Supplement on Pauli Spin Operators (Matrices) and the ϵ-Tensor

Einstein's Summation Convention EΣC: If an index appears exactly once as superscript and exactly once as subscript, then sum over this index. The index takes the values necessary in the given context.

Often, we are sloppy to write sub- and super-scripts: "Softened" EΣC: Sum over exactly 2 repeated indices.
Examples: $\vec{A}$ has **column/contra-variant components** A^a; **row/co-variant/dual components** A_a. Scalar product: $\vec{A}\circ\vec{B} = \vec{A}^T\vec{B} = A_aB^a = A^aB_a = \vec{B}^T\vec{A} = \vec{B}\circ\vec{A}$. Multiplication of two matrices: $(MN)^a{}_b = M^a{}_c\,N^c{}_b$. Transpose matrix: $(M^a{}_b)^T = M^b{}_a$. Trace: $\mathrm{tr}[M] = M^a{}_a$. Shorthand: $(A^a)^2 := A^a\,A_a$.
Not defined: $A^aA_aA^a$, and many other.

Definition totally anti-symmetric pseudo/axial unit tensor or rank 3/Levi-Civitá symbol:

$$\epsilon^{abc} := \begin{cases} 1 & \text{for } (abc) \text{ even permutation of } (123) \\ -1 & \text{for } (abc) \text{ odd permutation of } (123) \\ 0 & \text{otherwise (i.e. when 2 indices identical)} \end{cases} \tag{1}$$

Properties: $$\epsilon^{ab}{}_e\,\epsilon^{cde} = \delta^{ac}\,\delta^{bd} - \delta^{ad}\,\delta^{bc} \quad \text{with } \textbf{Kronecker-}\boldsymbol{\delta}\text{: } \delta^{ab} := \begin{cases} 1 & \text{for } a=b \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\text{for vectors } \vec{A},\,\vec{B},\,\vec{C}\text{:} \qquad \epsilon^{abc}\,A_b\,B_c = \left(\vec{A}\times\vec{B}\right)^a \quad , \quad \epsilon^{abc}\,A_a\,B_b\,C_c = \vec{A}\circ\left(\vec{B}\times\vec{C}\right) = \det(\vec{A}\,\vec{B}\,\vec{C}) \tag{3}$$

Definition Pauli spin operators $\boldsymbol{\sigma^a}$: Any set of 3 operators with the properties $(a = 1,2,3)$

$$\text{(i) Commutation relations: } \left[\sigma^a,\sigma^b\right] = 2\mathrm{i}\,\epsilon^{ab}{}_c\,\sigma^c \tag{4}$$

$$\text{(ii) Anti-commutation relations: } \left\{\sigma^a,\sigma^b\right\} = 2\,\delta^{ab} \tag{5}$$

Representation by matrices which generate all Hermitean, traceless 2×2 matrices with complex entries:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

This is hardly ever used. All relations ever necessary can be derived from the abstract definition above.

Properties of the Pauli operators derived from their definition:

$$\left(\sigma^1\right)^2 = \left(\sigma^2\right)^2 = \left(\sigma^3\right)^2 = 1 \quad , \quad \text{or } \ (\sigma^a)^2 = 1 \quad \text{(no sum over } a\text{!)} \tag{7}$$

$$(\sigma^a)^\dagger = (\sigma^a)^{-1} = \sigma^a \quad \text{(Hermitean, its own inverse)} \tag{8}$$

$$\mathrm{tr}[\sigma^a] = 0 \quad \text{(traceless)} \tag{9}$$

$$\det\sigma^a = 1 \quad \text{(unit determinant)} \tag{10}$$

$$\text{Orthogonality:} \qquad \mathrm{tr}[\sigma^a\sigma^b] = 2\,\delta^{ab} \tag{11}$$

$$\text{Completeness/Closure:} \qquad X = X_a\,\sigma^a \Longleftrightarrow \frac{1}{2}\mathrm{tr}[X\sigma^a] = X^a \tag{12}$$

The Pauli operators generate the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$ for $X^a\in\mathbb{C}$, and that of $\mathrm{SU}(2)$ for $X^a\in\mathbb{R}$. Relation to any element of the Lie groups $\mathrm{SL}(2,\mathbb{C})$ for $X^a\in\mathbb{C}$ and $\mathrm{SU}(2)$ for $X^a\in\mathbb{R}$:

$$\exp[\mathrm{i}X_a\frac{\sigma^a}{2}] = 1\,\cos\frac{X}{2} + \mathrm{i}\,\frac{X_a\sigma^a}{X}\sin\frac{X}{2} \quad , \quad \text{where } X = \sqrt{|X^aX_a|} \text{ is the "length of the vector" } X \tag{13}$$

$$U = b^0 + b_a\,\sigma^a \in \mathrm{SU}(2) \iff \left(b^0\right)^2 + b_ab^a = 1 \quad , \quad \text{i.e. } \mathrm{SU}(2)\simeq S^3 \tag{14}$$

$$\text{Mapping SU(2) onto S0(3): } R^{ab}[U] = \frac{1}{2}\,\mathrm{tr}[U^\dagger\sigma^aU\sigma^b] \in \mathrm{S0}(3) \ \ \forall U\in\mathrm{SU}(2) \iff \mathrm{S0}(3)\simeq\mathrm{SU}(2)/\mathbb{Z}_2 \tag{15}$$

$$\sigma^a\,\sigma^b = \delta^{ab} + \mathrm{i}\,\epsilon^{ab}{}_c\,\sigma^c \tag{16}$$

$$\left(\vec{A}\circ\vec{\sigma}\right)\left(\vec{B}\circ\vec{\sigma}\right) = \vec{A}\circ\vec{B} + \mathrm{i}\,\vec{\sigma}\circ\left(\vec{A}\times\vec{B}\right) \tag{17}$$

where $\vec{A}$, $\vec{B}$ are ordinary 3-dimensional vectors or commuting operators.