Problem Sheet 3

Due date: 21 September 2016 16:00

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.

I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.


1. ASYMPTOTICS OF THE ERROR FUNCTION (6P): As you know, the Gauss’ian distribution plays a prominent rôle in Statistics. The “complementary error function”

\[ \text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty dt \ e^{-t^2} \]

is the probability that an event which follows a Gauss’ian distribution has a value between \( x \) and \( \infty \).

a) (5P) Derive that for large \( x \to \infty \), the integral can be asymptotically approximated as

\[ \text{erfc}(x) = e^{-x^2} \frac{1}{\sqrt{\pi}x} \sum_{n=0}^m (-1)^n \frac{(2n-1)!!}{2^n x^{2n}} + R_m(x) , \]

where \( (2n-1)!! = 1 \times 3 \times 5 \cdots \times (2n-1) \). Determine the remainder as an integral and estimate its size.

b) (1P) Where is its “sweet-spot”?

2. PERTURBATION THEORY FOR MATRICES (8P): Consider the matrix of example (ii) in the lecture, split into un-perturbed and perturbed parts as:

\[ \begin{pmatrix} 7.5 & 0.4 & -0.7 \\ 0.4 & -3.1 & 0.1 \\ -0.7 & 0.1 & 2.6 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0.5 & 0.4 & -0.7 \\ 0.4 & -0.1 & 0.1 \\ -0.7 & 0.1 & 0.6 \end{pmatrix} \]

a) (4P) Perturbatively find two of the eigenvalues and eigenvectors to next-to-leading order (NLO). Treat corrections to one of the eigenvectors in detail, i.e. show the matrix multiplications involved. For the other one, you may sketch the way and only state the result. Of course, you should do this by hand, as practise in matrix multiplications. If you do, you have notes for all calculations anyway.

b) (2P) Perturbatively find two eigenvalues at next-to-next-to-leading order (N^2LO). Again, treat corrections to one of the eigenvalues in detail, i.e. show the matrix multiplications involved. For the other one, you may sketch the way and only state the result.

c) (2P) Provide a defendable estimate for each eigenvalue and eigenvector you calculated, including an error-bar.

Please turn over.
3. AN-HARMONIC OSCILLATOR (10P): A pendulum is released at rest from an amplitude $A$. Its motion is described by the non-linear equation of motion
\[ \ddot{x} + \Omega^2 x - \eta x^2 = 0. \]

a) (3P) You see that $\eta$ is not a dimension-less parameter, so we cannot pretend that $\eta \ll 1$. By considering the corresponding potential, derive a criterion for $\eta$ such that perturbation theory is useful. What is the dimension-less parameter which can be assumed to be small?

b) (7P) Assuming your parameter in a) is indeed small, solve for the oscillation pattern $x(t)$ to first order perturbation theory. Is it necessary to adjust the frequency of the leading-order solution? Give a reason!

4. PERIHELION SHIFT OF MERCURY, AGAIN (6P): Including the first post-Newtonian correction, the equation of motion of the distance $r$ of a mass $m$ as function of time $t$ in the gravitational field of a much heavier body $M \gg m$ is:
\[ \ddot{r}(t) = \frac{l^2}{m^2 r^3} - \frac{G_N M}{r^2} + \frac{6G_N^2 M^2}{c^2 r^3}, \]

where $l$ is the system’s angular momentum (cf. centrifugal barrier). We surmised in HW II.3 $\beta \sim \frac{G_N M}{r}$, now we know $\beta = \frac{3G_N M}{r}$ – so a tideous non-relativistic reduction only provides the factor “3”.

You will learn in the Mechanics lectures that one performs a change of variables from $r(t)$ to $u(\theta)$, where $\theta$ parametrises the planet’s orbit. The result is an equation which we need to solve by approximation:
\[ \frac{d^2 u}{d\theta^2} + \omega_0^2 u(\theta) = \frac{G_N M m^2}{l^2} - \eta u(\theta) \]

with $\eta := \frac{6G_N^2 M^2 m^2}{l^2 c^2} \ll 1$ the small parameter. The problem starts now.

a) (1P) Show: For $\eta = 0$ and with $\varepsilon$ the eccentricity of the orbit, the equation is solved by
\[ u_0(\theta) = \frac{G_N M m^2}{l^2} [1 + \varepsilon \cos \omega_0 \theta] \quad \text{when} \quad \omega_0^2 = 1. \]

Notice that $\omega_0$ is not really a frequency because $\theta$ does not have dimensions of time. We are however free to choose $\omega_0^2 = 1$, so that $\theta$ plays the rôle of an angle: $\cos \omega_0 \theta = 1$ for $\theta = 0, 2\pi, 4\pi, \ldots$.

b) (3P) Now expand the full equation to first order in $\eta$ (re-adjust “frequency”!):
\[ u = u_0 + \eta u_1 + O(\eta^2) \quad \text{and} \quad \omega^2 = \omega_0^2 + \eta \omega_1^2 + O(\eta^2) \]

Fix $\omega_1^2$ to cancel any problematic terms. Do not solve the equation of motion completely.

c) (2P) Consider finally one complete period with the perturbation. You see that $\omega$ is changed, so maxima of $\cos \omega \theta$ are not $2\pi$ apart. By how much does the distance $\Delta \theta$ of two maxima differ from the unperturbed case? That difference is called the perihelion shift. By recycling last week’s result, compare the perihelion shift of Mercury with LEVERRIER’S observation, $43''$ per century.

Alternatively, you can use: solar mass $M = 2.0 \times 10^{30}$ kg, Mercury mass $m = 3.3 \times 10^{23}$ kg, $l = 9.15 \times 10^{38}$ kg m$^2$ s$^{-1}$ for the Mercury-Sun system, Mercury’s rotation period $0.24885$ years.