Supplement: Comparison between Dirac's $\langle$Bra$|$Ket$\rangle$ Notation and Einstein's Σummation Convention

Quantity	**Dirac's $\langle$Bra$\|$Ket$\rangle$ Notation**	**Einstein Σummation Convention**
abstract vector in vector space $\mathbb{M}$	$\|x\rangle$ ket $\Updownarrow$ clear difference	x"="$\vec{x}$ column vector $\Updownarrow$ no clear distinction
abstract vector in dual space $\overline{\mathbb{M}}$	$\langle x\|$ bra	x"="$\vec{x}^T$ row vector (when x real; otherwise "="$\vec{x}^\dagger$)
basis of $\mathbb{M}$	$\|i\rangle$, $i=1,\dots,\dim\mathbb{M}$ no distinction betw. basis & vectors kets can be basis vectors	e_i, $i=1,\dots,\dim\mathbb{M}$ clear separation of vector components and basis
basis of $\overline{\mathbb{M}}$	$\langle i\|$, $i=1,\dots,\dim\overline{\mathbb{M}}$	ϵ^i, $i=1,\dots,\dim\overline{\mathbb{M}}$
ortho-normality	$\langle i\|j\rangle=\delta_{ij}\ \forall i,j=1,\dots,\dim\mathbb{M}$ if $\overline{\mathbb{M}}\simeq\mathbb{M}$	$\langle\epsilon^i\|e_j\rangle=\delta^i_j$
closure/completeness	$\sum\limits_i \|i\rangle\langle i\|=\mathrm{id}$ "inserting unity", used often	$\epsilon^i\otimes e_i=1$ tensor product, used infrequently
scalar product $\langle y,x\rangle$ of $x\in\mathbb{M}$ and $y\in\overline{\mathbb{M}}$	$\langle y\|x\rangle=(\langle x\|y\rangle)^*$ usually complex	$y_i\,x^i=y^i\,x_i$ usually real
metric $g_{ij}=$	???	$\langle e_i,e_j\rangle$
contravariant components	$\langle i\|x\rangle$	x^i
covariant/dual components	$\langle x\|i\rangle$	x_i
relation between the two	$\langle x\|i\rangle=(\langle i\|x\rangle)^*$	$x_i=g_{ij}\,x^j$
basis change	$\langle n\|x\rangle=\sum\limits_j\langle n\|j\rangle\,\langle j\|x\rangle$	$\tilde{x}^i=d^i{}_j\,x^j$
components of matrix M	$\langle i\|M\|j\rangle$	$M^i{}_j$ (index ordering important)
abstract matrix $M=$ from components	$\sum\limits_{ij}\|i\rangle\,\langle i\|M\|j\rangle\,\langle j\|$	$e_i\,M^i{}_j\,\epsilon^j$
Further typical features	prefers abstract vectors e.g. "states" in QM can deal with infinite-dim. space and continuous basis can deal with complex components non-orthogonal bases hard to describe order matters: $\|i\rangle\,\langle j\|\neq\langle j\|i\rangle$ no summation, e.g. projector on eigenstate: $\|i\rangle\,\langle i\|\neq\sum\limits_i\|i\rangle\,\langle i\|=\mathrm{id}$ ambiguous operator action $\langle y\|Mx\rangle\overset{?}{=}\langle My\|x\rangle\overset{?}{=}\langle y\|M\|x\rangle$ e.g. selfadjoint vs. Hermitean	prefers concrete representations e.g. locally orthog. coord., GRT prefers finite-dim. space prefers real components made for non-orthogonal bases & metrics components commute: $x_i\,y^j=y^j\,x_i$ summation "mandatory" operator action cannot be addressed