STRONG JUMP INVERSION

W. CALVERT, A. FROLOV, V. HARIZANOV, J. KNIGHT,
C. MCCOY, A. SOSKOVA, AND S. VATEV

Abstract. Turing jump of a structure and different forms of the jump inversion for a structure have been studied independently by Baleva, Soskov, and A. Soskova in Bulgaria, by Morozov, Stukachev, and Puzarenko in Russia, and by Montalbán in the United States. For a structure $\mathcal{A}$, the *jump* $\mathcal{A}'$ includes as part of its structure the relations from which we can compute all relatively intrinsically computably enumerable relations on $\mathcal{A}$. In this paper we investigate when strong jump inversion holds. We say that a structure $\mathcal{A}$ admits *strong jump inversion* if for every oracle X, if $\mathcal{A}$ has a copy that is *low* relative to X, then $\mathcal{A}$ has a copy computable in X. For example, while Jockusch and Soare proved in 1991 that there are low linear orderings without computable copies, Downey and Jockusch established in 1994 that every Boolean algebra admits strong jump inversion.

We establish a general result with sufficient conditions on a structure $\mathcal{A}$, expressed in terms of saturation and enumeration properties of sets of types having formulas of low arithmetic complexity, which guarantee strong jump inversion of $\mathcal{A}$. Our general result applies to structures from familiar algebraic classes, including certain classes of linear orderings, abelian p-groups, equivalence structures, and trees. When a structure $\mathcal{A}$ admits strong jump inversion, and $\mathcal{A}$ is low relative to an oracle X, we also consider the complexity of the isomorphisms between $\mathcal{A}$ and its X-computable copies. In the case of an infinite Boolean algebra with no 1-atom, such an isomorphism can be chosen to be Δ^0_3 relative to X. This is interesting because Knight and Stob established in 2000 that any low Boolean algebra has a computable copy and a corresponding Δ^0_4 isomorphism, and this bound has been proven to be sharp.

1. Introduction

We often identify a structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$. Recall that a relation R is *relatively intrinsically* Σ^0_α on a structure $\mathcal{A}$ if in all (isomorphic) copies $\mathcal{B}$ of $\mathcal{A}$, the image of R is Σ^0_α relative to $\mathcal{B}$. By a result

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of [1] and [7], these are the relations that are definable in $\mathcal{A}$ by computable Σ_α formulas, with parameters.

Since we can computably enumerate the computable Σ_1 formulas, we can enumerate, uniformly in $D(\mathcal{A})$, all relations that are relatively intrinsically Σ^0_1 on $\mathcal{A}$ (*r.i.c.e.*). Moreover, we can uniformly compute all of these relations from the jump of the diagram, $(D(\mathcal{A}))'$.

The *jump of a structure* $\mathcal{A}$, denoted $\mathcal{A}'$, is defined to include, as part of the new structure $\mathcal{A}'$, relations from which we can uniformly compute all r.i.c.e relations on $\mathcal{A}$, so that the r.i.c.e. relations on the jump of $\mathcal{A}$ are just those which are relatively intrinsically Σ^0_2 on $\mathcal{A}$ itself.

Definition 1.1 (Canonical jump)**.** *For a structure $\mathcal{A}$, the* canonical jump *is a structure $\mathcal{A}' = (\mathcal{A}, (R_i)_{i\in\omega})$, where from the index i of the relation R_i, we can compute the arity of R_i and a computable Σ_1 formula (without parameters) that defines it in $\mathcal{A}$.*

Note: The set $\emptyset'$ is included in the jump. We may give it by a family of relations $R_{f(e)}$, for a computable function f, where $R_{f(e)}$ is always true if $e \in \emptyset'$ and always false otherwise. If $\tau_{e,s}$ is $\top$ if e has entered $\emptyset'$ by step s and $\perp$ otherwise, then $\bigvee_s \tau_{e,s}$ is a computable Σ_1 definition of the desired relation $R_{f(e)}$.

The original definition of the *jump* of $\mathcal{A}$ appears in the Ph.D. thesis of Baleva, supervised by Soskov [4, 5]. The definition was later used by A. Soskova and Soskov [28] for some jump inversion theorems. The definition in [28] looks slightly different. Some arithmetic is added to the structure, and the sequence of relations is coded by a single relation. The domain of $\mathcal{A}'$ is the "Moschovakis extension" of $\mathcal{A}$ with an appropriate coding mechanism, and the added relation is one that codes the forcing relation of the computable infinitary Σ^0_1 formulas as an analogue of Kleene's set K. There is yet another notion of jump, which involves Σ-definability in the hereditarily finite sets over a base structure. This notion appears in work of Morozov [24], Stukachev [29, 30], Puzarenko [25], and others from the Novosibirsk school. It applies to base structures of arbitrary cardinality.

Montalbán [22] initially used relatively intrinsically Π^0_1 relations instead of r.i.c.e. relations. The definition given above is a modification of the one in [22], which was arrived at after some group discussions in Sofia in the summer of 2011. In the spring of 2012, Russian, Bulgarian, and U.S. researchers gathered in Chicago for further discussions of the notions of jump, at the workshop "Definability in computable structures", funded mainly by the Packard Foundation. Later Montalbán proved that the different-looking definitions are equivalent (see [23]).

For some structures, there is a smaller subset of the relations that is sufficiently powerful to replace the full set.

Definition 1.2. *A* structural jump *of $\mathcal{A}$ is an expansion $\mathcal{A}' = (\mathcal{A}, (R_i)_{i\in\omega})$ such that each R_i has a Σ_1 defining formula that we can compute from i, and every relation that is relatively intrinsically Σ^0_2 on $\mathcal{A}$ is r.i.c.e. on $\mathcal{A}' \oplus \emptyset'$.*

For certain classes of structures, there is structural jump formed by adding a finite set of such relations. In particular, the relation $atom(x)$ is sufficient for Boolean algebras, and the successor relation $succ(x, y)$ is sufficient for linear orders. See [22] for further examples.

There are different statements of "jump inversion". The well-known Friedberg Jump Inversion Theorem says that if $\emptyset' \leq_T Y$, then there is a set X such that $X' \equiv_T Y \equiv \emptyset' \oplus X$. We can easily produce a structure $\mathcal{B}$ such that $X \equiv_T \mathcal{B}$, and then $Y \equiv_T \mathcal{B}'$. This is one kind of jump inversion. A more interesting kind of jump inversion theorem was proved by Soskov and A. Soskova [27], [28], and later (independently) by Montalbán [22].

Theorem 1.3 (Soskov, A. Soskova, Montalbán)**.** *For any $\mathcal{A}$, if Y computes a copy of $\mathcal{A}'$, there exists X such that $X' \equiv_T Y$ and X computes a copy of $\mathcal{A}$.*

We are interested in the following notion of jump inversion.

Definition 1.4. *A structure $\mathcal{A}$* admits strong jump inversion *provided that for all sets X, if there is a copy $\mathcal{B}$ of $\mathcal{A}$ so that X' computes $(D(\mathcal{B}))'$, then X computes a copy of $\mathcal{A}$.*

Remark: The structure $\mathcal{A}$ admits strong jump inversion iff for all X, if $\mathcal{A}$ has a copy that is low over X, then it has a copy that is computable in X.

This definition of strong jump inversion was motivated by a result of Downey and Jockusch [8]. Let $\mathcal{A}$ be a Boolean algebra that is low over X. Then X' computes the set of atoms in $\mathcal{A}$. Now, $(\mathcal{A}, atom(x))$ is a structural jump of $\mathcal{A}$. Downey and Jockusch showed that if X' computes $(\mathcal{A}, atom(x))$, then X computes a copy of $\mathcal{A}$. The proof involves some non-uniformity. A Boolean algebra with only finitely many atoms obviously has a computable copy. Suppose $\mathcal{A}$ has infinitely many atoms. If $\mathcal{A}$ is low over X, then there is an X-computable Boolean algebra $\mathcal{B}$ with a function f, Δ^0_2 relative to X, which would be an isomorphism except that it may map a finite join of atoms in $\mathcal{B}$ to a single atom in $\mathcal{A}$. We convert f into an isomorphism by re-apportioning the atoms (see Vaught [32]). We have the following.

Theorem 1.5 (Downey-Jockusch)**.** *All Boolean algebras admit strong jump inversion.*

It is possible to express a version of strong jump inversion that more explicitly emphasizes copies of the canonical jump $\mathcal{A}'$ as structures, as opposed to the Turing jump of the diagram of copies, $(D(\mathcal{B}))'$. However, in all arguments throughout subsequent sections, we prefer to work with the definition given in 1.4. Therefore, the following proposition is stated as an aside, and only a proof sketch is provided.

Proposition 1.6. *For any structure $\mathcal{A}$, the following are equivalent:*

1. *For all sets X, if $\mathcal{A}$ has a copy that is low over X, then it has an X-computable copy (Definition 1.4).*
2. *For all sets X, if X' computes a copy of $\mathcal{A}'$, then X computes a copy of $\mathcal{A}$.*

Proof sketch. For (2) $\Rightarrow$ (1), assume $\mathcal{A}$ has a copy $\mathcal{B}$ with $(D(\mathcal{B}))' \leq X'$. Since $D(\mathcal{B}') \leq_T (D(\mathcal{B}))' \leq_T X'$, (2) implies that X computes a copy of $\mathcal{A}$.

For the other direction, let $\mathcal{B}'$ be a copy of $\mathcal{A}'$ and suppose $D(\mathcal{B}') \leq_T X'$. (Recall that the canonical jump computes $0'$.) We use forcing to construct a 1-generic permutation of ω, giving a copy $\mathcal{C}$ of $\mathcal{A}$. Computably in $\mathcal{B}'$, we produce a countable forcing sequence $(p_n)_{n\in\omega}$ deciding the atomic diagram of $\mathcal{C}$ and the statements $e \in (D(\mathcal{C}))'$. Given p_n taking $\bar{d}$ to $\bar{c}$, and wanting the next p_{n+1} to decide whether e is in the jump, we ask whether there is a forcing condition $q \supseteq p_n$ forcing $e \in (D(\mathcal{C}))'$. There is a computable Σ_1 formula that is true of $\bar{d}$ in $\mathcal{B}$ iff there is such a q. Using $\mathcal{B}'$, we know the answer. (See [2], Chapter 10 for more detailed examples of such forcing constructions.) $\square$

Not all countable structures admit strong jump inversion. For example, take a low completion T of PA. There is a model $\mathcal{A}$ whose complete diagram is computable in T. The jump of $\mathcal{A}$ is Δ^0_2. By a well-known result of Tennenbaum, since $\mathcal{A}$ is necessarily non-standard, there is no computable copy. Jockusch and Soare [15] showed that there are low linear orderings with no computable copy.

In Section 2, we give a general result with sufficient conditions for strong jump inversion. In Section 3, we bring together a number of examples of strong jump inversion.

2. General result

We would like to find general conditions on a structure $\mathcal{A}$ guaranteeing that $\mathcal{A}$ admits strong jump inversion. The result that we give is not difficult to prove. However, there are a number of examples where it applies. To state the result, we need some definitions.

Definition 2.1. *Let S be a countable family of sets. An* enumeration *of S is a set R of pairs (i, k) such that S is the family of sets $R_i = \{k : (i, k) \in R\}$. If $A = R_i$, we say that i is an R-*index *for A.*

Note: When we say that R is a *computable enumeration* of a family of sets, we mean that R is a computable set of pairs. This means that the sets R_i are *computable*, uniformly in i. Some researchers have used the term differently, saying that R is a *computable enumeration* if the sets R_i are uniformly *computably enumerable*.

Definition 2.2.

(1) *A* B_n-formula *is a finite Boolean combination of finitary* Σ_n*-formulas.*
(2) *A* B_n-type *is the set of* B_n*-formulas in a complete type. All of our types are assumed to be consistent.*

Definition 2.3. *Fix a structure* $\mathcal{A}$*. Let* S *be a set of* B_1*-types including all those realized in* $\mathcal{A}$*. Let* R *be an enumeration of* S*. An* R-labeling of $\mathcal{A}$ *is a function taking each tuple* $\bar{a}$ *in* $\mathcal{A}$ *to an* R*-index for the* B_1*-type of* $\bar{a}$*.*

Definition 2.4. *Suppose* $p(\bar{u})$ *and* $q(\bar{u}, x)$ *are* B_1*-types. We say that* $q(\bar{u}, x)$ *is* generated by the formulas of $p(\bar{u})$ and existential formulas *provided that for any universal formula* $\psi(\bar{u}, x)$*, we have* $\psi(\bar{u}, x) \in q(\bar{u}, x)$ *just in case there is a finite set of existential formulas in* $q(\bar{u}, x)$*, with conjunction* $\chi(\bar{u}, x)$*, such that* $(\exists x)[\chi(\bar{u}, x)\ \&\ neg(\psi(\bar{u}, x))]$ *is not in* $p(\bar{u})$*, where* $neg(\psi)$ *is the natural existential formula logically equivalent to* $\neg\psi$*.*

Definition 2.5. *The structure* $\mathcal{A}$ *is* weakly 1-saturated *provided that if* $p(\bar{u})$ *is the* B_1*-type of a tuple* $\bar{a}$*, and* $q(\bar{u}, x)$ *is a* B_1*-type generated by formulas of* $p(\bar{u})$ *and existential formulas, then* $q(\bar{a}, x)$ *is realized in* $\mathcal{A}$*.*

The following proposition is clear from the definitions. It explains the importance of the notion of weak 1-saturation.

Proposition 2.6. *Let* $p(\bar{u})$ *be a* B_1*-type. Suppose* $q(\bar{u}, x)$ *is a* B_1*-type* $q(\bar{u}, x)$ *that is generated by formulas of* $p(\bar{u})$ *and existential formulas. Then* $q(\bar{u}, x)$ *is consistent with all extensions of* $p(\bar{u})$ *to a complete type in variables* $\bar{u}$*.*

We can now state the general result.

Theorem 2.7. *Suppose* $\mathcal{A}$ *is weakly* 1*-saturated. Let* R *be a computable enumeration of a set of* B_1*-types including all those realized in* $\mathcal{A}$*. If* $\mathcal{A}$ *has an* R*-labeling that is* Δ^0_2 *relative to* X*, then it has an* X*-computable copy* $\mathcal{B}$*. Moreover, there is an isomorphism from* $\mathcal{B}$ *onto* $\mathcal{A}$ *that is* Δ^0_2 *relative to* X*.*

Proof. For simplicity, we suppose that $\mathcal{A}$ has a Δ^0_2 R-labeling; we base our construction on guesses at various portions of this labeling. Note that once we have guessed the label for a tuple $\bar{a}$ correctly, we computably know the entire B_1-type of that tuple. We build a computable copy $\mathcal{B}$ and a Δ^0_2 isomorphism f from $\mathcal{B}$ to $\mathcal{A}$. We have the following requirements.

R_{2a}: $a \in ran(f)$

R_{2b+1}: $b \in dom(f)$

We start with an R-index for the type of $\emptyset$, where this type is the B_1-theory of $\mathcal{A}$. At each stage s, we have a tentative partial isomorphism f_s mapping a tuple $\bar{d}$ from $\mathcal{B}$ to a tuple $\bar{c}$ in $\mathcal{A}$, where the R-indices of the types of $\bar{c}$ and all of its initial segments still look correct. (At a later stage t, we may see that some of the guesses at these indices are incorrect, and we retain only the portion of f_s satisfying an initial segment of requirements based on guesses at R-indices that all look correct.) Moreover, we have enumerated a finite part $\delta(\bar{d},\bar{b})$ of the atomic diagram of $\mathcal{B}$; this can never change, since $\mathcal{B}$ must be computable. We will have checked the consistency of $\delta(\bar{d},\bar{b})$ with our guesses at the R-indices of the B_1-types of the tuple $\bar{c}$ and its initial segments. Supposing that the function taking $\bar{c}$ to $\bar{d}$ satisfies the earlier requirements, we can satisfy the requirement R_{2a} once we guess the R-index for the B_1-type $p(\bar{u},x)$ of $\bar{c},a$. We map some b, either old or new, to a so that $\delta(\bar{u},\bar{v})$ is consistent with $p(\bar{u},x)$. (Recall that the B_1 types are computable.)

Suppose that the function taking $\bar{d}$ to $\bar{c}$ satisfies the requirement R_i for $i<2b+1$, and R_{2b+1} is least that is unsatisfied at this stage s. Again, we assume that we have correct guesses on the R-indices for the B_1 types of $\bar{c}$ and all of its initial segments; let $p(\bar{u})$ be the B_1-type of $\bar{c}$. Finally, we have put $\delta(\bar{d},b,\bar{b})$ in the atomic diagram of $\mathcal{B}$. Now, we determine, effectively in $p(\bar{u})$, a B_1-type $q(\bar{u},x)$ for $\bar{c}$ and a putative $f_s(b)$, where $q(\bar{u},x)$ is generated by formulas of $p(\bar{u})$ and existential formulas, including the formula $\varphi(\bar{u},x)$ asserting that there exists $\bar{v}$ such that $\delta(\bar{u},x,\bar{v})$ holds. We determine this B_1-type computably as follows. We start with $p(\bar{u})$ and $\varphi(\bar{u},x)$. We have a computable list $(\varphi_n(\bar{u},x))_{n\in\omega}$ of all existential formulas in variables $\bar{u},x$, in order of Gödel number. We consider these formulas, in order, and we put $\varphi_n(\bar{u},x)$ into $q(\bar{u},x)$ iff it is consistent with we have already put into $q(\bar{u},x)$. If we do not add $\varphi_n(\bar{u},x)$, then all tuples satisfying what did add must satisfy $neg(\varphi_n(\bar{u},x))$, so that is in $q(\bar{u},x)$. Knowing exactly which existential formulas are in $q(\bar{u},x)$, we can determine which B_1 formulas are in (using truth tables). We have described an effective procedure for determining $q(\bar{u},x)$. At step s, we can give a computable index for $q(\bar{u},x)$, but not an R-index.

By weak 1-saturation, if $p(\bar{u})$ really is the B_1-type of $\bar{c}$, then $q(\bar{c},x)$ will be realized in $\mathcal{A}$. We define $f_s(b)$ as follows. We find the first a such that, based on our guess at the R-index of the B_1 type of $\bar{c},a$, this type and $q(\bar{u},x)$ agree on the first s formulas; then $f_s(b)=a$. Of course, this guess at the element a is likely wrong. Therefore, in order to guarantee that this requirement is satisfied, at each subsequent stage t, we need to check that, based on our guess at the R-index of the B_1 type of $\bar{c},a$, this type and $q(\bar{u},x)$ agree on the first t formulas. If this is not the case, then we need to re-define $f_t(b)$, but always maintaining $q(\bar{u},x)$ as the guaranteed type of $q(\bar{c},f(b))$, so long as no earlier requirements become unsatisfied. (In particular, note that as we check consistency of the atomic diagram with the B_1 types associated with requirement R_{2b+1}, we use the computable index for $q(\bar{u},x)$.) By weak 1-saturation, there is a first a realizing $q(\bar{c},x)$, and eventually, we will have the R-index for the B_1 type of $\bar{c},a$. Then we will have $f_s(b)=f(b)=a$. $\square$

Remark. What weak 1-saturation does is to guarantee that if $\bar{c}$ has B_1-type $p(\bar{u})$, and $(\exists x)\varphi(\bar{u},x)\in p(\bar{u})$, where $\varphi(\bar{u},x)$ is existential, then we can find, effectively in $p(\bar{u})$, a B_1-type $q(\bar{u},x)$, including $\varphi(\bar{u},x)$, such that

$q(\bar{c}, x)$ must be realized in $\mathcal{A}$. If $\mathcal{A}$ is not weakly 1-saturated but satisfies this condition, then we would still have the conclusion of Theorem 2.7.

3. Examples

3.1. **Linear orderings.** The second author proved strong jump inversion for two special classes of linear orderings, with further results on complexity of isomorphisms. The results are given in [11], [12], [10]. Here is the result for the simpler class of orderings.

Theorem 3.1 (Frolov)**.** *Let $\mathcal{A}$ be a linear ordering such that each element lies on a maximal discrete set that is finite. Suppose there is a finite bound N on the sizes of these sets. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy with an isomorphism that is Δ^0_2 relative to X.*

Proof. We have a computable enumeration R of the B_1-types realized in linear orderings. The B_1-type of a tuple $\bar{a}$ is determined by the ordering and the sizes of the intervals. It is Δ^0_2 relative to $\mathcal{A}$ to say that the interval (a, b) has size n for some fixed n. It is Π^0_2 relative to $\mathcal{A}$ to say that the interval is infinite. We first show that $\mathcal{A}$ is weakly 1-saturated. For a tuple $\bar{a}$, we consider the possible B_1 types $q(\bar{a}, x)$. First, suppose $q(\bar{a}, x)$ locates x in a finite interval (a_i, a_{i+1}) so that the sizes of the intervals (a_i, x) and (x, a_{i+1}) add up properly. Then $q(\bar{u}, x)$ is generated by formulas of $p(\bar{u})$ and existential formulas, and $q(\bar{a}, x)$ must be realized. Next, suppose $q(\bar{a}, x)$ locates x in an infinite interval $(-\infty, a_0)$, (a_i, a_{i+1}), or (a_n, ∞). If $q(\bar{a}, x)$ is generated by formulas of $p(\bar{u})$ and existential formulas, then x must split the interval into two infinite parts. The ordering $\mathcal{A}$ has the feature that in any infinite interval, there is an element that splits the interval into two infinite parts. This implies that $\mathcal{A}$ is weakly 1-saturated.

Suppose that $\mathcal{A}$ is low over X. We can apply a procedure that is Δ^0_2 relative to X to assign an R-index to the type of any tuple $\bar{a} = (a_1, \ldots, a_n)$. The intervals $(-\infty, a_1)$ and (a_n, ∞) are infinite. The interval (a_i, a_{i+1}) is infinite if it has size $> N$. Using a procedure that is Δ^0_2 relative to X, we can determine whether the size is k, for $k \leq N$. We have an R-labeling of $\mathcal{A}$ that is Δ^0_2 relative to X. Then Theorem 2.7 gives an X-computable copy with an isomorphism that is Δ^0_2 relative to X. □

The next result says that another class of linear orderings admits strong jump inversion. Before we state it, we must review some well-known, basic concepts about linear orderings. Recall the block equivalence relation $\sim$ on a linear ordering $\mathcal{A}$, where $a \sim b$ iff $[a, b]$ is finite. For any linear ordering $\mathcal{A}$, each equivalence class under this relation is an interval that is either finite or of order type ω, ω^*, or $\zeta = \omega^* + \omega$. Furthermore, the quotient structure $\mathcal{A}/_\sim$ is itself a linear ordering, where each distinct point represents an equivalence class under $\sim$.

Note also in this result that for a given $\mathcal{A}$ low over X, it is not clear that $\mathcal{A}$ has an R-labeling Δ^0_2 relative to X, but there is a copy $\mathcal{B}$ with such an R-labeling.

Theorem 3.2 (Frolov)**.** *Let $\mathcal{A}$ be a linear ordering for which the quotient $\mathcal{A}/_\sim$ has order type η. Suppose also that in $\mathcal{A}$ every infinite interval has arbitrarily large finite successor chains. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy with an isomorphism that is Δ^0_3 over X.*

Proof. We have a computable enumeration R of all B_1-types realized in linear orderings such that from the index i of the type R_i, we can compute the sizes, including ∞, of the intervals. This was not important in the previous result. We may suppose that the index also tells us the tuple of variables. Every infinite interval $(-\infty, a)$, (a, a'), or (a', ∞) has an element that splits the interval into two infinite parts. This implies that $\mathcal{A}$ is weakly 1-saturated. Suppose $\mathcal{A}$ is low over X. We will prove the following.

Lemma 3.3. *There is a copy $\mathcal{B}$ of $\mathcal{A}$ with an R-labeling that is Δ^0_2 over X. Moreover, there is an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ such that f is Δ^0_3 relative to X.*

Assuming the lemma, we complete the proof of Theorem 3.2 as follows. Given $\mathcal{A}$, low over X, the lemma gives a copy $\mathcal{B}$ with an R-labeling that is Δ^0_2 relative to X, and an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ that is Δ^0_3 relative to X. By Theorem 2.7, there is an X-computable copy $\mathcal{C}$ with an isomorphism g from $\mathcal{C}$ to $\mathcal{B}$ that is Δ^0_2 relative to X. Then $f \circ g$ is an isomorphism from $\mathcal{C}$ to $\mathcal{A}$ that is Δ^0_3 relative to X.

Proof of Lemma. For simplicity, we suppose that $\mathcal{A}$ is low. We build a Δ^0_2 copy $\mathcal{B}$, and a Δ^0_3 isomorphism f. We suppose that the universe of $\mathcal{A}$ is ω, thought of as a set constants. The copy $\mathcal{B}$, also with universe ω, will have the intervals labeled by size. Throughout, we use the oracle Δ^0_2. Suppose $\mathcal{A}_n$ is the true ordering on the first n elements of $\mathcal{A}$, with the intervals correctly labeled by size. At stage s, we construct (using the Δ^0_2 oracle, of course) an approximation $\mathcal{A}_{n,s}$ in which intervals are either correctly labeled with a finite number at most s, or else carry the label ∞. We have a finite sub-ordering $\mathcal{B}_s$ of $\mathcal{B}$ in which the intervals are labeled by size, once and for all.

We want an isomorphism f from $\mathcal{B}$ onto $\mathcal{A}$. We must satisfy the following requirements.

R_{2a}: Put a into $ran(f)$.

R_{2b+1}: Put b into $dom(f)$.

At each stage s, we have a finite function f_s that seems to satisfy the first few requirements, so that our current labels on the intervals with endpoints in $ran(f_s)$ match the labels on the corresponding intervals in $dom(f_s)$. An interval that seemed infinite at stage s may be seen as finite at stage $s+1$. So, f_{s+1} may be defined on a smaller set.

Case 1: Suppose the next requirement is to put a into $ran(f)$. We have no problem finding an appropriate pre-image b.

Case 2: Suppose the next requirement is to put b into $dom(f)$. In the interesting sub-case, b lies in an interval (d, d'), where (d, b) and (b, d') are both infinite. We suppose that $f(d) = c$ and $f(d') = c'$, where (c, c') appears to be infinite. We need $a = f(b)$ such that (c, a) and (a, c') are both infinite, and whatever successor chain surrounds b is matched by one surrounding a. The naive strategy is to just look for a. This strategy may not work. Believing that we have found a, and seeing that a lies in a finite interval inside (c, c'), we may create a bigger successor chain around b, inside (d, d'). Eventually, we may discover that the interval (c, a) or (a, c') is finite. Now, we cannot map b to a. Moreover, we have made the search for $f(b)$ more difficult, in that it must lie in a larger finite interval matching the one we have created around b. This can keep happening. Our current guess at the appropriate $a = f(b)$ may keep attaching itself to a successor chain around c or c'.

We need a better strategy. Instead of looking for $a = f(b)$ directly, we first look for a "buffer pair" z, z' such that (c, z), (z, z') and (z', c') are all infinite. Once we find (z, z'), then we can immediately find a successor chain sufficient to match whatever one we may have created around b. We will eventually settle on the first good buffer pair (z, z'). Now, applying the hypothesis about $\mathcal{A}$, we will immediately succeed in finding a good $a = f(b)$, with a finite interval around a large enough to take care of the one we may have built around b.

In general, when we have f mapping b to a for some requirement, we vow not to locate b in a finite interval larger than the one we have seen around a. Following this procedure, we can eventually satisfy all requirements. $\square$

$\square$

3.2. **Models of a theory with few B_1-types.** Lerman and Schmerl [21] gave conditions under which an $\aleph_0$-categorical theory T has a computable model. They assumed that the theory is arithmetical and $T \cap \Sigma_{n+1}$ is Σ^0_n for each n. In [19], the assumption that T is arithmetical is dropped, and, instead, it is assumed that $T \cap \Sigma_{n+1}$ is Σ^0_n uniformly in n. The proof in [21] gives the following.

Lemma 3.4 (Lerman-Schmerl). *Let T be an $\aleph_0$-categorical theory that is Δ^0_N and suppose that for $1 \leq n < N$, $T \cap \Sigma_{n+1}$ is Σ^0_n. For any $n < N$, if $\mathcal{A}$ is a model whose B_{n+1}-diagram is computable in X', then there is a model $\mathcal{B}$ whose B_n-diagram is computable in X.*

We note that the Σ_1 diagram is computable in the jump of the model. Thus, the lemma implies that the model admits strong jump inversion. If T is an $\aleph_0$-categorical theory, then there are only finitely many types in each tuple of variables $\bar{x}$. Here is a variant of the result of Lerman and Schmerl.

Theorem 3.5. *Let T be an elementary first order theory, in a computable language. Suppose that for each tuple of variables $\bar{x}$, there are only finitely many B_1-types in variables $\bar{x}$ consistent with T. Suppose also that $T \cap \Sigma_2$ is Σ^0_1. Then every model $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_2 relative to X.*

Proof. First, we show that there is a computable enumeration of all the B_1-types. We have a computable enumeration R of B_1-formulas $(\varphi_n)_{n\in\omega}$. For each particular tuple of variables $\bar{x}$, we build a c.e. tree of types: at level n, the nodes represent the different finite sets of formulas $\pm\varphi_k$, for $k < n$, that we have seen to be consistent with T, using the fact that $T \cap \Sigma_2$ is c.e. By assumption, all of the B_1-types are isolated, in the sense that each B_1-type contains some formula not contained in any of the other B_1-types. In fact, there is some n such that the types in $\bar{x}$ are all generated by the conjunction of $\pm\varphi_k$ for $k < n$. (Therefore, each of these trees contains only finitely many infinite paths.) We now use all of these trees together to define the enumeration R. At stage 0, we assign the index 0 to a B_1-type in no variables that contains either the first B_1-sentence or its negation. At stage $s+1$, we include the indices for types from stage s, and, using the $(s+1)$-stage enumeration of the c.e. trees of types for all tuples of length $\leq s+1$, we add new indices of types whenever we see a splitting. That is, if at stage s, a node on a tree is already part of a type with an assigned index, and at stage $s+1$, it is first revealed that this node has two different consistent extensions, then one of those extensions retains the assigned index, and the other extension is assigned a new index. (Since there are only finitely many B_1-types in a given $\bar{x}$, we will use only finitely many indices for this fixed tuple.)

Next, we show that $\mathcal{A}$ is weakly 1-saturated. Suppose $q(\bar{u}, x)$ is a B_1-type (consistent, of course) generated by formulas true of $\bar{a}$ and existential formulas $\varphi(\bar{u}, x)$. Since $q(\bar{u}, x)$ is isolated, it is principal, with a generating formula $\gamma(\bar{u}, x)$, of the form $\rho(\bar{u})$ & $\chi(\bar{u}, x)$, where $\rho(\bar{u})$ is in the B_1-type of $\bar{a}$, and $\chi(\bar{u}, x)$ is a finite conjunction of existential formulas. We have $(\exists x)\chi(\bar{u}, x)$ true of $\bar{a}$ in $\mathcal{A}$, so the type is realized.

Lemma 3.6. *If $\mathcal{A}$ is low over X, then there is an R-labeling of $\mathcal{A}$ that is Δ^0_2 relative to X.*

Proof. For simplicity, we suppose $\mathcal{A}$ is low. For a tuple of variables $\bar{x}$, Δ^0_2 can find generating formulas for all of the types. Then Δ^0_2 can check which generating formula is true of a given tuple of elements $\bar{a}$. Then we have a Δ^0_2 R-labeling. $\square$

Finally, we apply Theorem 2.7 to get an X-computable copy $\mathcal{B}$ of $\mathcal{A}$ with an isomorphism from $\mathcal{B}$ to $\mathcal{A}$ that is Δ^0_2 relative to X. □

Note: There are non-$\aleph_0$-categorical theories satisfying the conditions of Theorem 3.5.

Proof. We write Θ for the ordering of type $\eta+2+\eta$. In [9], it was shown that for any linear ordering $\mathcal{A}$, $\Theta \cdot \mathcal{A}$ has a computable copy iff $\mathcal{A}$ has a Δ^0_2 copy. Let T_1 be a complete theory of linear orderings that is not $\aleph_0$-categorical. Let T be the complete theory whose models are exactly the orderings of the form $\Theta \cdot \mathcal{A}$, where $\mathcal{A}$ is a model of T_1. The theory T has a sentence saying that every element lies on an interval of type Θ. In addition, there are axioms guaranteeing that the restriction of our ordering to the set of elements that are the first in a successor pair satisfies all sentences φ in T_1.

We note that the B_1-types realized in models of T come from partitions into intervals of size 0 or ∞, with no two adjacent intervals of size 0. These are principal, so they are realized in all models of T. We note that if we replace T_1 by some other theory S_1 of infinite linear orderings, and form S in the same way, then the B_1-types realized in any and all models of S would be the same. Therefore, the Σ_2 theories are the same. If S_1 is decidable, then so is S. Thus, whether or not T_1 is decidable, $T \cap \Sigma_2$ is decidable. We chose T_1 not $\aleph_0$-categorical, so T is also not $\aleph_0$-categorical. □

3.3. **Abelian p-groups.** By Ulm's Theorem, a countable Abelian p-group G is determined up to isomorphism by its *Ulm* sequence. We recall what this is. Let $G_0 = G$, let $G_{\alpha+1} = pG_\alpha$, and for limit α, let $G_\alpha = \cap_{\beta<\alpha} G_\beta$. An element has *height* α if it is in $G_\alpha - G_{\alpha+1}$. Thus, for finite n, an element has height n if it is divisible by p^n and not by p^{n+1}. There is some first ordinal λ such that $G_\lambda = G_{\lambda+1}$. This is the *length* of the group. Let P be the set of elements of order p, plus 0. The quotient $(P \cap G_\alpha)/(P \cap G_{\alpha+1})$ is a vector space over Z_p. The dimension of this vector space is $u_\alpha(G)$. The *Ulm sequence* for G is the sequence $u_\alpha(G)$ for $\alpha < \lambda$. For more on Ulm invariants and Ulm's Theorem, see [16].

An Abelian p-group G of length ω has an Ulm sequence of the form $(u_n(G))_{n\in\omega}$. Such a group is a direct sum of copies of Z_{p^k}, for finite k, and the Prüfer group Z_{p^∞}. The following is well known (see, for instance [6]).

Lemma 3.7. *For each $n, k \in \omega$, we can find a computable Σ_2 sentence $\varphi_{n,k}$ that is true in G just in case $u_n(G) \geq k$.*

Proof Sketch. We take $\varphi_{n,k}$ saying that there exist $x_1, \ldots, x_k$ of order p such that all x_i are divisible by p^n and for all $z_1, \ldots, z_k \in Z_p$, not all 0, $z_1x_1 + \ldots + z_kx_i$ is not divisible by p^{n+1}. □

Proposition 3.8. *Let $\mathcal{A}$ be an Abelian p-group of length ω such that the divisible part has infinite dimension. Then $\mathcal{A}$ admits strong jump inversion.*

Proof. Suppose $\mathcal{A}$ is low over X. Then the set P of pairs (n, k) such that $u_n(G) \geq k$ is Σ^0_2 relative to X. Then we can produce a copy of $\mathcal{A}$ that is computable in X. We start with a computable group G that is a direct sum of infinitely many copies of the Prüfer group, such that for each direct summand, we have a labeled sequence of elements $(c_n^i)_{i\in\omega}$, where c_n^0 has order p and for all i, $pc_n^{i+1} = c_n^i$. To produce our X-computable copy of $\mathcal{A}$, we determine a subgroup H of G that is isomorphic to $\mathcal{A}$. The universe of H will be X-c.e. We get an X-computable copy by a standard trick, replacing each element $h \in H$ by the Gödel number of the pair (h, s), where s is the stage at which h is enumerated into H.

We enumerate elements into H, based on standard X-computable guesses at membership in P. Guessing that $(n, k) \in P$, we choose k of the elements $c_{n_j}^0$, and we add enough of the divisor chains to give the chosen elements $c_{n_j}^0$ height n. If we later decide that $(n, k) \notin P$, then, for the unwanted k^{th} element c_{n_j}, we vow to add the remainder of the divisor chain, giving $c_{n_j}^0$ infinite height. If, still later, we again decide that $(n, k) \in P$, we choose a new element $c_{n_{j'}}^0$, and we give it height n. If $(n, k) \in P$, then we will eventually designate at least k elements that will be given height n permanently. If $(n, k) \notin P$, then we may enumerate into H infinitely many elements with infinite chains. The full set H is the subgroup generated by the chosen c_n^i's. □

The proof of Proposition 3.8 is simple enough. The proposition does not give any information about the complexity of the isomorphism between $\mathcal{A}$ and H. Using X', we can say about a particular k-tuple $\bar{a}$ in $\mathcal{A}$ that the elements all have order p and height n, and that they are independent over the elements of order p having greater height. We will see that Proposition 3.8 also follows from a result on trees.

3.4. **Equivalence structures.** The following result is from [2]. (We give the result in relativized form, since that is how we plan to use it.)

Proposition 3.9. *Let $\mathcal{A}$ be a countable equivalence structure. Then $\mathcal{A}$ has an X-computable copy iff it satisfies the following conditions.*

(1) *The set of pairs (n, k) such that there are at least k classes of size n is Σ_2^0 relative to X.*
(2) *One of the following holds:*
    (a) *there is a bound on the sizes of finite classes,*
    (b) *there are infinitely many infinite classes,*
    (c) *there is an X-computable function $f(n, s)$ such that for each n, $f(n, s)$ is non-decreasing in s, with a finite limit $f^*(n) \geq n$ that is the size of some class.*

Using Proposition 3.9 we obtain the following.

Proposition 3.10. *Let $\mathcal{A}$ be an equivalence structure with infinitely many infinite classes. Then $\mathcal{A}$ admits strong jump inversion.*

Proof. If $\mathcal{A}$ is low over X, then the set P consisting of pairs (n,k) such that there are at least k classes of size n is Σ^0_2 relative to $\mathcal{A}$, so it is Σ^0_2 relative to X. Then Proposition 3.9 gives an X-computable copy of $\mathcal{A}$. $\square$

The proof of Proposition 3.10, on equivalence structures with infinitely many infinite classes, is simple enough without Theorem 2.7. Nonetheless, the conditions of Theorem 2.7 apply to these equivalence structures, as we show below.

(1) There is a computable enumeration R of the B_1-types that can be realized in equivalence structures of the kind we are considering.

The B_1 type of a tuple $\bar{x}$ gives the equality and equivalence relations, together with the sizes of the equivalence classes of the x_j. Thus, there is a computable enumeration R of the possible B_1-types such that we can effectively determine from an index i, whether the equivalence class of a given variable is infinite.

(2) The structure $\mathcal{A}$ is weakly 1-saturated.

To see this, consider a possible B_1-type $q(\bar{a},x)$ for x over $\bar{a}$, generated by formulas true of $\bar{a}$ and existential formulas. If $q(\bar{a},x)$ puts x in the class of some a_j, then it is realized. If $q(\bar{a},x)$ puts x in a new class, then the class must be infinite, so the type is realized.

(3) There is a copy $\mathcal{B}$ of $\mathcal{A}$ with an R-labeling that is Δ^0_2 relative to X.

Enumerating the set P, which is Σ^0_2 relative to X, we can (using X') build an equivalence structure $\mathcal{B}$ that has the appropriate number of classes of each size. Moreover, we can label the classes by size. From this we obtain an R-labeling of $\mathcal{B}$ that is Δ^0_2 relative to X.

3.5. **Trees.** We consider some special classes of subtrees of $\omega^{<\omega}$. For the language of trees, we use the predecessor function, where the top node is its own predecessor. The first special class of trees is related to Abelian p-groups. Given a tree $T \subseteq \omega^{<\omega}$, we can pass effectively to an Abelian p-group G_T, generated by the elements of T under the relations $\emptyset = 0$ and $pc = c'$, where c' is the predecessor of c. Non-isomorphic trees may give rise to isomorphic groups. Every countable Abelian p-group G is isomorphic to G_T for some T, although we do not have a uniform way to pass from the group G to an appropriate tree T.

For a computable Abelian p-group G of length less than ω^ω, there is a computable tree T with $G_T \cong G$. This is shown in unpublished work of Ash, Knight, and Oates [3] which re-proves results of Khisamiev [17], [18]. We consider a special kind of tree T such that G_T is an Abelian p-group of length ω with infinite-dimensional divisible part. The tree clearly displays the Ulm sequence.

Definition 3.11. *A* special tree $T \subseteq \omega^\omega$ *has infinitely many nodes at level* 1*, each with a chain below. Some of the chains may be finite, but for each* *of infinitely many nodes at level* 1*, the chain below it is infinite. There are no other nodes apart from these.*

Proposition 3.12. *Special trees admit strong jump inversion.*

Proof. We show, using our general result, that if T is a low special tree, then there is a computable copy. For a special tree T, the B_1-type of a tuple $\bar{a}$ is determined by the isomorphism type of the subtree generated by $\bar{a}$, together with the lengths of the chains below the terminal nodes in this subtree.

Weak 1-saturation. Let $\bar{c}$ be in T, and let $q(\bar{c}, x)$ be a B_1-type generated by B_1-formulas true of $\bar{c}$ and existential formulas. The type $q(\bar{c}, x)$ may locate x in the subtree generated by $\bar{c}$, or on the chain below some element of $\bar{c}$, apart from the top node. Then $q(\bar{c}, x)$ is clearly satisfied. The type $q(\bar{c}, x)$ may locate x on a new infinite chain. This is satisfied since we have infinitely many infinite chains.

We have a computable enumeration R of all finite subtrees of special trees, with added labels indicating the length of the chain below each node at level $k \geq 1$. This gives a computable enumeration R of the B_1-types realized in special trees.

Lemma 3.13. *If T is a low special tree, then there is a copy with a Δ^0_2 R-labeling.*

Proof. The set P of pairs (n, k) such that there are at least k nodes at level 1 with an infinite chain below is Σ^0_2. Then Δ^0_2 can enumerate P. Using this enumeration, Δ^0_2 builds a copy M of this tree, labeling the nodes at level 1 according to the length of the chain below. From this, it is clear how to give the R-labeling. □

Then by Theorem 2.7, there is a computable copy. □

We consider another simple class of trees.

Proposition 3.14. *Suppose $\mathcal{A}$ is a tree such that the top node is infinite (i.e., it has infinitely many successors), and each infinite node has only finitely many successors that are terminal, with the rest all infinite. Then $\mathcal{A}$ admits strong jump inversion.*

Proof. The B_1-type of a tuple $\bar{a}$ is determined by the subtree generated by $\bar{a}$ and labels "infinite" or "terminal" on the nodes, in particular, on the a_i. We have a computable enumeration of all possible labeled finite subtrees of trees of this kind. From this, we get a computable enumeration R of the B_1-types. Suppose that $\mathcal{A}$ is low. Then there is a Δ^0_2 R-labeling of $\mathcal{A}$.

Weak 1-saturation. Take $\bar{a}$ in $\mathcal{A}$. Consider a possible B_1-type $p(\bar{a}, x)$, generated by formulas true of $\bar{a}$ and existential formulas. The type may locate x in the subtree generated by $\bar{a}$. Then the type is realized. The type may locate x properly below some infinite a_i, or at some level not below any a_i. Again the type is realized by a new infinite element.

By Theorem 2.7, we get a computable copy of $\mathcal{A}$. □

We consider one last class of trees. We use some definitions and notation. If T is a sub-tree of $\omega^{<\omega}$, and $a \in T$, we write T_a for the tree consisting of a and all nodes below.

Definition 3.15. *For nodes a in a fixed tree T,*

(1) *we say that a is* finite *if T_a is finite,*
(2) *we say that a is* infinite *if T_a is infinite. (For the trees we consider below, if a is infinite, we will require not only that T_a is infinite, but also that a has infinitely many successors, so we will have agreement with the definition we used in Proposition 3.14.)*

Notation. Let a be finite, with T_a the subtree below a. Let T_a^1 be a possible re-labeling of the nodes in T_a in which the nodes in a subtree are labeled ∞. We write $(T_a^1)^*$ for the infinite tree that results from extending the labeled tree T_a^1 so that all new nodes in $(T_a^1)^*$ are labeled ∞, and each node labeled ∞ has infinitely many successors labeled ∞. (No finite node in T_a^1 acquires successors in $(T_a^1)^*$.)

We are ready to state the final result on trees.

Proposition 3.16. *Suppose T is a subtree of $\omega^{<\omega}$ such that the top node is infinite, and for any infinite node a, there are only finitely many finite successors. Suppose also that for any infinite node a, for any finite successor b, if T_b^1 is a possible re-labeling of T_b making all nodes in a certain subtree infinite, then there are infinitely many successors b_n of a such that $T_{b_n} \cong (T_b^1)^*$. Then T admits strong jump inversion.*

Proof. For simplicity, we suppose that T is low, and we apply Theorem 2.7 to produce a computable copy. For a tuple $\bar{a}$ in T, the B_1-type of $\bar{a}$ is determined by the subtree generated by $\bar{a}$ and formulas saying, for an element a of this subtree that it is infinite, or that it is finite with a specific finite tree T_a. We can show that T is weakly 1-saturated. Consider a B_1-type for $\bar{a}, x$, generated by B_1-formulas true of $\bar{a}$ and existential formulas. The type may put x in the subtree generated by $\bar{a}$, or in one of the trees T_{a_i}, where a_i (in the subtree) is finite. In either of these cases, the type is realized. Or, the type may put x below some infinite a_i (in the subtree). Again, the type is realized, since there is a copy of $\omega^{<\omega}$ below a_i. This shows that $\mathcal{A}$ is weakly 1-saturated.

We have a computable enumeration of the possible finite labeled subtrees, and, hence, of the B_1-types realized in trees of this kind. Let R be this computable enumeration of B_1-types. To apply Theorem 2.7, we need the following.

Lemma 3.17. *There is a copy $\mathcal{B}$ of T with a Δ_2^0 R-labeling.*

Proof. We build a Δ^0_2 copy $\mathcal{B}$ of T with nodes labeled as infinite, or with a specific finite tree below. We suppose that the ω-list of elements of T has the feature that the top element comes first, and any other element comes after its predecessor. This condition will also hold for the copy $\mathcal{B}$. For $\mathcal{B}$, we label the top node ∞. Having built a finite labeled subtree of $\mathcal{B}$, and determined a tentative partial isomorphism f from this to a subtree of T, we may find that some first node b labeled ∞ in $\mathcal{B}$ is mapped to a node a in T such that T_a is actually finite. The predecessor of b, say b', is labeled ∞, and we may still believe that the predecessor a' of a in T has an infinite tree below. In our $\mathcal{B}$, we vow to add no more terminal nodes to $\mathcal{B}_b$ and we look for a successor a'' of a with the appropriate $T_{a''}$. At a given stage, we take the first a'' that seems to work. Our first guess may not be correct—we may eventually see an unwanted finite node in $T_{a''}$. However, we will eventually find a good a'', with $T_{a''}$ matching our $\mathcal{B}_b$. $\square$

Applying Theorem 2.7, we get a computable copy of T. $\square$

3.6. **Special Boolean algebras.** By the result of Downey and Jockusch stated in the introduction, every low Boolean algebra has a computable copy. In [20], it is shown that for a low Boolean algebra $\mathcal{A}$, there is a computable copy $\mathcal{B}$ with a Δ^0_4 isomorphism. In unpublished work, Frolov proved that this is best possible, in the sense that there is a low Boolean algebra with no Δ^0_3 isomorphism taking $\mathcal{A}$ to a computable copy $\mathcal{B}$. Here we consider Boolean algebras with no 1-atom. This means that every infinite element splits into two infinite elements.

Proposition 3.18. *Suppose $\mathcal{A}$ is an infinite Boolean algebra with no 1-atom. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_3 relative to X.*

Proof. For a tuple $\bar{a}$ in a Boolean algebra, the B_1-type is determined by the sizes of the atoms in the algebra generated by the elements of $\bar{a}$. We have a computable enumeration R of all B_1-types realized in Boolean algebras. We may take R such that from i, we can determine what R_i says about the sizes of elements, whether finite or infinite. The fact that every infinite element of $\mathcal{A}$ splits into two infinite parts implies that $\mathcal{A}$ is weakly 1-saturated. We are assuming that $\mathcal{A}$ is low over X. To show that there is an X-computable copy, it is enough to show the following.

Lemma 3.19. *Let $\mathcal{A}$ be Boolean algebra with no 1-atom. If $\mathcal{A}$ is low over X, then there is a copy $\mathcal{B}$ with an R-labeling that is Δ^0_2 over X, with an isomorphism f that is Δ^0_3 over X.*

Proof. For simplicity, we suppose that $\mathcal{A}$ is low, and our entire construction uses a Δ^0_2 oracle. For notational convenience, when we write $\bar{a} \in \mathcal{A}$ or $\bar{b} \in \mathcal{B}$, we assume that $\bar{a}$ or $\bar{b}$ is the finite sub-algebra determined by a tuple. Since $\mathcal{A}$ is low, at stage s, if we guess that the R-labeling of an element of $\mathcal{A}$ is a particular finite $\leq s$, we are correct. But, we will incorrectly mark as infinite an element of $\mathcal{A}$ with true finite size $u > s$—that incorrect labeling will be corrected at stage u. We always guess the R-label correctly for elements that are truly infinite.

We must computably (relative to Δ^0_2) construct $\mathcal{B}$ with an R-labeling and an isomorphism f between $\mathcal{A}$ and $\mathcal{B}$ that is correct in the limit, so that f is Δ^0_3.

As usual, we have the following requirements.

R_{2a}: $a \in ran(f)$

R_{2b+1}: $b \in dom(f)$

At stage $s = 0$, we define $f(0_\mathcal{A}) = 0_\mathcal{B}$ and $f(1_\mathcal{A}) = 1_\mathcal{B}$; this will never change. We guess that $1_\mathcal{A}$ is labeled with ∞ (this will never be wrong), and we label $1_\mathcal{B}$ with ∞.

Assume that at stage s we have defined $\bar{b} \in \mathcal{B}$ with R-labels and $f_s : \bar{d} \to \bar{c}$ so that the following hold:

(1) the finite algebras $\bar{d}$ and $\bar{c}$ agree;
(2) if $f_s(d) = c$, then the R-label on d matches the stage s appproximation of the R-label on c;
(3) if $f_s(b) = a$, and the finite R-labels among those we have assigned to $\mathcal{B}_s$ imply that there are at least k atoms (of $\mathcal{B}$) below b, then by stage s, we have seen at least k atoms below a.

At stage $s+1$, we use our stage $(s+1)$-approximations of R-labelings of $\mathcal{A}$ to determine how big an "initial segment" (in the sense of the priority requirments) of f_s we can preserve while still satisfying conditions (1)-(3) above. Call this initial segment $f'_{s+1} : \bar{d}' \to \bar{c}'$. Consider the first requirement not satisfied by f'_{s+1}.

Consider when this requirement is R_{2a}. We see how the element a intersects each atom α of the subalgebra $\bar{c}'$, including approximate R-labelings when an α is split by a. (We assume that when an atom α is split by a, that does not reveal any R-labeling inconsistencies in f'_{s+1}. If it does, we have to use a smaller initial segment f''_{s+1}.) For each α that is split by a into α_1 and α_2 with certain R-labels, the corresponding β can be split—using the other elements of $\bar{b}$ or introducing new elements into $\mathcal{B}$ if necessary—into β_1 and β_2 with the same R-labels. Moreover, given the property about labeled atoms satisfied by α and β, the same property can be maintained for α_1 and β_1, and for α_2 and β_2. Using this work for a's intersection with each atom α, we can define f_{s+1} so that $f'_{s+1} \subseteq f_{s+1}$ and f_{s+1} satisfies R_{2a}.

Consider when this requirement is R_{2b+1}. Again, consider how b intersects each atom β of the subalgebra $\bar{d}'$, including R-labelings. (If b has not yet appeared among $\bar{b}$, this is trivial.) The only interesting case is when b splits β into two elements β_1, β_2, which both have R-label ∞. Because $\mathcal{A}$ contains no 1-atom, if α, the atom of $\bar{c}'$ corresponding to β, has a true R-label ∞, then we will discover, after a finite search, the least (in terms of the universe ω) element α_1 below α so that, at stage $s+1$, α_1 and $\alpha - \alpha_1$ both have R-label ∞. Moreover, if the stage s R-labels in $\bar{b}$ imply that β_1 has at least k_1 atoms (of $\mathcal{B}$) and β_2 has at least k_2 many, then by assumption we have seen at least $k_1 + k_2$ atoms (of $\mathcal{A}$) below α. Consider the element $\alpha_1\pm$ finitely many atoms below α so that this new element α_1' has at least k_1 atoms below it, and $\alpha - \alpha_1'$ has at least k_2 atoms below it. Map β_1, β_2 to these two elements, respectively. Using this work for b's intersection with each atom β, we can define f_{s+1} so that $f'_{s+1} \subseteq f_{s+1}$ and f_{s+1} satisfies R_{2b+1}.

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Department of Mathematics, Southern Illinois University, USA
E-mail address: wcalvert@siu.edu

Department of Mathematics, Kazan Federal University, Russia
E-mail address: Andrey.Frolov@ksu.ru

Department of Mathematics, George Washington University, USA
E-mail address: harizanv@gwu.edu

Department of Mathematics, University of Notre Dame, USA
E-mail address: knight.1@nd.edu

Department of Mathematics, University of Portland, USA
E-mail address: mccoy@up.edu

Department of Mathematical Logic, Sofia University, Bulgaria
E-mail address: asoskova@fmi.uni-sofia.bg

Department of Mathematical Logic, Sofia University, Bulgaria
E-mail address: stefanv@fmi.uni-sofia.bg