Problem Sheet 14

Special Due date: 10 December 2018 08:00

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. Why should I have to invent the problems? (**9P**): This time, you invent the problem – and its solution. Find a charge distribution on a hollow sphere of radius R such that only the total charge and the spherical multipoles with $l = 2$, $|m| = 1$ contribute. Provide a complete and concrete solution for the surface charge density ρ and the electrostatic potential outside the sphere, *in Cartesian coordinates*. Is your choice unique?

   Example of a similar problem: When $\rho(\vec{r}) = \dfrac{3xq}{4\pi R^3}\,\delta(R-r)$ with q some quantity with the dimension of a charge, then only $l = 1$, $|m| = 1$ contributes, the net charge is zero, and $\Phi(\vec{r}) = \dfrac{qxR}{(x^2+y^2+z^2)^3}$ for $r > R$ (if I did not mess up factors).

2. Even More Complex Integrations (**9P**): You can check your final results with an algebraic manipulation programme. If you use contour integration with neglecting an arc at infinity, ***discuss in detail that your function vanishes indeed on that arc.***

   a) (**1P**) Determine $\displaystyle\int\limits_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{e}^{\mathrm{i}kx}}{x+\mathrm{i}}$, $k > 0$.

   b) (**3P**) Determine $\displaystyle\int\limits_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{e}^{-\mathrm{i}ax}}{x^4+6x^2+8}$ for $a > 0$ and for $a < 0$.

   c) (**5P**) The following integral appears when one integrates the time-averaged power radiated into a unit solid angle at (θ,ϕ) from a charge which oscillates relativistically with frequency ω or time-dependent velocity $-l\omega\sin\omega t$. You will need it next semester. Show that

   $$\frac{1}{2\pi}\int\limits_0^{2\pi} \mathrm{d}\alpha\, \frac{\cos^2\alpha}{(1+a\sin\alpha)^5} = \frac{1}{8}\,\frac{4+a^2}{(1-a^2)^{\frac{7}{2}}}\ ,\ \text{where in Physics } \alpha = \omega t \text{ and } a = \frac{l\omega}{c}\cos\theta\ ,\ |a| < 1\ .$$

Please turn over.

3. Branch Points (**7P**) Given the function $(z-a)^{1/n}$, $n \in \mathbb{N} \setminus \{0;1\}$, $a \in \mathbb{R}$.

   a) (**2P**) Discuss under which conditions it is well-defined/analytic in the complex plane: If the phase of $z-a$ is defined in the interval $\theta \in]-\pi;\pi[$, locate the position of the branch cut and determine by how much the answers differ if one evaluates the function on either side of the branch cut.

   b) (**1P**) How many Riemann sheets do you need to represent the function?

   c) (**4P**) Calculate the integral of this function along a contour which encircles m times the branch point a and interpret your result.

4. Principal Value Integrals (**4P**)

   a) (**2P**) Find the principal value of $\displaystyle\int\limits_{-\infty}^{\infty} \mathrm{d}x \; \frac{\sin x}{(x^2+4)(x-1)}$.

   b) (**2P**) Determine $\displaystyle\int\limits_{-\infty}^{\infty} \mathrm{d}x \; \left(\frac{\sin x}{x}\right)^2$.

5. The Yukawa Potential, Again (**6P**) We come back to problem 10.3: If the photon had a mass m, the Poisson equation of Electrostatics would read:

$$\left[\Delta - m^2\right] \Phi(\vec{r}) = -4\pi \, \rho(\vec{r})$$

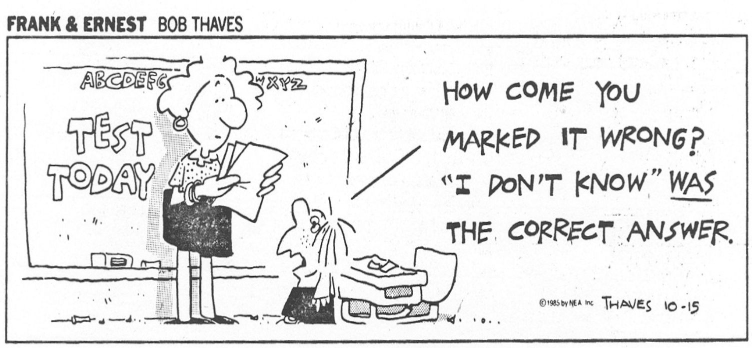
   Now, we calculate the Green's function for this problem via a Fourier transform.

   a) (**1P**) Construct the Green's function in momentum space: $\tilde{G}(\vec{k}) = \alpha \, \dfrac{1}{\vec{k}^2 + m^2}$ (determine $\alpha \in \mathbb{R}$).

   b) (**2P**) To alleviate the inverse Fourier transform to coordinate space, show: When $\tilde{f}(k)$ only depends on the magnitude of $\vec{k}$ but not on its angle, then in three dimensions:

$$f(r) = \gamma \int\limits_0^{\infty} \mathrm{d}k \; \frac{k}{r} \; \sin kr \; f(k) \quad , \quad \text{and determine the real number } \gamma.$$

   c) (**3P**) Find $G(\vec{r}, \vec{r}\,')$ in coordinate space using contour integration. You know the result.



Question of the Week (bonus 3P): How much wood would a woodchuck chuck if a woodchuck would chuck wood?