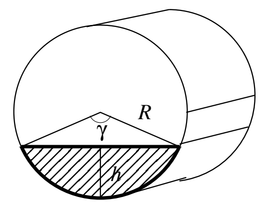
Problem Sheet 4

Due date: 26 September 2018 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. Cartalk (**6P**): "Cartalk" is a cult-NPR show about automotive problems. On 6 November 2010 (show #1045, `http://www.cartalk.com/ct/review/show.jsp?showid=201045`; plus follow-up next show), a caller faced Click and Clack with: "The fuel gauge of my gas tank is broken, but I can measure the height of the fluid with a stick. The tank is a lying cylinder. I know when half the volume is filled with gas – that is half the cylinder height. But what stickmark corresponds to the tank being $\frac{1}{4}$ filled?" The hosts winged that they would need "advanced calculus" for that. Prove them wrong.

   a) (**1P**) Show that in order to determine the $\frac{1}{4}$-full-mark in units of the cylinder radius, $\frac{h}{R} = 1 - \cos\frac{\gamma}{2}$, one must solve a transcendental equation for the opening angle γ in the figure:
   $$\gamma - \sin\gamma = \frac{\pi}{2} \quad .$$



   b) (**5P**) We now construct a perturbative solution, $\gamma = \gamma_0 + \epsilon\gamma_1 + \ldots$, where the formal expansion parameter ϵ is set to 1 at the end. A good start may be $\gamma_0 = \frac{2\pi}{3}$ [why?] and re-writing [why?]:
   $$\frac{\pi}{2} = \Delta_0 + \epsilon\Delta_1 = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) + \epsilon\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) \quad .$$
   Find the first correction and provide an error-assessment.

2. Scalar Product for Matrices (**5P**): The scalar product can come in surprising disguises. Given two arbitrary, quadratic matrices M and N with the same dimensions, which do not have to be symmetric, Hermitean or even diagonalisable. Show: The operation
   $$\langle M, N\rangle := \mathrm{tr}[M^\dagger N]$$
   defines a scalar product and is positive definite. For the latter, it helps to recall that a matrix consists of rows of column vectors (or vice versa).

3. Rayleigh-Ritz Variational Method (**7P**): The Hamilton operator of the harmonic oscillator is in appropriate units
   $$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 \quad .$$

   a) (**5P**) By varying the parameter c in the trial function
   $$\phi_0(x) = \begin{cases} (c^2 - x^2)^2 & \text{for } |x| < c \\ 0 & \text{for } |x| \geq c \end{cases} \quad ,$$
   obtain an upper bound for the ground-state energy and wave-function. Notice that ϕ_0 is not yet normalised and that it vanishes outside the interval $[-c; c]$, which makes the integrals easier.

   b) (**2P**) Compare to the exact result, also by calculating the "overlap" between the trial and exact wave-function. Interpret the overlap as the amount by which the two wave functions are "aligned" and provide a number for the "angle" between the two wave functions in the abstract space of functions. You may not be able to do the integral exactly.

Please turn over.

4. SOAP BUBBLES: THE CATENOID (**12P**): This is a special case of a "Classic", found in every textbook. But be careful to use *my* conventions for labelling variables, and not those of the book you get inspired from. Work through your own solution, especially when it comes to numerics and plots. No copying! Pay particular attention to plots! The numerical and plotting parts can be done with an algebraic manipulation programme or a small fortran or C code.

   Two rings with the same radius R are located concentric to the z-axis and parallel to the xy-plane, at distance $2L$ from each other and equi-distant from the origin. A thin film of soap stretches between them. Because of their surface tension, soap films want to minimise the area they cover. We now describe the form of this "minimal surface".

   a) (**2P**) Assume for brevity that you can parameterise the minimal surface by a monotonous function $z(x)$ plus rotational invariance. Show that for rings of different radii R_1 and R_2, the functional is

   $$J[z(x)] = \int\limits_{R_1}^{R^2} dx\; 2\pi\, x \sqrt{1 + (z'(x))^2} \quad .$$

   Show by solving an appropriate differential equation that one can invert $z(x)$ to obtain $x(z) = c \cosh \frac{z-b}{c}$, where cosh is the hyperbolic cosine. Derive an (implicit) equation for the constants b, c from the boundary conditions with $R_1 = R_2 = R$. Finally, show that the implicit equation can be recast using $\eta = L/R$ and $\xi = L/c$, as:

   $$\frac{\xi}{\eta} = \cosh\xi \quad .$$

   b) (**3P**) Now we solve this equation graphically. Sketch the left-hand and right-hand side as function of ξ, in the same graph. How many solutions (locally minimal surfaces) do you get for "large" values of η, and how many for "small" ones? From your sketch, estimate the "critical" value η_{crit} at which the number of solutions changes dramatically. Do this without switching on a computer.

   c) (**3P**) Sketch profiles of the surfaces of revolution for a "large" and "small" η, and for η_{crit}.

   d) (**1P**) For η_{crit}, the line ξ/η_{crit} is exactly tangential to the curve $\cosh\eta$, i.e. one finds exactly one solution. Show that this implies

   $$\xi(\eta_{\text{crit}}) \tanh\xi(\eta_{\text{crit}}) = 1 \quad \text{with} \quad \eta_{\text{crit}} = \frac{1}{\sinh\xi(\eta_{\text{crit}})} \quad .$$

   e) (**3P**) Solve this equation ($\xi \tanh\xi = 1$) for ξ to $\sim 4\%$ accuracy by perturbation about $\xi_0 = 1$.

Question of the Week (bonus 3P): How many cows provide the milk for Starbucks, daily?