'_{x,v'}$ in $t'$ and a $v''$-$y$ path $P'_{v'',y}$ in $t'$, respectively. If $v' = v''$, then the concatenation $P'_{x,v'}P'_{v'',y}$ is an $x$-$y$ path in $t'$, and otherwise $P'_{x,v'}eP'_{v'',y}$, for some orientation on $e$, is an $x$-$y$ path in $t'$.

If $\{x, y\} = \{v_1, v_2\}$, then the edge $v_1v_2$ itself provides the desired $x$-$y$ path. Suppose therefore that exactly one of $x$ and $y$ is either $v_1$ or $v_2$. Without loss of generality, suppose $y = v_1$. There is an $x$-$v$ path in $t_i$, and when the panel $P_1$ is detached at $v$ this path becomes either an $x$-$v_1$ path or an $x$-$v_2$ path in $t'$. In the latter case, concatenating with the edge $v_2v_1$ yields an $x$-$v_1$ path, so in either case we see that $x$ and $v_1$ are connected in $t'$. We thus conclude that $t'$ is connected.

Finally, up to renaming of vertices, no changes are made to $\beta'$. This process decreases the attachment weight by 1, and the adjusted $(T', \beta')$ still realizes a $K_m$ minor. Since this process can be repeated until $W(G) = 0$, the proof is complete. 

Since any minor of $LK_n$ is automatically a minor of $E_n$, the results of Section 2 readily imply that for $n \geq 5$ the graph $E_n$ contains a $K_{n+1}$ minor. This demonstrates that Hadwiger’s Conjecture does not imply the EFL conjecture.

There is also an interesting relationship between $E_n$ and $LK_n$ in the other direction.
Theorem 4.2. For each $m > 1$ and $n > 2$, if $K_m$ is a minor of $E_n$ then $K_m$ is a minor of $LK_n$.

Proof. The induced subgraph of $LK_n$ with vertex set $\{v_{1,j} \mid j \neq 1\}$ forms an $(n-1)$-clique, so certainly $K_m$ is a minor of $LK_n$ for $m < n$.

We can find $K_n$ as a minor of $LK_n$ by defining a pseudocomplete $n$-coloring $\alpha_n : V(LK_n) \rightarrow [n]$ as follows: Let

$$\alpha_n^{-1}(1) = \{v_{2,3}, v_{2,4}, \ldots, v_{2,n}\}$$

and for $i = 2, 3, \ldots, n$ let $\alpha_n^{-1}(i) = v_{1,i}$. Note that $\bigcup_{i=2}^{n} \alpha_n^{-1}(i)$ induces an $(n-1)$-clique in $LK_n$ and that for each $i$ there is an edge from $\alpha_n^{-1}(i)$ to $\alpha_n^{-1}(1)$. We see that contracting $\alpha_n^{-1}(1)$ to a single vertex yields the desired $K_n$ minor.

Suppose now that $m > n$ and that $K_m$ is a minor of $E_n$, but that $K_m$ is not a minor of $LK_n$. Then there is some vertex class $W \subseteq V(E_n)$ for the $K_m$ minor which contains a vertex $w = \hat{v}_j$ for some $j$. Since the degree of $\hat{v}_j$ is $n-1$ but $W$ contracts to a vertex of degree $m-1 \geq n$, there must be an additional vertex $v$ in $W$ which is adjacent to $w$. The vertex $v$ is necessarily adjacent to all neighbors of $w$ so any path in $E_n$ starting at $w$ passes through a neighbor of $v$. It follows that removing $w$ from $W$ leaves a suitable vertex class for the desired $K_m$ minor, and that no edges incident to $w$ are needed. Applying this reasoning to each $\hat{v}_j$ in each vertex class in $E_n$ for $K_m$ shows that $K_m$ is indeed a minor of $LK_n$. \hfill \Box

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