Additional Practise Sheet: Group Theory

Completely voluntary.

If you want, we can discuss your solutions in the **Final Question Time of the semster**.
No extra points are awarded – the values are only meant as grade of difficulty here.

1. GROUP OR NO GROUP (**3P**): Consider the following sub-sets of the group of invertible, real 3×3 matrices $M \in GL(3, \mathbb{R})$. Do these sets form groups under matrix multiplications? If yes, what is their identity element?

   a) (**1P**) The set of matrices which obeys $M^T = M$.

   b) (**2P**) The set of matrices which obeys $(M)_{ij} = 0$ for $i < j$ ("upper-triangular matrices").

2. AN ABSTRACT LIE ALGEBRA (**3P**): Show that the infinite-dimensional set of all first-order differential operators $\mathcal{L}[f] := f(x)\frac{\mathrm{d}}{\mathrm{d}x}$ forms an abstract Lie algebra, when the Lie bracket is the commutator $(\mathcal{L}[f_1]\mathcal{L}[f_2] - \mathcal{L}[f_2]\mathcal{L}[f_1])g(x)$ and $f(x), g(x)$ are arbitrary, infinitely often differentiable functions. Notice that $\mathcal{L}$ contains a derivative which acts on *all* functions to the right of it.

3. (**2P**) Show $\det M = \exp \operatorname{tr} \ln M$ when M is diagonalisable.

4. CAMPBELL-BAKER-HAUSDORFF FORMULA (**3P**) Assuming that X, Y are infinitesimal and elements of a Lie algebra, derive the displayed terms of the formula which relates products of exponentials to sms of Lie algebra elements:

$$\exp X \, \exp Y = \exp[X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \dots]$$

5. (**2P**) If $[A, B] = B$, calculate $\exp[\mathrm{i}\alpha A]\, B\, \exp[-\mathrm{i}\alpha A]$ with α commuting with A and B, e.g. a number. This relation is often used in QM.

6. (**1P**) Derive that *not all* elements of the Lorentz group can be written in the above form, i.e. with only positive elements on the diagonal. As an intermediate step, you could show that the determinant of the Lorentz transform is only constrained by $|\det \Lambda| = 1$, or re-examine your proof in a).