Problem Sheet 9

Due date: 31 October 2018 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. HYPERBOLIC COORDINATES, CONTINUED (**5P**): Return to the hyperbolic coordinates of the last HW.
   a) (**1P**) Determine the invariant volume element in these coordinates.
   b) (**4P**) Show that the most general function $\psi(u)$ which *only* depends on u and which satisfies LAPLACE'S EQUATION $\Delta\psi(u) = 0$ is
   $$\psi(u) = A + B \arctan \mathrm{e}^u \;\;, \text{ where } A, B \text{ are constants.}$$

2. INERTIA TENSOR (**5P**): With the least amount of effort, i.e. using the irreducible-tensor decomposition for rank-2 tensors, calculate the inertia tensor for the non-isotropic mass density $\rho(\vec{r}) = (\vec{r}\cdot\vec{p})^2 + \vec{r}^2\,\vec{p}^2$ inside a sphere of radius R, where $\vec{p}$ is a fixed vector. Outside the sphere, the mass density is zero. How many independent components does this particular inertia tensor have?

3. INTEGRATION THEOREMS (**7P**): The following problems are independent of each other.
   a) SOURCES AND CURLS (**3P**): Take an arbitrary vector field $\vec{A}$ without sources (or sinks) and an arbitrary scalar field Φ with Laplacean zero. Find the sources and curls of $\vec{A}\times\vec{\nabla}\Phi$.
   b) HELMHOLTZ' FUNDAMENTAL THEOREM (**4P**): Prove Helmholtz' theorem directly by applying "div" and "rot" to
   $$\vec{A}(\vec{r}) = \frac{1}{4\pi}\int \mathrm{d}^3r' \left[-\vec{\partial}_r\,\frac{\vec{\partial}_{r'}\cdot\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \vec{\partial}_r \times \frac{\vec{\partial}_{r'}\times\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] \;.$$

4. A STRANGE VECTOR POTENTIAL (**6P**): Consider the vector potential $\vec{A}(\vec{x}) = \vec{e}_\varphi\,\dfrac{\Phi}{2\pi r}$ in cylindrical coordinates (r,ϕ,z), where Φ is some constant. You can without fear of reprimand copy the form of differential operators in these coordinates from textbooks; they are orthogonal curvilinear.
   a) (**3P**) Calculate and discuss direction and strength of the magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$ and the flux it describes through a disk of arbitrary radius in the $(z=0)$-plane. As in the lecture for $\Delta\frac{1}{r}$, pay particular attention to those points where $\vec{A}$ is not well-defined, using Stokes' theorem.
   b) (**3P**) Under which condition can $\vec{A}$ be written as gradient of a scalar function which is well-defined (i.e. single-valued and finite) everywhere?

5. TENSORS AND PSEUDO-TENSORS (**3P**): We saw that the magnetic field $\vec{B}$ is a rank-1 pseudo tensor.
   a) (**1P**) The electric charge density $\rho(\vec{r})$ is obviously invariant under coordinate transformations, and also looks the same in a mirror. Using Gauß' law $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$, show that the electric field $\vec{E}$ must be a rank-one (polar) tensor, not a pseudo/axial tensor.
   b) (**2P**) A point charge q which moves with speed $\vec{v}$ is subject to the electro-magnetic LORENTZ FORCE $\vec{F}_\mathrm{L} = q[\vec{E} + \vec{v}\times\vec{B}]$. Determine whether $\vec{F}_\mathrm{L}$ is a "real" or pseudo-tensor, or a mixture, and find its rank.

Please turn over.

6. Dirac's δ-Distribution (**4P**): Prove the following properties. Where applicable, show that the proposition holds when multiplied with any square-integrable, suitable test function $f(x)$ and integrated over all space.

   a) (**2P**) $\delta(ax) = \dfrac{1}{|a|}\delta(x)$. Use the representation of the δ-distribution as limit of your favourite sequence of "true" functions.

   b) (**2P**) $\delta\left(f(x)\right) = \sum\limits_{i=1}^{N} \dfrac{\delta(x-x_i)}{|\frac{\mathrm{d}f}{\mathrm{d}x}(x_i)|}$ when $f(x)$ has only simple zeroes at $x = x_1, \dots, x_N$. You may use "heuristic" arguments by treating the distribution "like" a function, and assume that the Taylor expansion of $f(x)$ exists around all zeros x_i. Which conditions should $f(x)$ obey?

Question of the Week (bonus 3P): Estimate the weight of an Airbus A380.

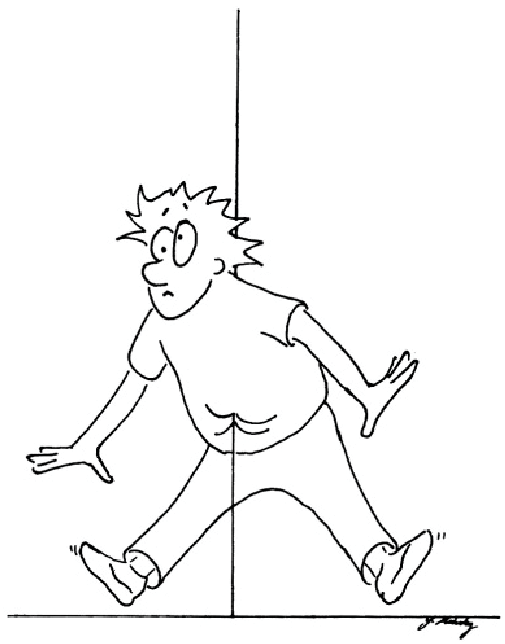

Figure P.1

Hapless Physicist Impaled on his own Delta Function
(Demonstrating the Perils of Insufficient Theoretical Rigor)