Supplement on Groups

Def. Group ($\mathcal{G}$,$\circ$): a set $\mathcal{G}$ with binary operator "$\circ$" such that

(G1) "$\circ$": $\mathcal{G} \circ \mathcal{G} \to \mathcal{G}$, i.e. $\forall g_1, g_2 \in \mathcal{G}: \; g_1 \circ g_2 \in \mathcal{G}$ — closure under group-operation

(G2) $\forall \, g_1, g_2, g_3 \in \mathcal{G}: \; g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$ — associativity

(G3) $\exists \; \mathrm{id} \equiv \mathrm{Id} \equiv 1 \equiv e \equiv E \equiv \dots \in \mathcal{G}: \; \mathrm{id} \circ g = g \; \forall g \in \mathcal{G}$ — identity/unit element

(G4) $\forall \, g \in \mathcal{G} \; \exists \, g^{-1} \in \mathcal{G}: \; g \circ g^{-1} = \mathrm{id}$ — inverse

Def. Abelian/Commutative Group: $g_1 \circ g_2 = g_2 \circ g_1 \; \forall \, g_1, g_2 \in \mathcal{G}$, i.e. all elements commute.

Def. Non-Abelian/Non-Commutative Group: a group which is not Abelian.

Def. Finite/Discrete Group of Order N: group with $N < \infty$ elements.

Def. Subgroup of $\mathcal{G}$: a set of elements of $\mathcal{G}$ which form a group by themselves.

Def. Representation: map $\mathcal{D} : (\mathcal{G}, \circ) \to (G, *)$ preserves group structure: $\mathcal{D}(g_1 \circ g_2) = \mathcal{D}(g_1) * \mathcal{D}(g_2) \; \forall g_1, g_2 \in \mathcal{G}$. $\mathcal{D}$ acts on vector space $\mathbb{M}$, dimension of a representation $\mathrm{dim}\mathcal{D} = \mathrm{dim}\mathbb{M}$.

Def. Linear Rep.: rep. on the set of invertible $n \times n$ matrices, $* \equiv$ matrix multiplications.

Def. Trivial Rep.: $\forall g \in \mathcal{G}: \; \mathcal{D}(g) = \mathrm{id}$: Al elements mapped into unity. Exists for every group.

Def. Faithful Rep.: $\mathcal{D}$ is bijective (i.e. 1-to-1 and onto): invertible iso-morphism.

Def. Fundamental Rep.: $\mathcal{D}$ is faithful and defines $\mathcal{G}$.

Def. Reducible Rep.: A linear rep. for which one can find one matrix S which simultaneously brings all elements of $\mathcal{G}$ into block-diagonal form. Then, $\mathcal{D}$ is the direct sum of invariant subspaces.

Def. Irreducible Rep./Irrep: $\mathcal{D}$ has no invariant subspaces except $\mathbb{M}$: no proper block-diagonal structure.

Def. Unitary Rep.: $\forall g \in \mathcal{G}: \; \mathcal{D}(g^{-1}) = \mathcal{D}^\dagger(g)$.

Def. Lie/Continuous Group (Version I): A group with at least one continuous parameter.

Def. Lie/Continuous Group (Version II): A group which is also a manifold (closed, smooth hyper-surface without boundaries), parameterised by at least one continuous coordinate.

Def. Dimension of a Lie Group: $\mathrm{dim}\mathcal{G}$ = dim hyper-surface = number of independent continuous parameters = number of coordinates necessary to specify a point on $\mathcal{G}$.

Def. Compact Group: Volume of manifold is finite. Equivalent: Every $g \in \mathcal{G}$ is bounded.

Def. Connected Component of $\mathcal{G}$: All g for which a path $g(t)$ exists, parameterised by t, such that $g(0) = \mathrm{id}$, $g(t=1) = g$, and $\forall t \in [0;1]: \; g(t) \in \mathcal{G}$; i.e. g and id can be joined by a path entirely in $\mathcal{G}$.

Def. Lie Bracket: $\forall x, y, z \in L[\mathcal{G}]$ and $\alpha, \beta \in \mathbb{C}$:

(1) $[x, y] \in$ tangent space — closure

(2) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$ — bilinear

(3) $[x, y] = -[y, x]$ — anti-symmetric

(4) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ — Jacobi-/Bianchi-identity

Def. Lie Algebra $L[\mathcal{G}]$ (Version I): Tangent space of $\mathcal{G}$ in id, with "Lie Bracket". $\mathrm{dim}L[\mathcal{G}] = \mathrm{dim}\mathcal{G}$

Def. Lie Algebra $L[\mathcal{G}]$ (Version II): vector space L with "Lie Bracket" $L \times L \to L$: $[.,.] : (x, y) \mapsto \mathrm{i}[x, y]$.

Def. Basis of $L[\mathcal{G}]$/Generators of $\mathcal{G}$: a CONS $\{t^a\}$, $a = 1, \dots, \mathrm{dim}L[\mathcal{G}]$ which spans L, ortho-normalised by matrix scalar product $2\mathrm{tr}[t^a t_b] = \delta^a_b$.

Exp-Map $\exp : L[\mathcal{G}] \to \mathcal{G}: \; X \in L[\mathcal{G}] \mapsto g \in \mathcal{G}$: $g = \exp \mathrm{i}X$.
Generates all g in connected component of $\mathcal{G}$; bijective locally around id, but not globally.