Hadronic Reactions (Part 3)

- Principles of 2-body scattering
  - Unitarity,
  - crossing symmetry,
  - causality,
  - analyticity
- Partial wave decomposition
- Applications: Pion-pion scattering, Photoproduction
- Nuclear reactions in a small box (Dr. Raquel Molina)
- Project work: photoproduction at low energies: Multipole decomposition of recent data from the MAMI facility (Mainz, Germany): Combining statistical techniques, Feynman diagrams, and real-world experiment
Introduction and Motivation


### Elementary particles of the Standard Model

#### Leptons
- \( e \) (electron), mass 0.511 MeV, \( \nu_e \), mass < 0.000003 MeV,
- \( \mu \) (muon), mass 106 MeV, \( \nu_\mu \), mass < 0.2 MeV,
- \( \tau \) (tau), mass 1777 MeV, \( \nu_\tau \), mass < 20 MeV,

#### Quarks (each in 3 “colors”)
- \( d \), mass 7 GeV, \( u \), mass 3 GeV,
- \( s \), mass 120 GeV, \( c \), mass 1200 GeV,
- \( b \), mass 4300 GeV, \( t \), mass 175,000 GeV,

- Charge: -1, 0, -1/3, 2/3

#### Particles like the electron
- Fermions, spin 1/2

#### Particles like the photon
- Bosons, spin 1
- \( \gamma \) (photon), mass 0 GeV
- \( g \) (gluon), mass 0 (8 “colors”)
- \( W^\pm \) (weak interaction), mass 80.420 GeV, \( Z^0 \) (weak interaction), mass 91.188 GeV

(Gravity is negligible.)
Q: How many quarks or gluons have ever been directly observed?

A: 0 (zero)

Q: The mass of a down quark is 7 MeV and that of an up quark is 3 MeV. Then, the mass of the proton \((uud)\) should be \(m_P \sim 13\) MeV, right?

A: \(m_P = 938.272\) MeV

It is obviously a long way from our “periodic table” of quarks and gluons to matter and its properties as we know them.
Even these 4% not well understood

- Dark Energy: 74%
- Everything else, including all stars, planets, and us: 4%
- Dark Matter: 22%
$e^+ e^- \rightarrow q\bar{q} \rightarrow $ many particles
Quark-gluon interaction: QCD

Remember from Mechanics:
\[ L = T - V \]
describes your physical system \([T: \text{Kinetic energy}, V: \text{potential energy}]\)

\[
\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f \left[ D_\mu \gamma_\mu + m_f \right] \psi_f + \frac{1}{4} \sum_a G^a_{\mu\nu} G^{a\mu\nu}
\]

\[ \text{anti-quark} \quad \bullet \quad \text{quark} \]

\[ \text{gluons} \]

\[ \text{e.g.:} \quad G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu \]

One fundamental coupling strength \( \alpha_S = \frac{g^2}{4\pi} \ll 1. \)

For small \( \alpha_S \), one can solve QCD in a controlled way (perturbation theory).
The running coupling

\[ \alpha_s(Q)/\pi \]

\[ \text{JLab CLAS} \]
\[ \text{JLab PLB 650 4 244} \]
\[ \alpha_{s,gl}/\pi \text{ world data} \]
\[ \alpha_{s,F}/\pi \]
\[ \text{GDH limit} \]
\[ \text{pQCD evol. eq.} \]
\[ \alpha_{s,\pi}/\pi \text{ OPAL} \]

\[ \leftarrow \text{Energy decreases} \]

Deur, Burkert, Chen, Korsch, PLB665 (2008)

No easy solution of QCD at lower energies!
The full complexity: Parton shower and hadronization

Only colorless final states $\leftrightarrow$ confinement

Electroweak Interaction

Strong Interaction; QCD regime

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Lattice QCD for hadrons

- Simulate the complexity of QCD at low energies with the help of supercomputers
- Ab-initio approach: QCD → hadron masses
- Discretization in space and time, in a finite volume, to make the problem numerically treatable
The baryon spectrum: $N^*$ and $\Delta$ resonances

- Many resonances predicted in lattice calculations

$m_\pi = 396 \, MeV$ (!)
- Search for these states in dedicated experimental programs
Photoproduction experiments: Jefferson Lab, MAMI, ELSA, ...
Photoproduction cross sections

\[ \sigma / \mu b \]

\[ E [\text{GeV}] \]

\[ \gamma + p \rightarrow X \]
\[ \gamma + p \rightarrow p + \pi^- + \]
\[ \gamma + p \rightarrow p + \pi^0 + \]
\[ \gamma + p \rightarrow K^+ + \Lambda \]
\[ \gamma + p \rightarrow p + \eta \]

[Data: JLab, ELSA, MAMI]
Partial wave analysis from many observables: $\gamma N \rightarrow \pi N$

Differential cross section $\gamma p \rightarrow n\pi^+$

- Photon: Spin 1
- Nucleon: Spin $\frac{1}{2}$
- Single, double, triple polarization observables
- Order principle in the chaos:
  Conserved quantum numbers, e.g.,

$J^P$: Total angular momentum with Parity

Partial wave analysis

Decompose experimental data with respect to conserved quantum numbers.
Resonances have a certain, conserved $J^P$. 

[CLAS measurements, PRC 79 (2009); Solid (dashed) lines: SAID (MAID) analysis; filled: CLAS, triangles: MAMI]
Symmetries of the strong interaction

- Electric charge $Q$
- Baryon charge (baryon number conservation)
  
  $+1$ for $p, n, \Lambda, \Sigma, \Xi, \ldots$
  
  $-1$ for anti-particles
  
  $0$ for mesons $(\pi, K, \rho, \omega, \varphi, \ldots)$
- Isotopic spin (isospin) $I$ approximately
  
  \[
  \frac{m_n - m_p}{m_p} \sim \frac{m_{\pi^0} - m_{\pi^+}}{m_{\pi^+}} \sim \alpha \approx \frac{1}{137}
  \]

- Strangeness $S$: Lambda's and Kaons always produced together:
  
  $\pi^- + p \rightarrow \Lambda + \bar{K}^0$

  but never observed: $\pi^- + p \rightarrow n + K^0$, or $\pi^- + p \rightarrow \Lambda + \pi^0$

  Gell-Mann-Nishijima
  
  \[Q = I_3 + \frac{B}{2} + \frac{S}{2} \]
Nuclear effective force – finite range

\[ D_\pi(q) = \frac{1}{\mu^2 - q^2} \]

in analogy to

\[ \frac{g^2}{q^2} \left( \bar{u}\gamma^\mu u \right) \left( \bar{u}\gamma_\mu u \right) \]

\[ 1/q^2 \]

→ Scattering amplitude:

\[ A = \frac{g^2}{\mu^2 - q^2} \]

What is this in the non-relativistic picture?

\[ f = -\frac{2m}{4\pi} \int e^{i\mathbf{k}\cdot\mathbf{r}} V(r) \psi(r) \, d^3r \quad \text{Born approximation} \quad \rightarrow \quad f_B = -\frac{2m}{4\pi} \int e^{i\mathbf{q}\cdot\mathbf{r}} V(r) \, d^3r \]

\[ \mathbf{q} \text{ is the momentum transfer, } \mathbf{q} = \mathbf{k}' - \mathbf{k} \]

\[ E = \frac{k^2}{2m} \]

\[ |q_0| \sim \frac{q^2}{m} \ll |\mathbf{q}|, \quad \text{so that} \quad q^2 = q_0^2 - q^2 \sim -q^2. \quad \rightarrow \quad A \sim \frac{g^2}{\mu^2 + q^2} \cdot e^{-\mu r} \]

position space:

\[ V(r) = -\frac{4\pi}{2m} \int e^{-i\mathbf{r}\cdot\mathbf{q}} \frac{g^2}{\mu^2 + q^2} \frac{d^3q}{(2\pi)^3} = \frac{g^2}{2m} \cdot e^{-\mu r} \]

→ Effective interaction characterized by a finite radius \( r_0 = 1/\mu \) EVEN with point-like interactions
The S-matrix

- Transition from some initial state $a$ to some final state $b$:
  \[ S = I + i \mathcal{T}; \quad S_{ab} = \delta_{ab} + i \mathcal{T}_{ab} \]

- With incoming particle $i$ and outgoing particles $j$:
  \[ T_{ab} = (2\pi)^4 \delta^4 \left( \sum_{i \in a} p_i - \sum_{j \in b} k_j \right) \prod_{i \in a} \frac{1}{\sqrt{2p_{0i}}} \prod_{j \in b} \frac{1}{\sqrt{2k_{0j}}} \cdot \mathcal{M}_{ab} \]
  
  - energy momentum conservation
  - wave function renormalization: factors of $1/\sqrt{2p_0}$
  - “$T$”-matrix and Lorentz invariant amplitude $\mathcal{M}_{ab}$

- Reaction probability: square $T$!
\[ d\sigma(a \rightarrow b) \equiv \frac{1}{j} |\mathcal{M}_{ab}|^2 (2\pi)^4 \delta^4 \left( p_1 + p_2 - \sum_{j \in b} k_j \right) \cdot \frac{1}{[n!]} \prod_{j \in b} d\Gamma(k_j) \]

\[ d\Gamma_j = \left( \frac{1}{\sqrt{2k_{0j}}} \right)^2 \frac{d^3k_j}{(2\pi)^3} = \frac{d^3k_j}{2(2\pi)^3 k_{0j}} = \frac{d^4k_j}{(2\pi)^4} \cdot 2\pi \delta_+ (k_j^2 - m_j^2) \]

• Lorentz invariant flux \( J = 4p_c(s) \sqrt{s} \)

• 2 particles incoming, \( p_1 = -p_2 = (0, 0, p_c) \)

\[ p_c = p_c(s) = \sqrt{(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)} \]

\[ \frac{1}{[n!]} = \prod_s \frac{1}{n_s!}, \quad \sum_s n_s = n \]

• Eliminate multiple counting of physically indistinguishable configuration produced by permutation of identical particles; If in the final state there are \( n_s \) particles of type \( s \),

This slide is merely for your information and not derived in detail. More details: Gribov, Sec. 1.7, Peskin Schroeder Sec. 4.5

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Unitarity

- The S-matrix is unitary (additional explications on blackboard):
  \[ S S^\dagger = 1 \implies T_{ab} - T_{ab}^\dagger = i \left( T T^\dagger \right)_{ab} \]

- In matrix notation:
  \[ \frac{1}{i} (T_{ab} - T_{ba}^*) = \sum_c T_{ac} T_{cb}^* \]

  Intermediate states \( c \): Sum over all possible quantum numbers, momenta, and even particle species
  \[ \rightarrow \text{Concept of coupled channels (blackboard)} \]

- Time reversal invariance: \( T_{ab} = T_{ba} \)
  \[ \frac{1}{i} (T_{ab} - T_{ab}^*) = 2 \text{Im} T_{ab} = \sum_c T_{ac} T_{bc}^* \]
A typical coupled-channel problem

- Pion-nucleon scattering

Table 11. Angular momentum structure of the coupled channels in isospin $I = 1/2$ up to $J = 9/2$. The $I = 3/2$ sector is similar up to obvious isospin selection rules.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$J^P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi N$</td>
</tr>
<tr>
<td>2</td>
<td>$\rho N (S = 1/2)$</td>
</tr>
<tr>
<td>3</td>
<td>$\rho N (S = 3/2,</td>
</tr>
<tr>
<td>4</td>
<td>$\rho N (S = 3/2,</td>
</tr>
<tr>
<td>5</td>
<td>$\eta N$</td>
</tr>
<tr>
<td>6</td>
<td>$\pi \Delta (</td>
</tr>
<tr>
<td>7</td>
<td>$\pi \Delta (</td>
</tr>
<tr>
<td>8</td>
<td>$\sigma N$</td>
</tr>
<tr>
<td>9</td>
<td>$K \Lambda$</td>
</tr>
<tr>
<td>10</td>
<td>$K \Sigma$</td>
</tr>
</tbody>
</table>

[D. Ronchen, M. Doring et al., EPJA 49, 44 (2013)]
2--> 2 Scattering and the Mandelstam plane

- On-mass-shell external particles:
  \[ p_i^2 = m_i^2 \]

- Two independent kinematical variables (e.g., scattering angle and energy):
  Three four-vectors (12 components)
  Four on-mass-shell conditions
  Three rotations and three Lorentz boosts. Altogether: 12-4-3-3=2

Two independent kinematic variables to characterize the invariant amplitude in 2--> 2 scattering

- Similarly: for a 2 → 3 process, 5 independent kinematic variables
Mandelstam variables

- Characterize kinematics through Mandelstam variables:

\[
\begin{align*}
  s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\
  t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\
  u &= (p_1 - p_4)^2 = (p_2 - p_3)^2
\end{align*}
\]

- Using on-mass-shell condition, one gets immediately

\[
s + t + u = \sum_{i=1}^{4} m_i^2
\]

- Can be visualized in the Mandelstam plane:
Meaning of Mandelstam variables

- Choose center-of-mass (cm) frame: \( p_1 + p_2 = 0 \)

\[
s = (p_{1\mu} + p_{2\mu})^2 \equiv (p_{10} + p_{20})^2 - (p_1 + p_2)^2 = (E_{1c} + E_{2c})^2 = E_c^2
\]

Square of the energy of total energy of colliding particles

- Express \( t \) and \( u \) through scattering angle:

\[
t = (p_{3\mu} - p_{1\mu})^2 \equiv (E_3 - E_1)^2 - (p_3 - p_1)^2
\]

\[
= (E_3 - E_1)^2 - (p_3 - p_1)^2 - 2p_1 p_3 (1 - \cos \Theta)
\]

\[
\cos \Theta = \frac{p_1 \cdot p_3}{|p_1| |p_3|}
\]

\[
p_i = |p_i|
\]

\[
p_1 = p_2 = p_c \quad p_3 = p_4 = p'_c
\]
(continued)

- Set all masses equal (eg: pion-pion scattering). Then:

\[ t = -2p_c^2(1 - \cos \Theta_c) \quad u = -2p_c^2(1 + \cos \Theta_c) \]

- Physical scattering amplitude is complex (see unitarity). Can we see this from a Feynman diagram? → Indeed, we have just calculated the imaginary part of the two-particle propagator.