Problem Sheet 13

Due date: 05 December 2018 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

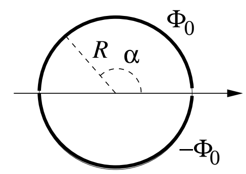
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.

I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.

News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. Repeating the Lecture (**6P**): For this, you can very closely follow the example in class. A ball with radius R holds a charge density $\rho(\vec{r}) = \theta(R-r)\; W\,(\vec{n}\cdot\vec{r})^2$. Determine W such that the total charge is Q, and find all spherical charge multipoles and the scalar potential $\Phi(r,\theta,\phi)$ outside the ball.

2. A Guillotine'd Cylinder (**8P**): An infinitely long, infinitesimally thin cylinder of radius R is separated into halves by cutting it along the cylinder axis. The potentials on the halves are constant and opposite, Φ_0 and $-\Phi_0$, see figure. Let's neglect any effects very close to the cuts.



   a) (**4P**) Determine the elementary solution of the Laplace equation $\Delta\Phi = 0$ in cylindrical coordinates. Show that the ansatz $\Phi(r,\phi,z) = U(r)\;\chi(\phi)$ leads to a separation of variables:

$$\frac{r}{U(r)}\;\frac{\partial}{\partial r}\;r\;\frac{\partial}{\partial r}U(r) = -\frac{1}{\chi(\phi)}\;\frac{\partial^2}{\partial\phi^2}\chi(\phi) = \mu^2$$

   Pay particular attention to the elementary solution for the value $\mu = 0$ of the separation constant, i.e. construct it and then show that is irrelevant for the problem at hand.

   b) (**4P**) Construct the potential *inside and outside* the cylinder. The solution is an infinite series. Discuss where the solution converges well, and where it does not.

   Note: The problem has actually an exact solution (and if you find it, you are very, very good at sums – but that's not a part of the problem):

$$\Phi(r,\alpha) = \frac{2\Phi_0}{\pi}\;\arctan\left[\frac{2a}{1-a^2}\;\sin\alpha\right] \quad \text{with} \quad a = \begin{cases} \frac{r}{R} & \text{for } r < R \\ \frac{R}{r} & \text{for } r > R \end{cases},$$

   where r,α are the polar coordinates when the cylinder is centred at the origin, and the cut separating the halves is along the x-axis.

Please turn over.

3. Complex Functions (**7P**): Let $z = x + \mathrm{i}y$ be complex, with x (y) its real (imaginary) part.

   a) (**1P**) Decompose the following complex function into real and imaginary part, $u + \mathrm{i}v$: $\dfrac{1+z^2}{\mathrm{i}-z^2}$

   b) (**2P**) Let $u(x, y) = 2x(1-y)$ be the real part of an analytic function $f(z)$. Construct its imaginary part $v(x, y)$.

   c) (**4P**) Decompose $\ln z$ into its real and imaginary part, $u + \mathrm{i}v$. Determine whether it obeys the Cauchy-Riemann condition $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -\partial v/\partial x$ everywhere, or at nearly all points (and if so, state at which it does not). Determine the nature of each of its singularities.

   Hint: Sometimes, it helps to derive the form of u and v using the detour via the polar representation of a complex number, $z = |z|\,\mathrm{e}^{\mathrm{i}\phi}$. But that is a question of taste.

4. Complex Integration (**4P**):

   a) (**2P**) Evaluate the following integrals, with $\mathcal{C}$ the circle $|z| = 3$:

$$\oint\limits_{\mathcal{C}} \frac{\mathrm{d}z}{z^3(z+5)} \quad , \quad \oint\limits_{\mathcal{C}} \mathrm{d}z \, \frac{\mathrm{e}^z}{(z-1)(z-2)}$$

   b) (**2P**) Turn the following integral into a contour integral around the unit circle and evaluate:

$$\int\limits_0^{2\pi} \mathrm{d}\vartheta \, \frac{\sin^2\vartheta}{a+\cos\vartheta} \quad , \quad a > 1$$

5. More Complex Integrations (**5P**): You can check your final results with an algebraic manipulation programme. If you use contour integration with neglecting an arc at infinity, ***discuss in detail that your function vanishes indeed on that arc.***

   a) (**2P**) Evaluate the integral $\displaystyle\int\limits_0^\infty \mathrm{d}x \, \frac{2x^2+1}{x^4+5x^2+6}$

   b) (**3P**) Evaluate the integral $\displaystyle\int\limits_0^\infty \frac{\mathrm{d}x}{x^4+4x^2+4}$