Additional Practise Sheet; Green's Functions and PDEs **Completely voluntary.**

If you want, we can discuss your solutions in the **Final Question Time of the semster**.
No extra points are awarded – the values are only meant as grade of difficulty here.

1. FROBENIUS' METHOD (**6P**): We will find a solution to the differential equation

$$4x\, f''(x) + 2(1-x) f'(x) - f(x) = 0 \text{ by Frobenius' ansatz } f(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n \quad a_0 \neq 0 \ .$$

   a) (**2P**) Show that the indicial equation dictates that $\alpha = 0$ or $\alpha = \frac{1}{2}$.
   b) (**4P**) For the case $\alpha = 0$, derive the recurrence relation for the coefficients a_j and construct the closed answer.

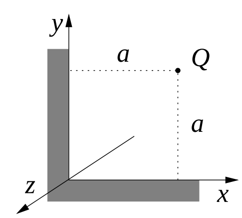
   Note: The case $\alpha = \frac{1}{2}$ is straight-forward and therefore boring. Do not do it.

2. A TWO-DIMENSIONAL GREEN'S FUNCTION (**4P per correct, independent way**) Show that the Green's function to the *two*-dimensional Poisson equation "without boundaries at infinity" is

$$G(\vec{r}, \vec{r}') = \alpha \ln \frac{|\vec{r} - \vec{r}'|}{C} \quad ,$$

   where C is an arbitrary constant. Determine the constant α. There are several ways to do this problem, given the toolchest developed in the last semester. **Each correct way gives** 4 **points.**

   You might want to recall also some Green's functions of one- and three-dimensional "empty" space.



3. IMAGE CHARGES AND GREEN'S FUNCTION FOR PLATES AT AN ANGLE (**4P**) A point charge q is located at $\vec{r}_0 = (a, a, 0)$ in front of two infinitely long, perfectly conducting, grounded plates which meet at the origin in an angle of 90°; see figure for details. The problem is three-dimensional. Use the method of image charges (you need 3 images).

   a) (**2P**) Determine the potential everywhere.
   b) (**2P**) Derive the induced surface charge density on the plates at $z = 0$. Sketch!

4. SPHERICAL HARMONICS (**7P**):

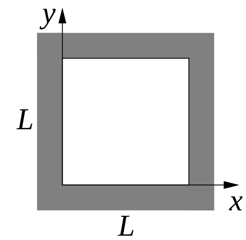
   a) (**2P**) Determine the parity of $Y_{lm}(\theta, \phi)$.
   b) (**3P**) Inspired by a good book, prove the addition theorem for spherical harmonics:

$$P_l(\cos\alpha) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega') = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}(\alpha, 0) \ ,$$

   where α is the angle between (θ, ϕ) and (θ', ϕ'), i.e. $\cos\alpha = \cos\theta\ \cos\theta' + \sin\theta\ \sin\theta'\ \cos(\phi - \phi')$.
   c) (**2P**) Construct the first three Legendre polynomials from the generating function.

Please turn over.

5. SQUARE WITH BOUNDARY CONDITIONS IN ELECTROSTATICS (**8P**): The sides of a square (length of sides L) are made of some material such that the boundary conditions on the potential are



$$\Phi(x,y) = \begin{cases} \Phi_0 \sin \frac{3\pi x}{L} & \text{on the upper side, } y = L \\ 0 & \text{on all other sides.} \end{cases} \quad .$$

The problem is two-dimensional. There are no charges inside the square; see figure.

a) (**1P**) Are these boundary condition of the Dirichlet or von-Neumann type?

b) (**5P**) Determine the potential $\Phi(x,y)$ everywhere inside the square.

Hint: Show that you need to solve differential equations (you need to determine the constant of separation α):

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = -\alpha^2 \, X(x) \ , \ \frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = \alpha^2 \, Y(y) \ .$$

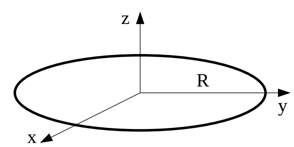
with the boundary conditions $X(x=0) = 0 = X(x=L)$ and $Y(y=0) = 0$.

Fail-safe point: $\Phi(x,y) \propto \sin \frac{3\pi x}{L} \left[\mathrm{e}^{\frac{3\pi y}{L}} - \mathrm{e}^{\frac{-3\pi y}{L}} \right]$.

c) (**2P**) Using a pillbox construction, determine the charge density on the lower plate, $y = 0$.

6. LEGENDRE POLYNOMIALS (**2P**) Prove that $\int\limits_{-1}^{1} \mathrm{d}x \; P_l(x) = 0$ for l a positive integer.

7. ELECTROSTATIC MULTIPOLE MOMENTS (**7P**): An infinitesimally thin, conducting, circular ring of radius R carries the homogeneous line charge density μ. It is centred at the origin in the xy plane, see figure.



Hints: Determine all spherical multipole moments with respect to the origin.
If multipole moments vanish, you can substitute a calculation by a short argument.

Recall that $Y_{l0}(\theta,\phi) \propto P_l(\cos\theta)$, so that some results can also be written using Legendre polynomials at a given value, which you need to look up.

a) (**1P**) Convince yourself that the charge density can take the form $\rho(r,\theta,\phi) = \frac{\mu}{r}\, \delta(r-R)\, \delta(\cos\theta)$.
Is this a Dirichlet or von-Neumann boundary problem?

b) (**5P**) Determine all spherical multipole moments for $r \gg R$. Give explicit answers for the monopole, dipole and quadrupole moments.

c) (**1P**) Determine the scalar potential far from the origin as expansion in a suitable, small parameter.

d) (**3P**) Determine the scalar potential close to the origin as expansion in a suitable, small parameter.