III. Descriptions

3. Lattice QCD

**Or: Using Large Computers for Fun**

References: [(Path Integral: Ryd 5; Sakurai: Modern QM 2.5); CL 10.5; PDG 17; Wagner [arXiv:1310.1760 [hep-lat]]; Alexandru, Lee, Lujan, . . .]
Historic Note

Thirty-one years ago, Dick Feynman told me about his ‘sum over histories’ version of quantum mechanics. ‘The electron does anything it likes’, he said. ‘It goes in any direction at any speed, forward or backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function.’ I said to him, ‘You’re crazy’. But he wasn’t.

Kleinert/Duru solved the H atom using Feynman’s Path Integral formalism in 1979 (!).
(b) Path Integrals on a Computer

Sample action of very many paths: Approximate Path Integral by Monte Carlo!

(1) Euclidean Time (Analytic Continuation/Wick Rotation)

Phase factor $e^{iS}$: rapid oscillations, numerically bad. $\implies$ use Euclidean Time

$$iS[\Phi] = + \frac{1}{2} \int dt \, d^3r \left[ \left( \frac{\partial \Phi(t, \vec{r})}{\partial t} \right)^2 - V(t, \vec{r}) \right]$$

$$\implies \text{Euclideanise } t \to \tau_E \quad \implies - \frac{1}{2} \int d\tau_E \, d^3r \left[ \left( \frac{\partial \Phi(\tau_E)}{\partial \tau_E} \right)^2 + V(\tau_E, \vec{r}) \right] =: -S_E[\Phi]$$

$$e^{\frac{i}{\hbar} S} \to e^{-\frac{S_E}{\hbar}}: L = T - V \text{ in } 3 + 1 \text{ spacetime dim.}$$

$\implies$ QM Partition Function $e^{-\beta H}$ with $H = T + V$ in 4 spatial dim. at temperature $\beta = \frac{i}{\hbar} (t_f - t_i)$.

Disadvantages:

$-\tau_E$ is not a time -- just a parameter.

$-$ No guarantee that analytic continuation to real time $t$ describes same QM system -- but so far, all explored cases do (proofs perturbative).

Advantages: Lattice QCD in $3 + 1$ dimensions is Statistical Mechanics in 4 dim.: Same techniques!
(2) Discretisation

Simplest: 4-dim. lattice with spacing $a$, $N$ points, length $L = Na$ in each direction.

Computer probes large but finite number if paths: Particle “hops” from point to point, calculate action, weight with $e^{-S_E}$, sum over many configurations:

$$\mathcal{A}_E = \langle x_{E,f}, \vec{r}_f \rangle \sum_{\text{paths}} e^{-S_E |x_{E,i}, \vec{r}_i\rangle}$$

cf. Ising Model of Statistical Mechanics
(3) Taking The Limits

**Continuum Limit** \(a \rightarrow 0\)

**Thermodynamic/Infinite-Volume Limit** \(L \rightarrow \infty\) (esp. \(\Delta t = N a \hat{\beta} = \frac{1}{\text{temperature}} \rightarrow \infty\))

Set dimension-ful **scale** \(\Rightarrow\) later...

**In Practise**

All of hadron comfortably inside box: \(L \gg \sqrt{\langle r^2 \rangle} \approx 1\text{fm}\) – today's standard: \(L > 3\text{fm}\).

Discretisation error: \(0 \leftarrow a \ll \text{typ. short-distance scale} \sim \frac{1}{2\text{GeV}} \approx 0.1\text{fm}\).

Asymptotic Freedom of QCD helps: \(g \downarrow 0\) for large \(q\)

\(\Rightarrow\) asymptotically non-interacting theory “shields” \(a \rightarrow 0\) details.

Lattice QCD: perturbation hard \(\Rightarrow\) find intermediate match.

**Today’s Lattices**

\(N = \frac{L}{a} > 30\) per dimension \((d = 4)\), each has spin \((4\)-dim. spinor\), colour \((3 \times 3)\), \(\geq 3\) flavours \((uds)\).

\((N = 64)^4 \approx 10^7\) doable, \(32^4 \approx 10^6\) standard.

**Huge** matrix describes how particles hop from one point to next (usually sparse).
(d) Free Fields on the Lattice

Points $\rightarrow$ Fields: $x(t) \rightarrow \Phi(x^\mu)$ For A Massive Real Scalar Field

Consider 1-Dimensional Case: only time direction, nothing else – generalisation straightforward.

$$iS[\Phi] = \frac{1}{2} \int dt \left( \left( \frac{\partial \Phi(t)}{\partial t} \right)^2 - m^2 \Phi^2(t) \right)$$

Euclideanise $it \rightarrow \tau_E$

$$\quad \rightarrow -\frac{1}{2} \int d\tau_E \left[ \left( \frac{\partial \Phi(\tau_E)}{\partial \tau_E} \right)^2 + m^2 \Phi^2(\tau_E) \right] =: -S_E[\Phi]$$

Rewrite as 2nd derivative

$$\quad \rightarrow -\frac{1}{2} \int d\tau_E \Phi(\tau_E) \left[ -\frac{\partial^2}{\partial \tau_E^2} + m^2 \right] \Phi(\tau_E)$$

Discretise à la Runge-Kutta RK2

$$\quad \rightarrow -\frac{a}{2} \sum_{\text{lattice sites } n} \left[ \Phi_n \left( -\frac{1}{a^2} (\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}) + m^2 \Phi_n^2 \right) \right]$$

Convert to matrix on vector $\bar{\Phi} = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_N \end{pmatrix}$ → $\propto \bar{\Phi}^T \begin{pmatrix} -\frac{2}{a^2} + m^2 & \frac{1}{a^2} & 0 & 0 & \cdots \\ \frac{1}{a^2} & -\frac{2}{a^2} + m^2 & \frac{1}{a^2} & 0 & \cdots \\ 0 & \frac{1}{a^2} & -\frac{2}{a^2} + m^2 & \frac{1}{a^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \bar{\Phi}$

This is a Linear Chain of Coupled Harmonic Oscillators:

Dislocation $\Phi_n$ at point $n$ by spring with constant $\propto \frac{1}{a^2}$, nearest-neighbour interactions, $m$ provides additional “drag”.

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PHYS 6610: Graduate Nuclear and Particle Physics I, Spring 2023
H. W. Grießhammer, INS, George Washington University III.3.5
Momentum Restriction: Brillouin Zone and Minimum Momentum

\[ S_E[\Phi] = \frac{a}{2} \sum_{\text{lattice sites } n} \left[ \Phi_n \left( \frac{-1}{a^2} (\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}) + m^2 \Phi_n^2 \right) \right] \]

Solve by **Discrete Fourier Transform** \( \Phi_n = \int \frac{dk}{2\pi} e^{ika} \Phi(k) \) at momentum \( k \). → HW

**Result:** Correct continuum limit for relativistic \( E-p \) relation:

\[
\text{propagator} \xrightarrow{a \to 0} m^2 + k^2 + O(a^2).
\]

**Resolution** \( a \) cannot resolve high momenta/high-frequency oscillations.

\[ \Rightarrow \text{Useful momenta must be inside Brillouin Zone } -\frac{\pi}{a} \leq k \leq \frac{\pi}{a}. \]

In *finite* lattice volume, there is also a **smallest nonzero momentum**.

**Example hypercube with Periodic Boundary Condition** \( \Phi_n = \Phi_{n+N} \):

\[ k_{\text{min}} = \pm \frac{2\pi}{L} \]

\[ \Rightarrow \text{Momenta are discretised: } k = 0, \pm \frac{2\pi}{L}, \pm \frac{4\pi}{L}, \ldots, \pm \frac{\pi}{a} \]
(e) QCD on the Lattice (pretend no matrices: simpler – results hold for matrices)

\[
\mathcal{L}_{QCD} = \bar{\Psi} i \gamma^\mu \left[ \partial_\mu - ig A_\mu \right] \Psi - m \bar{\Psi} \Psi - \frac{1}{2} \text{tr}[F_{\mu \nu} F^{\mu \nu}]
\]

Discretisation: Quarks \( \Psi \) on lattice sites, derivative connects adjacent sites.

\( A_\mu \) has direction, \( A_\mu \xrightarrow{\text{G.T.}} U[A_\mu + \frac{i}{g} \partial_\mu] U^\dagger \) \( \implies \) How to keep gauge invariance for \( a \neq 0 \)?

**Idea:** gauge-covariant derivative \( D_\mu := \partial_\mu - ig A_\mu \implies A_\mu \text{ links} \) adjacent lattice points in \( \mu \)-direction.

**Link Variable** \( U[x + ae_\mu \leftrightarrow x] \equiv \exp ig \int_0^a d\tau A_\mu (x + \tau e_\mu) \)

\( e_\mu \): unit vector in \( \mu \)-direction
QCD on the Lattice

\( \mathcal{L}_{QCD} = \bar{\Psi} i \gamma^\mu \left[ \partial_\mu - ig A_\mu \right] \Psi - m \bar{\Psi} \Psi - \frac{1}{2} \text{tr}[F_{\mu\nu} F^{\mu\nu}] \)

Discretisation: Quarks \( \Psi \) on lattice sites, derivative connects adjacent sites.

\( A_\mu \) has direction, \( A_\mu \xrightarrow{\text{G.T.}} U[A_\mu + \frac{i}{g} \partial_\mu] U^\dagger \implies \) How to keep gauge invariance for \( a \neq 0 \)?

Idea: gauge-covariant derivative \( D_\mu := \partial_\mu - ig A_\mu \implies A_\mu \) links adjacent lattice points in \( \mu \)-direction.

Link Variable \( U[x + ae_\mu \leftarrow x] \equiv \exp ig \int_0^a d\tau A_\mu(x + \tau e_\mu) \)

Gauge Transformation \( \Psi(x) \rightarrow e^{ig\alpha(x)} \Psi(x) \equiv U(x) \Psi(x) \): again: no matrices to simplify!

\[ U[x + ae_\mu \leftarrow x] \xrightarrow{\text{G.T.}} \exp ig \int_0^a d\tau \left[ A_\mu(x + \tau e_\mu) + \partial_\mu \alpha(x + \tau e_\mu) \right] \]

integral of derivative \( = \exp[ig \alpha(x + ae_\mu) + \left( ig \int_0^a d\tau A_\mu(x + \tau e_\mu) \right) - ig \alpha(x)] \)
(e) QCD on the Lattice (pretend no matrices: simpler – results hold for matrices)

$$\mathcal{L}_{QCD} = \bar{\Psi} i\gamma^{\mu} \left[ \partial_{\mu} - ig A_{\mu} \right] \Psi - m \bar{\Psi} \Psi - \frac{1}{2} \text{tr} [F_{\mu\nu} F^{\mu\nu}]$$

Discretisation: Quarks $\Psi$ on lattice sites, derivative connects adjacent sites.

$A_{\mu}$ has direction, $A_{\mu} \xrightarrow{\text{G.T.}} U[A_{\mu} + \frac{i}{g} \partial_{\mu}] U^\dagger \implies$ How to keep gauge invariance for $a \neq 0$?

Idea: gauge-covariant derivative $D_{\mu} := \partial_{\mu} - ig A_{\mu} \implies A_{\mu}$ links adjacent lattice points in $\mu$-direction.

**Link Variable** $\mathcal{U}[x + ae_{\mu} \leftarrow x] \equiv \exp ig \int_{0}^{a} d\tau A_{\mu}(x + \tau e_{\mu})$

**Gauge Transformation** $\Psi(x) \rightarrow e^{ig \alpha(x)} \Psi(x) \equiv U(x) \Psi(x)$: again: no matrices to simplify!

$$\mathcal{U}[x + ae_{\mu} \leftarrow x] \xrightarrow{\text{G.T.}} \exp ig \int_{0}^{a} d\tau \left[ A_{\mu}(x + \tau e_{\mu}) + \partial_{\mu} \alpha(x + \tau e_{\mu}) \right]$$

integral of derivative

$$= \exp [ig \alpha(x + ae_{\mu}) + \left[ ig \int_{0}^{a} d\tau A_{\mu}(x + \tau e_{\mu}) \right] - ig \alpha(x)]$$

split exponents

$$= e^{ig \alpha(x + ae_{\mu})} \left[ \exp ig \int_{0}^{a} d\tau A_{\mu}(x + \tau e_{\mu}) \right] e^{ig \alpha(x)}$$

$e_{\mu}$: unit vector in $\mu$-direction
\( \mathcal{L}_{QCD} = \bar{\Psi} i \gamma^\mu [\partial_\mu - ig A_\mu] \Psi - m \bar{\Psi} \Psi - \frac{1}{2} \text{tr}[F_{\mu \nu} F^{\mu \nu}] \) \( \times \times \)

**Discretisation: Quarks** \( \Psi \) on lattice sites, derivative connects adjacent sites. \( \times \times \)

\( A_\mu \) has direction, \( A_\mu \xrightarrow{\text{G.T.}} U[A_\mu + \frac{i}{g} \partial_\mu] U^\dagger \implies \) How to keep gauge invariance for \( a \neq 0 \)?

**Idea:** gauge-covariant derivative \( D_\mu := \partial_\mu - ig A_\mu \implies A_\mu \text{ links} \) adjacent lattice points in \( \mu \)-direction.

**Link Variable** \( \mathcal{U}[x + ae_\mu \leftarrow x] \equiv \exp ig \int_0^a d\tau A_\mu (x + \tau e_\mu) \)

\( e_\mu \): unit vector in \( \mu \)-direction

**Gauge Transformation** \( \Psi(x) \rightarrow e^{ig\alpha(x)} \Psi(x) \equiv U(x) \Psi(x) \):

\( U[x + ae_\mu \leftarrow x] \xrightarrow{\text{G.T.}} \exp ig \int_0^a d\tau [A_\mu (x + \tau e_\mu) + \partial_\mu \alpha(x + \tau e_\mu)] \)

**integral of derivative**

\[ = \exp[ig \alpha(x + ae_\mu) + \left( ig \int_0^a d\tau A_\mu (x + \tau e_\mu) \right)] - ig \alpha(x) \]

**split exponents**

\[ = e^{ig \alpha(x + ae_\mu)} \left[ \exp ig \int_0^a d\tau A_\mu (x + \tau e_\mu) \right] e^{ig \alpha(x)} \]

**use definitions**

\[ = U(x + ae_\mu) \mathcal{U}[x + ae_\mu \leftarrow x] U^\dagger(x) \]

\( \Psi(x + ae_\mu) \rightarrow \Psi(x + ae_\mu) \times \times \) interaction by link between sites and gives correct \( a \rightarrow 0 \) limit.
(e) QCD on the Lattice (pretend no matrices: simpler – results hold for matrices)

\[ \mathcal{L}_{QCD} = \bar{\Psi} i \gamma^\mu \left[ \partial_\mu - ig A_\mu \right] \Psi - m \bar{\Psi} \Psi - \frac{1}{2} \text{tr}[F_{\mu\nu} F^{\mu\nu}] \]

Discretisation: Quarks \( \Psi \) on lattice sites, derivative connects adjacent sites.

\( A_\mu \) has direction, \( A_\mu \xrightarrow{\text{G.T.}} U[A_\mu + \frac{i}{g} \partial_\mu] U^\dagger \implies \) How to keep gauge invariance for \( a \neq 0 \)?

**Idea:** gauge-covariant derivative \( D_\mu := \partial_\mu - ig A_\mu \implies A_\mu \) links adjacent lattice points in \( \mu \)-direction.

**Link Variable**
\[
U[x + ae_\mu \leftarrow x] \equiv \exp ig \int_0^a d\tau A_\mu(x + \tau e_\mu)
\]

**Gauge Transformation**
\[
\Psi(x) \rightarrow e^{ig\alpha(x)} \Psi(x) \equiv U(x) \Psi(x): \text{ again: no matrices to simplify!}
\]

\[
U[x + ae_\mu \leftarrow x] \xrightarrow{\text{G.T.}} = U(x + ae_\mu) \ U[x + ae_\mu \leftarrow x] \ U^\dagger(x)
\]

\[
\implies \bar{\Psi}(x + ae_\mu) \ U[x + ae_\mu \leftarrow x] \Psi(x) \quad \text{is gauge-invariant (def. & property holds also for matrices)!}
\]
QCD on the Lattice (pretend no matrices: simpler – results hold for matrices)

\[ \mathcal{L}_{QCD} = \bar{\Psi} i \gamma^\mu \left[ \partial_\mu - ig A_\mu \right] \Psi - m \bar{\Psi} \Psi - \frac{1}{2} \text{tr}[F_{\mu\nu} F^{\mu\nu}] \]

Discretisation: Quarks \( \Psi \) on lattice sites, derivative connects adjacent sites.

\( A_\mu \) has direction, \( A_\mu \xrightarrow{\text{G.T.}} U[A_\mu + \frac{i}{g} \partial_\mu] U^\dagger \) \( \xrightarrow{\text{How to keep gauge invariance for } a \neq 0?} \)

Idea: gauge-covariant derivative \( D_\mu := \partial_\mu - ig A_\mu \) \( \xrightarrow{\text{links}} \) adjacent lattice points in \( \mu \)-direction.

Link Variable \( U[x + ae_\mu \leftarrow x] \equiv \exp \left[ ig \int_0^a d\tau A_\mu(x + \tau e_\mu) \right] \)

Gauge Transformation \( \Psi(x) \rightarrow e^{ig\alpha(x)}\Psi(x) \equiv U(x) \Psi(x): \)

\[ \Psi(x) \rightarrow e^{ig\alpha(x)}\Psi(x) \]

again: no matrices to simplify!

\[ U[x + ae_\mu \leftarrow x] \xrightarrow{\text{G.T.}} U(x + ae_\mu) U[x + ae_\mu \leftarrow x] U^\dagger(x) \]

\[ \xrightarrow{\text{is gauge-invariant (def. & property holds also for matrices)!}} \]

\[ ga \xrightarrow{\text{midpoint approximation for integral}} \]

\[ \bar{\Psi}(x + ae_\mu) \left[ 1 + ig a A_\mu(x + \frac{a}{2} e_\mu) + O(ga)^2 \right] \Psi(x) \]
QCD on the Lattice (pretend no matrices: simpler – results hold for matrices)

\[ \mathcal{L}_{\text{QCD}} = \bar{\Psi} i \gamma^\mu \left[ \partial_\mu - i g A_\mu \right] \Psi - m \bar{\Psi} \Psi - \frac{1}{2} \text{tr}[F_{\mu \nu} F^{\mu \nu}] \]

Discretisation: Quarks \( \Psi \) on lattice sites, derivative connects adjacent sites.

\[ A_\mu \text{ has direction, } A_\mu \xrightarrow{\text{G.T.}} U[A_\mu + \frac{i}{g} \partial_\mu] U^\dagger \implies \text{How to keep gauge invariance for } a \neq 0? \]

Idea: gauge-covariant derivative \( D_\mu := \partial_\mu - i g A_\mu \implies A_\mu \text{ links adjacent lattice points in } \mu\text{-direction.} \)

Link Variable \( U[x + ae_\mu \larrow x] \equiv \exp i g \int_0^a \text{d} \tau A_\mu (x + \tau e_\mu) \)

Gauge Transformation \( \Psi(x) \rightarrow e^{ig\alpha(x)} \Psi(x) \equiv U(x) \Psi(x) \):

\[ \Psi(x + ae_\mu) \text{ is gauge-invariant (def. & property holds also for matrices)!} \]

\[ \frac{ga \rightarrow 0}{= \Psi(x + ae_\mu) \left[ 1 + ig a A_\mu (x + \frac{a}{2} e_\mu) + O(ga)^2 \right] \Psi(x)} \]

\[ \text{midpoint approximation for integral} \]

\[ \rightarrow \Psi(x + ae_\mu) \Psi(x) + \Psi(x + ae_\mu) \text{ ig } a A_\mu (x + \frac{a}{2} e_\mu) \Psi(x) \]

\[ \rightarrow \text{interaction by link between sites} \]

\[ \implies \text{Use link variables } U \in SU(N): \text{ gauge invariance for all } a \neq 0, \text{ no direct reference to } A_\mu! \]
Plaquette:

4 adjacent sites with their links

\[
\begin{pmatrix}
\, 0 \\
\a 
\end{pmatrix}
\quad \rightarrow
\quad \begin{pmatrix}
\, a \\
\a 
\end{pmatrix}
\]

\[
\begin{pmatrix}
\, 0 \\
\a 
\end{pmatrix}
\quad \rightarrow
\quad \begin{pmatrix}
\, a \\
\a 
\end{pmatrix}
\]

\[
\begin{pmatrix}
\, a \\
\, 0 
\end{pmatrix}
\quad \rightarrow
\quad \begin{pmatrix}
\, a \\
\, 0 
\end{pmatrix}
\]

\[
\begin{pmatrix}
\, 0 \\
\, 0 
\end{pmatrix}
\quad \rightarrow
\quad \begin{pmatrix}
\, 0 \\
\, a 
\end{pmatrix}
\]

\[
\begin{pmatrix}
\, 0 \\
\, a 
\end{pmatrix}
\quad \rightarrow
\quad \begin{pmatrix}
\, a \\
\, 0 
\end{pmatrix}
\]

\[
\begin{pmatrix}
\, a \\
\, a 
\end{pmatrix}
\quad \rightarrow
\quad \begin{pmatrix}
\, a \\
\, a 
\end{pmatrix}
\]

\[
\begin{pmatrix}
\, a \\
\, 0 
\end{pmatrix}
\quad \rightarrow
\quad \begin{pmatrix}
\, 0 \\
\, a 
\end{pmatrix}
\]

\[
\begin{pmatrix}
\, a \\
\, 0 
\end{pmatrix}
\quad \rightarrow
\quad \begin{pmatrix}
\, 0 \\
\, a 
\end{pmatrix}
\]

Looks like part of Field Strength Tensor

\[ F_{xy} = \partial_x A_y - \partial_y A_x \]

Combine with the other 2 parallel links for gauge-invariant expression (holds also with matrices):

Plaquette Variable

\[
U_{\square} = U_{\begin{pmatrix}
\, 0 \\
\a 
\end{pmatrix}} U_{\begin{pmatrix}
\, 0 \\
\a 
\end{pmatrix}} U_{\begin{pmatrix}
\a \\
\, 0 
\end{pmatrix}} U_{\begin{pmatrix}
\, 0 \\
\, 0 
\end{pmatrix}} = 1 + i g a \left[ A_x \left( \begin{pmatrix}
\, a \\
\, 0 
\end{pmatrix} \right) - A_x \left( \begin{pmatrix}
\, a \\
\, 0 
\end{pmatrix} \right) \right] \]

\[ \approx F_{xy} + O(ga)^2 \]
Gluon Action: Field Strength Tensor on the Lattice (again: non-matrix case)

Plaquette:

4 adjacent sites with their links

\[
\begin{pmatrix} 0 \\ a \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = U \left[ \begin{pmatrix} 0 \\ a \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right],
\]

Compare two parallel links 1 lattice spacing apart in \((xy)\)-plane:

\[
U[\begin{pmatrix} a \\ 0 \end{pmatrix} \leftarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}] = \exp ig \int_0^a d\tau A_x \begin{pmatrix} \tau \\ 0 \end{pmatrix} \xrightarrow{a \rightarrow 0} 1 + ig a A_x \left( \frac{a}{2} \right) + O(ga)^2
\]

Like Field Strength Tensor

\[
F_{xy} = \partial_x A_y - \partial_y A_x
\]
Gluon Action: Field Strength Tensor on the Lattice (again: non-matrix case)

Plaquette:

4 adjacent sites with their links

Compare two parallel links 1 lattice spacing apart in (xy)-plane:

\[
U\left[ \begin{pmatrix} a \\ 0 \end{pmatrix} \leftarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] = \exp ig \int_0^a d\tau A_x \begin{pmatrix} \tau \\ 0 \end{pmatrix} \leftarrow \begin{pmatrix} a \\ 0 \end{pmatrix} 1 + ig a A_x \begin{pmatrix} a \\ 2 \end{pmatrix} + \mathcal{O}(ga)^2
\]

\[
U\left[ \begin{pmatrix} 0 \\ a \end{pmatrix} \leftarrow \begin{pmatrix} a \\ a \end{pmatrix} \right] = U^\dagger\left[ \begin{pmatrix} a \\ 0 \end{pmatrix} \leftarrow \begin{pmatrix} 0 \\ a \end{pmatrix} \right] \quad \text{reverse integration direction}
\]

\[
= \exp -ig \int_0^a d\tau A_x \begin{pmatrix} \tau \\ a \end{pmatrix} \leftarrow \begin{pmatrix} a \\ a \end{pmatrix} 1 - ig a A_x \begin{pmatrix} a \\ 2 \end{pmatrix} + \mathcal{O}(ga)^2
\]
**Plaquette:**

4 adjacent sites with their links

\[
\begin{array}{cccc}
(0) & U[(0) \leftarrow (a)] & (a) \\
\downarrow & \downarrow & \downarrow \\
(0) & U[(a) \leftarrow (0)] & (0)
\end{array}
\]

Compare two parallel links 1 lattice spacing apart in (xy)-plane:

\[
U[(a) \leftarrow (0)] = \exp \left[ ig \int_0^a d\tau A_x(\tau) \right] \left[ a \to 0 \right] 1 + ig a A_x \left( \frac{a}{2} 0 \right) + O(ga)^2
\]

\[
U[(0) \leftarrow (a)] = U^\dagger[(a) \leftarrow (0)]
\]

reverse integration direction

\[
= \exp \left[ -ig \int_0^a d\tau A_x(\tau) \right] \left[ a \to 0 \right] 1 - ig a A_x \left( \frac{a}{2} a \right) + O(ga)^2
\]

Combine:

\[
U[(0) \leftarrow (a)] U[(a) \leftarrow (0)] = 1 + ig a \left[ A_x \left( \frac{a}{2} 0 \right) - A_x \left( \frac{a}{2} 0 \right) \right] + O(ga)^2
\]

Looks like part of Field Strength Tensor

\[
F_{xy} = \partial_x A_y - \partial_y A_x \quad \left[ a \to 0 \right] - a \partial_y A_x \left( \frac{a}{2} \frac{a}{2} \right)
\]
Gluon Action: Field Strength Tensor on the Lattice (again: non-matrix case)

**Plaquette:**
4 adjacent sites with their links

\[
\begin{array}{c}
(0) \\
\uparrow \\
0 \\
\downarrow \\
(0) \\
\end{array}
\begin{array}{c}
U[(0) \leftrightarrow (a)] \\
\otimes \\
U[(a) \leftrightarrow (0)] \\
\otimes \\
(0) \\
\downarrow \\
(0) \\
\end{array}
\]

Compare two parallel links 1 lattice spacing apart in (xy)-plane:

\[
U[(0) \leftrightarrow (a)] = \exp ig \int_0^a d\tau A_x(\tau) \left. \right|_0^{a/2} 1 + ig a A_x\left(\frac{a}{2}\right) + O(ga^2)
\]

\[
U[(0) \leftrightarrow (a)] = U^\dagger[(a) \leftrightarrow (0)]
\]

\[
= \exp -ig \int_0^a d\tau A_x(\tau) \left. \right|_0^{a/2} 1 - ig a A_x\left(\frac{a}{2}\right) + O(ga^2)
\]

Combine:

\[
U[(0) \leftrightarrow (a)] U[(a) \leftrightarrow (0)] = 1 + ig a \left[ A_x\left(\frac{a}{2}\right) - A_x\left(\frac{a}{2}\right) \right] + O(ga^2)
\]

Looks like part of Field Strength Tensor

\[
F_{xy} = \partial_x A_y - \partial_y A_x \left. \right|_{a/2}^{a/2} - a \partial_y A_x\left(\frac{a}{2}\right)
\]

Combine with the other 2 parallel links for gauge-invariant expression (holds also with matrices):

**Plaquette Variable**

\[
U_\square := U[(0) \leftrightarrow (a)] U[(a) \leftrightarrow (0)] U[(a) \leftrightarrow (0)] U[(0) \leftrightarrow (0)]
\]

\[
= 1 + ig a^2 (\partial_x A_y - \partial_y A_x) + O(ga^2)
\]

\[
= F_{xy}
\]

Looks like Field Strength Tensor
QCD Action on the Lattice

In QCD, $\mathcal{U}[x + ae_\mu \leftarrow x]$, $\mathcal{U} \in SU(3)$, $F_{xy} = \partial_x A_y - \partial_y A_x - ig [A_x, A_y]$ contains self-interactions.

But same relation between $F$ and $\mathcal{U}$ holds:

$$F_{xy} = \frac{1}{ia^2} \left[ \mathcal{U} - 1 \right] + \mathcal{O}(ga)^0$$

Gluon action

$$\int d^4x \mathcal{L}_{\text{glue}} = -\frac{1}{2g^2} \sum \text{tr}[\mathcal{U}^\square]: \sum \text{um over all elementary plaquettes on lattice}$$

→ 0 for $g \to \infty$: strong-coupling limit is easy on lattice! (hard in perturbation: complementing)
QCD Action on the Lattice

In QCD, $U[x + ae_\mu \leftarrow x], U^\square \in SU(3)$, $F_{xy} = \partial_x A_y - \partial_y A_x - ig [A_x, A_y]$ contains self-interactions.

But same relation between $F$ and $U^\square$ holds: $F_{xy} = \frac{1}{ia^2} \frac{1}{g} [U^\square - 1] + O(ga)^0$

Gluon action $\int d^4x \mathcal{L}_{\text{glue}} = -\frac{1}{2g^2} \sum \text{tr}[U^\square]: \sum \text{um over all elementary plaquettes on lattice}$

$\to 0$ for $g \to \infty$: strong-coupling limit is easy on lattice! (hard in perturbation: complementing)

$\implies$ Insert QCD action on lattice into Euclidean Path Integral:

$$ \mathcal{Z} = \int \mathcal{D}[\Psi] \mathcal{D}[\Psi] \mathcal{D}[U] \exp - \sum_{\text{sites,links}} \mathcal{L}[\Psi, \Psi; U, U^\square[U]] $$

$A_\mu$ replaced by matrices $U \in SU(3)$. Plaquette variables $U^\square[U]$ are functions of link variables $U$.

To solve, go to Monte Carlo & play dice for random links and sites on 4-dimensional hypercube:

$\mathcal{Z}$ is partition function of Statistical Mechanics in 4-dimensional space (no time).
Infinitely heavy quarks $\Rightarrow$ action $m_q \bar{q}q \rightarrow \infty \Rightarrow$ quarks classical $\Rightarrow$ quarks not in PI.

$q\bar{q}$ pair on site, drag $\bar{q}$ away $\Rightarrow$ gauge inv. $\bar{q}(y)U(y \leftarrow x)q(x)$, close loop $C(\tau, l)$, annihilate $q\bar{q}$.

Wilson Loop $W[C(\tau, l)] = \prod_{\text{links } i \text{ along } C} U_i$ (index $i = 1, \ldots$ labels links between sites of lattice)

is extension of plaquette $U_{\square}$ to rectangle $C(\tau, l)$ with sides $(\tau, l)$ in lattice units;

represents static $q\bar{q}$ pair at distance $l$ existing for (Euclidean) "time" $\tau$.

Expectation value for energy of static $q\bar{q}$ pair (static loop $C$) (kin.energy $T = 0$) only from gluon action:

$$\langle q\bar{q}(\tau, l)|e^{-H_{\text{IE}}}q\bar{q}(0, l)\rangle = \int \prod_{\text{all links } i} dU_i \left[ \prod_{\text{links } j \text{ along } C} U_j \right] e^{-\left[-\frac{1}{2g^2} \sum_{\text{plaquettes}} \text{tr} U_{\square}\right]} \propto e^{-(T+V)\tau} \overset{T=0}{=} e^{-V(l)\tau}$$
Calculate the Path Integral in the Strong Coupling Limit of $U(1)$

$$\langle q\bar{q}(\tau, l)|e^{-Ht}|q\bar{q}(0, l) \rangle = \int \prod_{\text{all links } i} dU_i \left[ \prod_{\text{links } j \text{ along } C} U_j \right] e^{\frac{1}{2g^2} \sum \text{tr} U} \propto e^{-(T+V)x} \xrightarrow{T=0} e^{-V(l)\tau}$$

$U(1)$ gauge group: each $U_i = e^{i\alpha_i}$ on link $i$ is number on unit circle, with some angle $\alpha_i \in [0; 2\pi[$.

Integration over angles: $\int_0^{2\pi} d\alpha e^{in\alpha} = \delta_{n0}$ for $n \in \mathbb{Z}$ nonzero only if no phase (average on circle is 0).

$$e^{-V(l)\tau} \propto \langle q\bar{q}(\tau, l)|e^{-Ht}|q\bar{q}(0, l) \rangle \propto \int \prod_{\text{all links } i \text{ on lattice}} d\alpha_i \left[ \prod_{\text{only links } j \text{ along } C} e^{i\alpha_j} \right] \exp \left[ \frac{1}{2g^2} \sum_{\text{all links } klmn \text{ of all lattice plaquettes}} e^{i(\alpha_k + \alpha_l + \alpha_m + \alpha_n)} \right]$$

Strong Coupling Limit $g \to \infty$: expand $\exp$ of gluon action:

$$\mathcal{O}(g^0): \int \prod_{\text{all links } i \text{ on lattice}} d\alpha_i \left[ \prod_{\text{only links } j \text{ along } C} e^{i\alpha_j} \right] = 0$$

since for lattice link $i = \text{link } j$ of Wilson loop: $\int_0^{2\pi} d\alpha_i \ e^{i\alpha_i=j} = 0$. 
Calculate the Path Integral in the Strong Coupling Limit of \( U(1) \)

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\[
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\[
e^{-V(l)\tau} \propto \langle \bar{q}q(\tau, l)|e^{-H\tau}|\bar{q}q(0, l)\rangle \propto \int \prod_{i \text{ on lattice}} d\alpha_i \left[ \prod_{\text{only links } j \text{ along } C} e^{i\alpha_j} \right] \exp \left[ \frac{1}{2g^2} \sum_{\text{all links } klmn \text{ of all lattice plaquettes}} e^{i(\alpha_k+\alpha_l+\alpha_m+\alpha_n)} \right]
\]

\[
\mathcal{O}(g^{-2}): \frac{1}{2g^2} \int \prod_{i \text{ on lattice}} d\alpha_i \left[ \prod_{\text{only links } j \text{ along } C} e^{i\alpha_j} \right] \sum_{\text{all links } klmn \text{ of all lattice plaquettes}} e^{i(\alpha_k+\alpha_l+\alpha_m+\alpha_n)}
\]

Match one link \( k \) of plaquette to link \( j \) in \( C \): \( \alpha_j = -\alpha_k \) equal and opposite angles.

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$O(g^{-2})$: $\frac{1}{2g^2} \int \prod_{\text{all links } i \text{ on lattice}} d\alpha_i \left[ \prod_{\text{only links } j \text{ along } C} e^{i\alpha_j} \right] \sum_{\text{all links } klmn \text{ of all lattice plaquettes}} e^{i(\alpha_k + \alpha_l + \alpha_m + \alpha_n)}$

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$\implies$ Need at least $\tau \times l$ terms! $\implies$ First nonzero integral:

$$O \left( \frac{1}{2g^2} \right)^{\tau l} e^{-V(l)\tau} \propto \langle q\bar{q}(\tau, l)|e^{-Ht}|q\bar{q}(0, l)\rangle \propto \left( \frac{1}{2g^2} \right)^{\tau l} = e^{-l \ln[2g^2]} \tau$$
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Linear potential $V(l) = \sigma l$ of $q\bar{q}$ pair for strong $g \to \infty$, $\sigma \propto \ln g^2$: Flux tube/string potential!

$\implies$ Every Gauge Theory has Confinement, in any $d \geq 2$! “Christmas”! Even in $U(1)$ (QED)??

Weak-coupling limit $g \to 0$: the more plaquettes, the more important. $\implies$ Unsolvable.

Lattice QCD “simple” for $g \gg 1$, complicated for $g \ll 1$: complements perturbative QCD.
(g) Very Rough Outline of Lattice “Computations”


**(1) Create “Pure Glue” Ensemble:** Throw dice for values of $U$ at each link, weighted with action in PI.

- **Relaxation/Updating** to equilibrate system ($\rightarrow$ Ising model).
- **Importance Sampling:** prefer configurations with small action.
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Euclidean lattice \( \Rightarrow \) Partition Function in 4 Euclidean dimensions. \( \Rightarrow \) Heavily borrow from Stat. Phys.

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**(2) Throw in appropriate quark sources at \( t_E = 0 \) & \( T_E \):**

e.g. \( (uud) \) with quantum numbers of proton; but does not need to be very accurate representation!

\[
\int \left[ \mathcal{D}\Psi \right] \left[ \mathcal{D}\overline{\Psi} \right] \left[ \mathcal{D}\mathcal{U} \right] \frac{\partial}{\partial \mathcal{U}} \mathcal{S}_{E} \left[ \overline{\Psi}, \Psi, \mathcal{U} \right] \mathcal{Z} \propto \langle uud(T_E) | e^{-H t_E} | uud(t_E = 0) \rangle
\]

\[
\text{insert complete set of phys. states} \quad \sum_n \langle uud | e^{-H t_E} | n \rangle \langle n | uud \rangle
\]

phys. states are eigenstates to \( H_E \) with energies \( E_n \)

\[
\sum_n \langle uud | e^{-E_n t_E} | n \rangle \langle n | uud \rangle \quad \text{let system relax in Euclidean time } t_E:
\]

\[
\propto \sum_n a_n e^{-E_n t_E} \rightarrow e^{-E_0 t_E} \text{ for } 0 \ll t_E \ll T_E
\]

\( \Rightarrow \) At “intermediate times” far away from source/sink \( 0 \ll t_E \ll T_E \), exponential decay guarantees only lowest physical state with quantum numbers of source & sink survives: **Filter out ground state.**

\( \Rightarrow \) Read off energy \( E_0 \equiv M \) mass of lowest state in correlation plots: **“Plateau Plots”**.
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3. **Set Scale:** All quantities in units of \(a\) \(\Longrightarrow\) pick few observables to fix \(a, m_q, \ldots\); then predict rest.
Everything (masses, energies, even input $m_q$) is given in units one dimension-ful quantity: lattice spacing $a$.

$$\lim_{\Delta t_E \to \infty} \langle B\text{-meson}(\Delta t_E) | e^{-H \Delta t_E} | B\text{-meson}(t_E = 0) \rangle \propto e^{-\Delta t_E M_{B\text{-meson}}}$$

![Graphs showing temporal correlation function and effective mass](image)

**Figure 3:** **left:** the temporal correlation function of a $B$ meson creation operator as a function of the temporal separation (taken from [12]); **right:** the effective mass of the kaon as a function of the temporal separation (taken from [13]).
A More Realistic Example – And Some People’s Fantasies

Effective-mass shift $\Delta E = 2M_N - M(\text{deuteron})$ in $32^3 \times 96$ lattice, using lattice units.

Fit-error construction: At least 3 different people use different algorithms to identify plateaus independently, each providing an error estimate. Total error is statistical sum of all.

Watch out for strong correlation of points: same lattice data!

[HALQCD [arXiv:1502.04182v2 [hep-lat]]]
Eff. shift $\Delta E = 4M_N - M(\text{He})$ in $(4.3\,\text{fm})^3 (48^4 \text{ lattice})$, in lattice units.
Quote: “Fit result with one standard deviation error band and total error including the systematic one is expressed by solid and dashed lines, respectively.”
QCD will be solved by Christmas. [Wilson 1974 in seminars] (Hence name “Christmas Paper”.)

1. **Lattice Discretisation Breaks (Euclidean) Rotational Invariance**

   \[ \Rightarrow \] No angular momentum conservation or partial waves, but can use discrete cubic symmetries.

   Conceptually largely under control but can be numerically quite expensive; gets better as \( a \to 0 \).
A Few Selected Problems in Lattice QCD

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2. **Need to Fit Physics into Box** i.e. “have all meson clouds inside box”

   Lightest hadron: $m_\pi = 140\text{MeV} \implies \lambda_\pi^{\text{Compton}} \sim \frac{\pi}{m_\pi} \sim 4\text{fm}$,

   but resolution $a \ll \frac{1}{1\text{GeV}} \sim 0.1\text{fm}$.

   $\implies$ Need $L \gtrsim 4\text{fm}$, $a \ll 0.1\text{fm}$, $N^4 \gg (40)^4 \approx 2.5\text{M}$ quite expensive.

   **Way out:** Compute at larger $m_\pi \implies$ smaller $L$; then extrapolate to physical $m_\pi$ using model-independent, well-understood $m_\pi$-dependence of **Chiral Effective Field Theory**. $\rightarrow$ next section
(h) A Few Selected Problems in Lattice QCD

(3) Light Quarks Computationally Extremely Expensive  Big problem for pion Physics from lattice.

Small energy/action penalty $e^{-S_E[m_q]} \quad \Longrightarrow \quad$ easy to create light $q\bar{q}$ pairs in vacuum fluctuations.

Such quark-pair vacuum fluctuations are called “disconnected”: lines not connected to sources (cf. valence quarks). But they are still moving in gluon background: These are not Feynman diagrams!

Historically, such “disconnected” quark lines were just dropped: Quenching now only for tests.
(4) Can Only “Measure” Exponential Decays & Static Observables

\[ e^{-H_E t_E} \implies \text{static statistical system in 4 spatial dimensions} \]

\[ \implies \text{Scattering phase shifts only by induced energy-shift (Lüscher’s method } \rightarrow \text{ later).} \]

Especially challenging for shallow bound states, like deuteron:

\[ M_{\text{deuteron}} = M_p + M_n - E_B \approx [2000 - 2]\text{MeV} \implies e^{-(2M_N - E_B) t_E} \longleftrightarrow e^{-2M_N t_E} : 0.1\%\text{-effect!} \]

\[ \implies \text{Need to measure tiny energy difference to two-free-nucleons!} \]

Terrible signal-to-noise for Nuclear systems.
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(7) Insatiable Hunger for Bigger and Faster Computers

In my view, learn about Nature by combining lattice, perturbative and \( \chi \) EFT:

well-defined, systematically improvable descriptions of QCD, estimating theory uncertainties, in complementing and partially overlapping energy regimes.
(i) Very Few (Even More Selected) Lattice Results

Extrapolation to Physical Masses

Use known low-energy Nuclear Physics (Chiral EFT) to cut down on computational cost.

Not just a linear extrapolation!

[Duerr et al. *Science* (2008)]
Static Potential between Infinitely Heavy Quarks (Quarkonium)

Infinitely heavy \(\implies\) no recoil \(\implies\) no retardation or colour radiation \(\implies\) Potential makes sense.

\[ [V(r) - V(r_0)] \times r_0 \]

\[ \overline{\Xi}: \kappa_{\text{sea}} = 0.1370 \]
\[ \Phi: \quad \kappa_{\text{sea}} = 0.1380 \]
\[ \overline{\Xi}: \quad \kappa_{\text{sea}} = 0.1390 \]
\[ \Xi: \quad \kappa_{\text{sea}} = 0.1395 \]
\[ \Lambda: \quad \kappa_{\text{sea}} = 0.1398 \]

Appears quite linear.
Energy Density: Flux Tube for a Heavy Meson [Leinweber et al. 2003, click here for homepage]
Action Density: Flux Tube for a Heavy Baryon

[Leinweber et al. 2003, click here for homepage]
Action Density: “Pure-Glue” Vacuum Fluctuations

Leinweber et al. 2003, click here for homepage
QCD Precision Spectroscopy: Quarkonia & Heavy-Light Mesons

[PDG 2013 Fig. 14.8]

η_c, ψ', ψ set scales of \( m_c, m_b, \alpha_s(Q_0^2) \)

\[ \begin{align*}
\eta_c, \psi', \psi & \text{ set scales of } m_c, m_b, \alpha_s(Q_0^2) \\
\eta_b, B_c & \\
B_s, B_s^* & \\
\eta_b, J/\psi & \\
\eta_c, \chi_{c0}, \chi_{c1}, \chi_{c2}, h_c, \chi_{b0}, \chi_{b1}(1P), \chi_{b2}(1P), \chi_{b2}(2P), \chi_{b2}(2P) & \\
\text{expt, \ 'fix parameters, postdictions, predictions} & \\
\end{align*} \]
Approaching physical pion masses, good accuracy.

[Duerr et al. Science (2008), from PDG 2013 Fig. 14.7]
Colour-neutral bound states of glue are *unique signal* of Non-Abelian Gauge Theories.

**Glueballs:** Any state dominated by glue. In particular when glue dictates quantum numbers.

Discovery would allow direct test of QCD – Does not exist in Constituent Quark model!

**Problem:** Light-quark admixture

⇒ **Lattice computation:**

Bad signal-to-noise, quark loops give huge corrections!

**GlueX:** Glueball search in Hall D is major motivation for JLab 12GeV-upgrade. Unique experimental signal difficult.

[“Quenched” computation: no disconnected quark lines.]

Hadron Polarisabilities: GW Leads Connecting Data & QCD

GW focus

Needs to be phrased as energy-difference: \( \Delta E = -2\pi \alpha^{(N)}_E \bar{E}^2 \).

Neither Approach Uses The Other To Fit!

[lattice: Lujan/Alexandru/Freeman/Lee [arXiv:1411.0047 [hep-lat]]; chiral extrapolation: hgie/McGovern/Phillips [arXiv:1511.01952 [nucl-th]]; Downie/Feldman take data at HI\gamma S, MAMI, ...]
Problem: Lattice QCD gave up time-dependence by rotation to Euclidean time.

Solution: can still “feel” interactions in finite volume (cf. \( t \)-independent scattering theory)

e.g. \( \pi\pi \) scattering: compute \( \Delta E = E - 2m_\pi = 2\sqrt{k_n^2 + m_\pi^2} - 2m_\pi \implies \) get \( k_n \), insert into

\[
\text{Lüscher’s formula } k_n \cot \delta(k_n) = \frac{1}{\pi L} \lim_{\Lambda \to \infty} \sum_{j} \frac{1}{\sqrt{j^2 - (k_n L/2\pi)^2}} - 4\pi\Lambda \text{ (with error bars!)}
\]

Valid for: below first inelasticity, \( L \gg \text{interaction range } r_0 \sim \frac{1}{m_\pi} \)

but can have scatt. length \( a \sim L \)

Many extensions available and being worked on:
3-body, box with different lengths, coupled channels, etc.

[Döring/Mai/..., hg...]

\( NN \) scattering at \( m_\pi = 805\text{MeV} \)  

[NPLQCD PRC88 (2013) 124003]
\( \pi \pi \) phase shifts identify \( \rho \) resonance; unphysical \( m_\pi = 316 \text{MeV} > m_\pi^{\text{phys}} = 140 \text{MeV} \).
Merger of EFT and lattice has started exploring how few-nucleon systems emerge from QCD.

Surprisingly little change in few-nucleon systems – but \( nn \) becomes bound when \( m_\pi \) increased!

[J. Kirscher [arXiv:1509.07697 [hep-lat]] (got his PhD in GW’s EFT group)]
Next: 4. Pions and 0, 1, 2,... Nucleons

Familiarise yourself with: [(Goldstone: CL 5; Ryd 8.1-3);
CL 5; Ryd 8.1-2; Ber 2, 3;
Scherer/Schindler: Primer $\chi$ EFT;
Ericson/Weise: Pions and Nuclei Chap. 9—see me!]