Problem Sheet 10

Due date: 7 November 2018 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. DIRAC'S δ-DISTRIBUTION (CONTINUED) (**8P**): Prove the following properties. Where applicable, show that the proposition holds when multiplied with any square-integrable, suitable test function $f(x)$ and integrated over all space.

   a) (**3P**) It seems odd that we can construct $\delta(x)$ as limit of functions: Didn't we require that the space of functions be complete, i.e. that every Cauchy sequence converges to a function in the Hilbert space of functions? Show that your favourite sequence of "true" functions whose limit is the δ-distribution is *not* a Cauchy-sequence. Why does this imply that $[\delta(x)]^2$ does not exist?

   b) (**2P**) Derive a representation of HEAVISIDE'S STEP-"FUNCTION" as limit of a sequence of functions using its definition $\theta(x) := \int\limits_{-\infty}^{x} \mathrm{d}y\ \delta(y) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$ and the "rectangle-representation" of the sequence whose limit is the δ-distribution.

   c) (**1P**) Calculate the "derivative of the δ-distribution", $\int\limits_{-\infty}^{\infty} \mathrm{d}x\ f(x)\ \frac{\mathrm{d}}{\mathrm{d}x}\delta(x)$.

   d) (**2P**) Derive from the Cartesian version expressions for Dirac's δ-distribution in cylindrical and spherical coordinates. It may be useful to recall that $\int \mathrm{d}^3 r\ \ \delta^{(3)}(\vec{r}) = 1$, whatever coordinate system you use.

2. PRACTISING WITH DIRAC'S δ-DISTRIBUTION (**3P**):

   a) (**2P**) Show that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}|x| = 2\,\delta(x)$ in the "distributional sense".

   b) (**1P**) Watching the integration limits, calculate:
   $$\int\limits_{0}^{\infty} \mathrm{d}x\ \delta(x^2 - x - 2)\ \mathrm{e}^{-x^2}\ \left(1 - \cos\frac{5\pi x}{2}\right)$$

3. PROPERTIES OF FOURIER TRANSFORMS (**7P**): Prove for functions $f(x)$, $g(x)$ with Fourier transforms $F(\omega)$, $G(\omega)$ and $\mathcal{N}$ the normalisation convention of the Fourier transform:

   a) FALTUNG/CONVOLUTION THEOREM (**2P**): $\mathcal{N}\int \mathrm{d}y\ g(y)\ f(x-y) = \int \frac{\mathrm{d}\omega}{2\pi\,\mathcal{N}}\,F(\omega)\,G(\omega)\,\mathrm{e}^{+\mathrm{i}\omega x}$.

   b) (**1P**) The Fourier transform of a real function $f(x)$ obeys $F(-\omega) = F^*(\omega^*)$, ω *complex.*

   c) (**2P**) The Fourier transform of an even function $f(x) = f(-x)$ is even. When $f(x)$ is also real, then so is its Fourier transform.

   d) (**2P**) As we expect to recover Fourier series from Fourier integrals, proving the following result is very informative: When $f(x)$ is periodic with period L, then its Fourier transform is zero, except when $kL = 2\pi n$, where $n \in \mathbb{Z}$ is an integer. Can you write this using Dirac's δ-distribution?

Please turn over.

4. Practising with Fourier Transforms (**5P**):

   a) (**2P**) Find the Fourier transforms of $\mathrm{e}^{-a|x|}$ and $\dfrac{1}{a^2+x^2}$, where $\mathrm{Re}[a] > 0$.

   b) (**2P**) Construct the solution to $\vec{\nabla}^2 G(\vec{r}) = \delta^{(d)}(\vec{r})$ in momentum space. Does $\tilde{G}(\vec{k})$ depend on the number of dimensions d you consider?

   c) (**1P**) Combine this result with the result of 2a above to calculate the Fourier transform of $1/k^2$ in one dimension.

5. Helmholtz' Fundamental Theorem (**3P**) Fill in the dots in the Fourier back-transformation in our "proof" of Helmholtz' Fundamental Theorem, i.e. calculate $\mathcal{F}^{-1}$ of:

$$\Phi(\vec{k}) = \mathrm{i}\frac{\vec{k}\cdot\vec{A}(\vec{k})}{|\vec{k}|^2} \quad , \quad \vec{a}(\vec{k}) = -\mathrm{i}\frac{\vec{k}\times\vec{A}(\vec{k})}{|\vec{k}|^2}$$

6. Hermitean Operator Theory in Quantum Mechanics (**4P**): Consider now the radial part of the Laplace operator in three dimensional, spherical coordinates: $\dfrac{1}{r}\dfrac{\partial^2}{\partial r^2}r$. It appears, for example, in the Schrödinger equation for the hydrogen atom. For bound states, we obviously want that the wave-function f is finite everywhere and disappears for large distances (it's called "bound state", after all): $f(r=0)$ finite, $f(r=\infty) = f'(r=\infty) = 0$. Is the operator Hermitean on these functions? Is it self-adjoint? Discuss in particular functions which diverge as $r\to 0$, but for which $\lim\limits_{r\to 0} r^n\, g(r) = 0$ for suitable $n\in\mathbb{N}$.

   Caveat: Recall $\int \mathrm{d}^3r \;= \int \mathrm{d}r\;\mathrm{d}\cos\theta\;\mathrm{d}\phi\; r^2$.

Question of the Week (bonus 3P): Mammals in the arctic are pretty big. Explain this by comparing their heat production (proportional to the number of cells the animal has) to their heat loss.