Π^0_1 CLASSES AND STRONG DEGREE SPECTRA OF RELATIONS

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Abstract. We study the weak truth-table and truth-table degrees of the images of subsets of computable structures under isomorphisms between computable structures. In particular, we show that there is a low c.e. set that is not weak truth-table reducible to any initial segment of any scattered computable linear order. Countable Π^0_1 subsets of 2^ω and Kolmogorov complexity play a major role in the proof.

1. Introduction, notation, and preliminaries

While in classical mathematics isomorphic structures are often identified, in computable mathematics they can have very different algorithmic properties. One of the important questions in computable model theory is how a specific substructure or a relation on a computable structure may change if the structure is isomorphically transformed to another computable structure. A structure $\mathcal{A}$ is *computable* if its domain is a computable set and its relations and functions are uniformly computable; or equivalently, if the atomic diagram of $\mathcal{A}$ is computable. Let R be an additional relation on the domain of a computable structure $\mathcal{A}$; that is, R is not named in the language of $\mathcal{A}$. The study of such relations began with the work of Ash and Nerode [3], who found a syntactic condition under which, for every isomorphism f from $\mathcal{A}$ onto a computable structure $\mathcal{B}$, the image $f(R)$ is computably enumerable (c.e.). Such relations are called *intrinsically c.e.* on $\mathcal{A}$. This notion generalizes to *intrinsically* $\mathcal{P}$ on $\mathcal{A}$ for a computability theoretic class $\mathcal{P}$ of relations. Barker [5] studied intrinsically Σ^0_α (where α is a computable ordinal) relations and their syntactic characterizations.

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Another approach to the study of relations on computable structures is to look at the set of degrees of images of such a relation under isomorphisms from the original structure to its computable copies. More precisely, Harizanov [16] defined the *Turing degree spectrum* of R on $\mathcal{A}$, in symbols $\mathrm{DgSp}_{\mathcal{A}}(R)$, to be the set of all Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ to computable models. For a computable structure $\mathcal{B}$ such that $\mathcal{B} \cong \mathcal{A}$, the *Turing degree spectrum of R on $\mathcal{A}$ with respect to $\mathcal{B}$*, in symbols $\mathrm{DgSp}_{\mathcal{A},\mathcal{B}}(R)$, is the set of all Turing degrees of the images $f(R) \subseteq |\mathcal{B}|$ (where $|\mathcal{B}|$ is the domain of $\mathcal{B}$) under all isomorphisms f from $\mathcal{A}$ to $\mathcal{B}$. The study of Turing degree spectra has been continued in many papers by Ash, Cholak, Downey, Goncharov, Harizanov, Hirschfeldt, Khoussainov, Knight, Shore, White, and others (for example, see [1, 2, 11, 15, 17, 18, 19, 20, 21, 22, 23, 28]).

Our computability-theoretic notions and notation are standard and as in [35] and [37]. In particular, we use $\langle\ \rangle$ to denote a computable coding of a finite string of natural numbers. Our emphasis is on degree structures based on "strong reducibilities" that imply Turing reducibility.

We say that X is *weak truth-table reducible* to Y ($X \leq_{\mathrm{wtt}} Y$) if there exist e and a computable function h such that $X(x) = \varphi_e^{(Y \upharpoonright h(x))}(x)$ for all $x \in \omega$. That is, $X = \varphi_e^Y$, with use of the oracle Y in this computation bounded by the computable function h. When this occurs we shall say that $X \leq_{\mathrm{wtt}} Y$ *via φ_e and h*.

Similarly, X is *truth-table reducible* to Y ($X \leq_{\mathrm{tt}} Y$) if there exist e and a computable function h such that $X = \varphi_e^Y$ and for each n, the computation $\varphi_e^\sigma(n)$ converges for every $\sigma \in 2^{h(n)}$. When this occurs we shall say that $X \leq_{\mathrm{tt}} Y$ *via φ_e and h*. (Strictly speaking, it is not necessary to mention h here, but we do so to maintain parallelism with the wtt-case.)

It is well known that

$$X \leq_{\mathrm{tt}} Y \implies X \leq_{\mathrm{wtt}} Y \implies X \leq_{\mathrm{T}} Y.$$

We write $\deg(X)$ for the Turing degree of X, $\deg_{\mathrm{tt}}(X)$ for the truth-table degree of X, and $\deg_{\mathrm{wtt}}(X)$ for the weak truth-table degree of X. The set of all Turing degrees is denoted by $\mathcal{D}$, the set of all truth-table degrees by $\mathcal{D}^{\mathrm{tt}}$, and the set of all weak truth-table degrees by $\mathcal{D}^{\mathrm{wtt}}$. We define the *truth-table* (*weak truth-table*) *degree spectrum* of R on $\mathcal{A}$, in symbols $\mathrm{DgSp}_{\mathcal{A}}^{\mathrm{tt}}(R)$ ($\mathrm{DgSp}_{\mathcal{A}}^{\mathrm{wtt}}(R)$), to be the set of all truth-table (weak truth-table) degrees of the images of R under all isomorphisms from $\mathcal{A}$ to computable models.

Harizanov [16] described natural conditions under which the Turing degree spectrum of a relation R on a structure $\mathcal{A}$ realizes all Turing degrees. Here we show that these conditions similarly ensure that all tt-degrees appear in the truth-table degree spectrum $\mathrm{DgSp}^{\mathrm{tt}}_{\mathcal{A}}(R)$.

We also investigate the wtt-spectrum of relations on computable linear orderings $\mathcal{L}$ of order type $\omega + \omega^*$. In particular, we consider the set $\omega_{\mathcal{L}}$, the ω-part of $\mathcal{L}$, containing the elements in the universe of $\mathcal{L}$ having only finitely many predecessors. We find that, although $\mathrm{DgSp}_{\mathcal{L}}(\omega_{\mathcal{L}})$ is exactly the collection of Δ^0_2 degrees, it is not true that $\mathrm{DgSp}^{\mathrm{wtt}}_{\mathcal{L}}(\omega_{\mathcal{L}}) = \{\mathbf{a} \in \mathcal{D}^{\mathrm{wtt}} : \mathbf{a} \leq_{\mathrm{wtt}} \mathbf{0}'\}$. In fact, we construct a c.e. set D that is not wtt-reducible to any initial segment of any computable scattered linear ordering. To do this, we show that D is not wtt-reducible to any set that belongs to any countable Π^0_1 subset of 2^ω. We further show that D may be chosen to be of low Turing degree, and we discuss the degrees of such sets D.

Although this result is a negative one, we also obtain a partial positive result: For any Δ^0_2 set A, there exists a computable linear ordering $\mathcal{L}$ of type $\omega + \omega^*$ so that $A \leq_{\mathrm{T}} \omega_{\mathcal{L}} \leq_{\mathrm{tt}} A$.

2. Uncountable Turing degree spectra

In [16], Harizanov studied uncountable degree spectra, and established natural conditions that are sufficient for $\mathrm{DgSp}_{\mathcal{A}}(R)$ to contain all Turing degrees. In [18], Harizanov proved that these conditions are also necessary. Another proof was obtained independently by Ash, Cholak, and Knight in [1].

Let $\mathcal{A}$ be a computable structure and let R be an extra relation on $\mathcal{A}$. We will assume, without loss of generality, that R is unary, since it is well-known that the general case can easily be reduced to that of unary relations. Let $\mathcal{B}$ be a computable structure such that $\mathcal{A} \cong \mathcal{B}$.

We say that a partial function p from $|\mathcal{A}|$ to $|\mathcal{B}|$ is a *finite partial isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if p is 1-1, $\mathrm{dom}(p)$ is finite, and for every atomic formula $\psi = \psi(x_0, \ldots, x_{n-1})$ and every $a_0, \ldots, a_{n-1} \in \mathrm{dom}(p)$, we have

$$\mathcal{A} \vDash \psi[a_0, \ldots, a_{n-1}] \iff \mathcal{B} \vDash \psi[b_0, \ldots, b_{n-1}],$$

where $b_0 = p(a_0), \ldots, b_{n-1} = p(a_{n-1})$. We denote the set of all finite partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$ by $\mathcal{I}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$. In [16], the following equivalence relation $\sim_R$ on $\mathcal{I}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ is defined:

$$\begin{aligned} q \sim_R r &\iff (\forall b \in \mathrm{ran}(q) \cap \mathrm{ran}(r))\ [q^{-1}(b) \in R \Leftrightarrow r^{-1}(b) \in R] \\ &\iff (\forall b \in \mathrm{ran}(q) \cap \mathrm{ran}(r))\ [b \in q(R) \Leftrightarrow b \in r(R)]. \end{aligned}$$

Theorem 2.1 (Harizanov [16]). *Let $\mathcal{A}$ be a computable structure and R an additional computable relation on $\mathcal{A}$.*

(1) *The following are equivalent:*

(a) $\mathrm{DgSp}_{\mathcal{A}}(R)$ *is uncountable.*
(b) *For every computable model* $\mathcal{B}$ *isomorphic to* $\mathcal{A}$*, the degree spectrum* $\mathrm{DgSp}_{\mathcal{A},\mathcal{B}}(R)$ *is uncountable.*
(c) *For every computable model* $\mathcal{B}$ *isomorphic to* $\mathcal{A}$*, the degree spectrum* $\mathrm{DgSp}_{\mathcal{A},\mathcal{B}}(R)$ *has cardinality* 2^{ω}*.*
(d) *For every computable model* $\mathcal{B}$ *isomorphic to* $\mathcal{A}$*, there is a nonempty set* $\mathbb{S} \subseteq \mathcal{I}_{fin}(\mathcal{A},\mathcal{B})$ *such that the following two conditions are satisfied:*
   (i) $(\forall p \in \mathbb{S})(\forall a \in A)(\forall b \in B)(\exists q \in \mathbb{S})$ $[q \supseteq p \wedge a \in \mathrm{dom}(q) \wedge b \in \mathrm{ran}(q)]$*, and*
   (ii) $(\forall p \in \mathbb{S})(\exists q, r \in \mathbb{S})$ $[q \supseteq p \wedge r \supseteq p \wedge \neg(q \sim_R r)]$.

(2) *Let a family* $\mathbb{S} \subseteq \mathcal{I}_{fin}(\mathcal{A},\mathcal{B})$ *satisfy conditions* (i) *and* (ii) *of part* (d)*. Then for every set* $C \subseteq \omega$ *such that* $C \geq_T \mathbb{S}$*, there is an isomorphism* f *from* $\mathcal{A}$ *to* $\mathcal{B}$ *for which we have*

$$C \equiv_T f(R) \oplus \mathbb{S} \equiv_T f \oplus \mathbb{S}.$$

In particular, if $\mathbb{S}$ *is computable (or even just c.e.), then we have* $\mathrm{DgSp}_{\mathcal{A},\mathcal{B}}(R) = \mathcal{D}$ *and, moreover, for every set* $C \subseteq \omega$*, there is an isomorphism* f *from* $\mathcal{A}$ *to* $\mathcal{B}$ *such that*

$$C \equiv_T f(R) \equiv_T f.$$

For example, if $\mathcal{Q} = (\mathbb{Q}, \leq)$, where $\mathbb{Q}$ is the set of all rational numbers, and $R = \{q \in \mathbb{Q} : q < \sqrt{2}\}$, then $\mathrm{DgSp}_{\mathcal{Q},\mathcal{Q}}(R) = \mathcal{D}$. In [16], there are also examples of uncountable Turing degree spectra with $\mathrm{DgSp}_{\mathcal{A},\mathcal{B}}(R) \neq \mathcal{D}$.

Theorem 2.2 (Ash-Cholak-Knight [1], Harizanov [18])**.** *Let* $\mathcal{A}$ *be a computable structure with an additional computable relation* R*.*

(1) *The following are equivalent:*
   (a) $\mathrm{DgSp}_{\mathcal{A}}(R) = \mathcal{D}$ *and, moreover, for every set* $C \subseteq \omega$*, there is an isomorphism* f *from* $\mathcal{A}$ *onto a computable* $\mathcal{B}$ *such that*

$$C \equiv_T f(R) \equiv_T f.$$

   (b) *For every set* C*, there is an automorphism* f *of* $\mathcal{A}$ *such that*

$$C \equiv_T f(R) \equiv_T f.$$

   (c) *There is a nonempty computable (or even c.e.) family* $\mathbb{S} \subseteq \mathcal{I}_{fin}(\mathcal{A},\mathcal{A})$ *such that the conditions* (i) *and* (ii) *from part* (d) *of Theorem 2.1 are satisfied.*

(2) *Let* $\mathcal{B}$ *be a computable structure isomorphic to* $\mathcal{A}$*. If there is a Turing degree* $\mathbf{d}$ *that cannot be obtained in* $\mathrm{DgSp}_{\mathcal{A},\mathcal{B}}(R)$ *via an isomorphism of degree* $\mathbf{d}$*, then there is such a degree that is* Δ^0_3*.*

3. Countable and uncountable strong degree spectra

We aim to show that if the Turing degree spectrum of a computable relation on a computable structure is as large as possible, then under certain conditions the same is true of the truth-table degree spectrum, and hence also of the wtt-degree spectrum. In particular, we show that when the conditions in Theorem 2.2(1) hold, the images of R under isomorphisms to computable copies of $\mathcal{A}$ realize all tt-degrees.

Theorem 3.1. *Let $\mathcal{A}$ be a computable structure, and suppose that R is a computable relation on $\mathcal{A}$. Assume further that the conditions of part* (1) *of Theorem 2.2 are satisfied. Then* $\mathrm{DgSp}^{tt}_{\mathcal{A}}(R) = \mathcal{D}^{tt}$.

Proof. Assume that condition (c) of Theorem 2.2(1) holds, that is, that there is a c.e. $\mathbb{S} \subset I_{\text{fin}}(\mathcal{A}, \mathcal{A})$ satisfying the conditions of Theorem 2.1(1)(d). We will show that for every $X \in 2^\omega$ there is an automorphism $f_X : \mathcal{A} \to \mathcal{A}$ such that $X \equiv_{\text{tt}} f_X(R)$.

To do this, we will construct computable maps $\mathcal{T} : 2^{<\omega} \to \mathbb{S}$ and $\mathcal{N} : 2^{<\omega} \to \omega$, via the following recursive procedure.

Initialize by setting $\mathcal{T}(\langle\rangle) = \emptyset$. Then, given $\mathcal{T}(\sigma) = r$, produce $\mathcal{N}(\sigma)$, $\mathcal{T}(\sigma\hat{\ }0)$, and $\mathcal{T}(\sigma\hat{\ }1)$ as follows.

Search effectively for the least $n \in \omega$ and first $r_0, r_1 \in \mathbb{S}$ extending r such that $r_1^{-1}(n) \in R$ and $r_0^{-1}(n) \notin R$. Such n, r_0, r_1 must exist by property (1)(d)(ii) of Theorem 2.1. Let $\mathcal{N}(\sigma) = n$.

Then, for $i = 0, 1$, effectively find the first $q_i \supseteq r_i$ in $\mathbb{S}$ so that $\mathrm{dom}(q_i)$ contains the least element of $\mathcal{A}$ not in $\mathrm{dom}(r)$, and $\mathrm{ran}(q_i)$ contains the least element of $\mathcal{A}$ not in $\mathrm{ran}(r)$. Note that q_i must exist by property (1)(d)(i) of Theorem 2.1. Let $\mathcal{T}(\sigma\hat{\ }i) = q_i$.

Now, given $X \in 2^\omega$, let

$$f_X = \bigcup_l \mathcal{T}(X \upharpoonright l).$$

By construction, f_X is an automorphism of $\mathcal{A}$.

Lemma 3.2. *For each $X \in 2^\omega$, we have $f_X(R) \leq_{tt} X$.*

Proof. To decide whether $m \in f_X(R)$, find the least l such that $m \in \mathrm{ran}(\mathcal{T}(\sigma))$ for every $\sigma \in 2^l$. Then

$$m \in f_X(R) \iff (\mathcal{T}(X \upharpoonright l))^{-1}(m) \in R.$$

From this observation we conclude that $f_X(R) \leq_{\text{tt}} X$ via the algorithm described above and $h(m) = l$. □

Lemma 3.3. *For each $X \in 2^\omega$, we have $X \leq_{tt} f_X(R)$.*

Proof. Given $X \upharpoonright l = \sigma$, we have

$$X \upharpoonright (l+1) = \sigma\hat{\ }1 \iff \mathcal{T}(\sigma\hat{\ }1) \subset f_X \iff \mathcal{N}(\sigma) \in f_X(R).$$

Thus X (and indeed f_X) may be reconstructed in a tt-way given knowledge of $f_X(R)$. $\square$

The previous two lemmas establish that $f_X(R) \equiv_{\mathrm{tt}} X$ so that all tt-degrees are attained in the tt-degree spectrum of R on $\mathcal{A}$.

Additionally, the proof of Lemma 3.3 shows that $f_X \leq_{\mathrm{tt}} f_X(R)$. We may remark that $f_X(R)$ is tt-reducible to f_X within the class of total functions. That is, there is an e so that φ_e^f is a total functional on the class of total functions, and $\varphi_e^{f_X}$ is the characteristic function of $f_X(R)$. $\square$

We now consider countable strong degree spectra. Let $\mathcal{L}_0$ denote some standard computable linear ordering of order type $\omega + \omega^*$. For example, let $|\mathcal{L}_0| = \{a_n\}_{n \in \omega}$ and let $\prec$ denote the ordering relation of $\mathcal{L}$ with

$$a_0 \prec a_2 \prec a_4 \prec \cdots \prec a_5 \prec a_3 \prec a_1.$$

For any linear ordering $\mathcal{L} \cong \omega + \omega^*$, let $\omega_{\mathcal{L}}$ denote the ω-part of $\mathcal{L}$. For the linear ordering $\mathcal{L}_0$ defined above, $\omega_{\mathcal{L}_0} = \{a_0, a_2, a_4, \ldots\}$.

Note that

$$\mathrm{DgSp}_{\mathcal{L}_0}(\omega_{\mathcal{L}_0}) = \{\mathbf{a} : \mathbf{a} \leq \mathbf{0}'\}$$

(Proposition 3.1 of [18]). To see that $\mathrm{DgSp}_{\mathcal{L}_0}(\omega_{\mathcal{L}_0}) \subseteq \{\mathbf{a} : \mathbf{a} \leq \mathbf{0}'\}$, let $\mathcal{L}$ be any computable linear ordering of order type $\omega + \omega^*$, and note that $\omega_{\mathcal{L}}$ is the set of elements of L with finitely many predecessors and hence is a Σ^0_2 set. The complement of this set is also Σ^0_2 by a similar argument, and so $\omega_{\mathcal{L}} \leq_{\mathrm{T}} \emptyset'$. The reverse inclusion is shown in Theorem 5.2 of [24], which establishes that each nonzero Turing degree $\mathbf{a} \leq \mathbf{0}'$ contains an immune, coimmune semirecursive set A. It is shown in Theorem 4.1(iii) of [24] (due to K. Appel and T. McLaughlin) that the semirecursive sets are precisely the initial segments of computable linear orderings. It is easily seen that if a computable linear ordering $\mathcal{L}$ has an immune, coimmune initial segment A, then $\mathcal{L}$ has order type $\omega + \omega^*$ and $\omega_{\mathcal{L}} = A$. (For example, each element of A has only finitely many predecessors, since otherwise the set of its predecessors would be an infinite c.e. subset of A.) One might hope, in analogy to Theorem 3.1, that the wtt-spectrum of $\omega_{\mathcal{L}}$ contains all wtt-degrees below that of the halting set, $\emptyset'$. This is not true, as we shall see, but the following result goes halfway in that direction.

Theorem 3.4. *For every Δ^0_2 set A, there exists a computable linear ordering $\mathcal{L}$ of order type $\omega + \omega^*$ such that $A \leq_T \omega_{\mathcal{L}} \leq_{tt} A$. Furthermore, if A is c.e., we may require in addition that $\omega_{\mathcal{L}}$ be c.e.*

Proof. We first apply the construction used to prove Theorem 5.2 of [24], which asserts that every nonzero degree below $\mathbf{0}'$ contains a semirecursive set that is immune and coimmune. We review this construction here for the convenience of the reader. Define $r_A = \sum_{n\in A} 2^{-n}$, and for each x define $r_x = \sum_{n\in D_x} 2^{-n}$, where D_x is the nonempty finite set with canonical index x. Let $X_0 = \{x : r_x \leq r_A\}$. By the proof of Theorem 3.6 of [24], $A \equiv_{\mathrm{tt}} X_0$.

We consider first the case where neither X_0 nor its complement is c.e., so A is *strongly non-c.e.* in the sense of [24]. Since A is Δ^0_2, there is a computable function f such that $\lim_{n\to\infty} r_{f(n)} = r_A$. Define

$$X = \{n : f(n) \in X_0\} = \{n : r_{f(n)} \leq r_A\}.$$

Then $X \leq_{\mathrm{m}} X_0 \leq_{\mathrm{tt}} A$. Also, by Lemma 5.5 of [24], $A \leq_{\mathrm{T}} X$. (To apply this lemma, we need the assumption that A is strongly non-c.e.) By the same lemma, X is semirecursive and bi-immune. Thus, as explained just before the statement of Theorem 3.4, there is a computable linear ordering $\mathcal{L}$ of order type $\omega + \omega^*$ such that $X = \omega_{\mathcal{L}}$. This concludes the proof of the theorem in the case where A is strongly non-c.e.

Suppose now that X_0 or its complement is c.e. Since each of these sets is tt-equivalent to A (as mentioned just above), we may use either of these sets in place of A, and thus we may assume without loss of generality that A is c.e. Thus, the entire proof of the theorem will be complete if we prove it for the c.e. case. Note that we may further assume that A is not computable, since otherwise the theorem is obviously true.

Assuming now that A is c.e. and noncomputable, we let B be the Dekker deficiency set of A, that is $B = \{s | (\exists a \in A_s)(\exists t > s)(\exists b < a)[b \in A_t \wedge b \notin A_s]\}$ (see [37], Theorem V.2.5). Recall that B is hypersimple. Then the standard proof that $B \equiv_{\mathrm{T}} A$ in fact shows that $A \leq_{\mathrm{T}} B \leq_{\mathrm{tt}} A$. Thus, it suffices to construct a computable linear ordering $\mathcal{L}$ of order type $\omega + \omega^*$ such that $B = \omega_{\mathcal{L}}$. It follows from Theorem 3.2 of [24] that B is semirecursive, and thus, by Theorem 4.1(iii) of [24], that B is an initial segment of some computable linear ordering $\mathcal{L}_0$. Note that the order type of the restriction of $\mathcal{L}_0$ to $\overline{B}$ is ω^*, since $\overline{B}$ is immune. We produce $\mathcal{L}$ from $\mathcal{L}_0$ by 'rearranging' the elements of B so that they will have order type ω, while keeping all elements of B below all elements of $\overline{B}$ and leaving the ordering of elements of $\overline{B}$ unchanged.[1] In more detail, suppose that we have

[1]There is a related result in Theorem 4 of [4], but without degree theoretic considerations. There Barmpalias established that a noncomputable c.e. set is hypersimple and semirecursive if and only if it is the ω-part of a computable linear ordering of type $\omega + \omega^*$.

defined $\mathcal{L}$ on all numbers $< n$. We now place n among those numbers in the ordering $\mathcal{L}$. Let

$$I(B_n) = \{i < n : (\exists k < n)\ [k \in B_n \ \wedge\ i \leq_{\mathcal{L}} k]\},$$

where $\{B_s\}_{s \in \omega}$ is a computable enumeration of B. If $i \in I(B_n)$, decree that $i <_{\mathcal{L}} n$. Suppose now that $i < n$ and $i \notin I(B_n)$. Then let the ordering of i and n in $\mathcal{L}$ be the same as their ordering in $\mathcal{L}_0$.

To see that this construction works, verify by induction on n that if $i, j < n$ for $j \in B$, and $i <_{\mathcal{L}} j$, then $i \in B$. If we assume this for n, it follows that $I(B_n) \subseteq B$, since each element of $I(B_n)$ is $<_{\mathcal{L}}$ some element of B. The construction now places n above all elements of $\mathcal{L}$ that are in B, and in the same relative position to elements of $\overline{B}$ that are less than n, thus ensuring that if $n \in B$ then all elements of $\overline{B}$ appear above it in the ordering. So this downward closure property holds for $n + 1$.

Transitivity is also easily verified by showing by induction on n that $\mathcal{L}$ is transitive on $\{i : i < n\}$. Finally, if $i \in B$, then $i \in B_n \subseteq I(B_n)$ for all sufficiently large n, so, by construction, i has only finitely many predecessors in $\mathcal{L}$. □

4. Scattered orderings, ranked sets, and Kolmogorov complexity

A linear ordering $\mathcal{L}$ is *scattered* if it fails to contain a copy of $\mathbb{Q}$. That is, there is no subset $S \subset |\mathcal{L}|$ so that $(\mathbb{Q}, <_{\mathbb{Q}}) \cong (S, <_{\mathcal{L}})$.

We showed in Theorem 3.4 that for every Δ^0_2 set A, there exists a computable linear ordering $\mathcal{L}$ of order type $\omega + \omega^*$ such that $A \leq_{\mathrm{T}} \omega_{\mathcal{L}} \leq_{\mathrm{tt}} A$. We will now use results from algorithmic information theory to show that this result fails if Turing reducibility is replaced by wtt-reducibility. In fact we prove a far stronger theorem: there is a c.e. set D that is not wtt-reducible to any initial segment of any scattered linear ordering. The results in this section and the next section were originally proved without using algorithmic information theory. However, the use of algorithmic information theory has greatly simplified the arguments.

The following notion is central to our proof.

Definition 4.1 ([7])**.** A real $f \in 2^\omega$ is said to be *ranked* if $f \in P$ for some countable Π^0_1 class $P \subseteq 2^\omega$.

There is an extensive literature on ranked sets. See, for example, [7, 8, 10, 29]. It was shown by Kreisel [29] that every ranked set is hyperarithmetic, and it is known that every hyperarithmetic set is Turing reducible to some ranked set. The ranked sets behave particularly well under tt-reducibility, since every set tt-reducible to a ranked set is ranked (see [8], Lemma 1.2(b)). Since $\emptyset'$ is not ranked, by [8], Corollary 6.2, it follows from these results of Cenzer and Smith that $\emptyset'$ is not tt-reducible to any ranked set. We extend this result in Corollary 4.8 from tt-reducibility to wtt-reducibility. However, it will follow from Theorem 5.13 that not every set that is wtt-reducible to a ranked set is itself ranked. The following lemma is known, and is used in the proof of Proposition 4.3 relating ranked sets to initial segments of scattered linear orderings.

Lemma 4.2. *Let $\mathcal{L}$ be a countable linear ordering. Then $\mathcal{L}$ is scattered iff $\mathcal{L}$ has only countably many initial segments.*

Proof. Assume first that $\mathcal{L}$ is not scattered, and let $S \subseteq |\mathcal{L}|$ be such that $(S, <_{\mathcal{L}}) \cong (\mathbb{Q}, <)$. Then S has uncountably many initial segments since $\mathbb{Q}$ has. For each initial segment I of S, let $D(I)$ be its downward closure in $\mathcal{L}$. Then $D(I)$ is an initial segment of $\mathcal{L}$, and D is an injection, so $\mathcal{L}$ has uncountably many initial segments.

For the converse, assume that $\mathcal{L}$ has uncountably many initial segments. Assume the universe of $\mathcal{L}$ is ω. We must show that $\mathcal{L}$ is not scattered. Let $\mathcal{I}$ be the set of initial segments of $\mathcal{L}$. Then $\mathcal{I}$ is an uncountable closed set in the Cantor space 2^ω, and so has a perfect subset $\mathcal{J}$. Let $T \subseteq 2^{<\omega}$ be a perfect tree such that $[T] = \mathcal{J}$, where $[T]$ is the set of paths through T. Define an extension-preserving map $U : 2^{<\omega} \to T$ by recursion in the standard fashion: let $U(\langle\,\rangle) = \langle\,\rangle$, and for all σ, let $U(\sigma\hat{\ }\langle 0\rangle)$ and $U(\sigma\hat{\ }\langle 1\rangle)$ be the least incompatible extensions of $U(\sigma)$ in T. For each $\sigma \in 2^{<\omega}$, choose a number a_σ on which the strings $U(\sigma\hat{\ }\langle 0\rangle)$ and $U(\sigma\hat{\ }\langle 1\rangle)$ are defined and disagree. Let $S = \{a_\sigma : \sigma \in 2^{<\omega}\}$. Then S has the order type of $\mathbb{Q}$ under $\mathcal{L}$. First, S has no least or greatest element because a_σ lies strictly between $a_{\sigma\hat{\ }\langle 0\rangle}$ and $a_{\sigma\hat{\ }\langle 1\rangle}$ in $\mathcal{L}$. To show that S is dense, let μ and τ be distinct strings in $2^{<\omega}$. If μ and τ are incompatible, let σ be the longest string extended by both μ and τ. The reader may easily verify that a_σ is strictly $\mathcal{L}$-between a_μ and a_τ. If μ is a proper extension of τ, then either $a_{\mu\hat{\ }\langle 0\rangle}$ or $a_{\mu\hat{\ }\langle 1\rangle}$ is strictly $\mathcal{L}$-between a_μ and a_τ. □

Proposition 4.3. *Let $\mathcal{L}$ be a scattered computable linear ordering. Then every initial segment of $\mathcal{L}$ is ranked.*

Proof. Let P be the class of initial segments of $\mathcal{L}$. It is easy to see that P is a Π^0_1 class. Since $\mathcal{L}$ is scattered, it has only countably many initial segments. Thus P is countable, and every element of P is ranked. □

We denote the (plain) Kolmogorov complexity of a string σ by $C(\sigma)$. For a definition of Kolmogorov complexity and a thorough discussion of its properties, see [31]. We now introduce the key notion from algorithmic information theory that we will employ.

Definition 4.4. An *order* is a computable nondecreasing unbounded function. A set A is *complex* if there is an order g such that

$$\forall n\, C(A \upharpoonright n) \geq g(n).$$

Recall that a function f is *diagonally non-computable* (abbreviated by DNC) if for each $e \in \omega$, the value of $\varphi_e(e)$ differs from $f(e)$ when the former is defined.

Theorem 4.5 (Kjos-Hanssen, Merkle, and Stephan [30])**.** *A set A is complex iff there is a DNC function $f \leq_{wtt} A$.*

The following corollary can also be proved directly.

Corollary 4.6. *The complex sets are closed upwards under wtt-reducibility.*

Theorem 4.7. *Let P be a Π^0_1 class with a complex element A. Then P has a perfect Π^0_1 subclass Q with $A \in Q$.*

Proof. Let g be an order such that $\forall n\, C(A \upharpoonright n) \geq g(n)$. Let

$$Q = \{X \in P : \forall n\, C(X \upharpoonright n) \geq g(n)\}.$$

Then Q is a Π^0_1 subclass of P, and is nonempty, since $A \in Q$. However, every element of Q is complex, so Q cannot have any isolated elements, since any such element would be computable, and hence not complex. Thus Q is perfect. □

Call a set D *superlow* if $D' \leq_{\mathrm{wtt}} \emptyset'$. This definition is due to Mohrherr [33] (see also Bickford and Mills [6] and Mohrherr [34]), and the concept has been investigated by a number of authors.

Corollary 4.8.

(1) *If $\emptyset' \leq_{wtt} A$, then A is not ranked, and, in fact, every Π^0_1 class P with $A \in P$ has a perfect Π^0_1 subclass Q with $A \in Q$.*
(2) *If $\emptyset' \leq_{wtt} A$, then A is not an initial segment of any computable scattered linear ordering.*
(3) *There is a superlow set D such that* (1) *and* (2) *hold with $\emptyset'$ replaced by D. Thus, D cannot be wtt-reduced to any ranked set or to any initial segment of any computable scattered linear ordering.*

Proof. To prove the first part, simply note that there is a DNC function $f \leq_{\mathrm{wtt}} \emptyset'$. The second part follows at once from the first part and Proposition 4.3. To prove the third part, recall that the $\{0,1\}$-valued

DNC functions form a nonempty Π^0_1 class in 2^ω. The proof of the Low Basis Theorem (see [27]) shows that every such class has a superlow element. □

We now prove a strengthened version of part (2) of Corollary 4.8.

Corollary 4.9. *If $\emptyset'$ is wtt-reducible to an initial segment of a computable linear ordering $\mathcal{L}$, then $|\mathcal{L}|$ has a subset $S \leq_T \emptyset'$ such that $(S, <_{\mathcal{L}}) \cong (\mathbb{Q}, <)$.*

Proof. Let $\mathcal{L}$ be a computable linear ordering and suppose that $\emptyset' \leq_{\mathrm{wtt}} A$, where A is an initial segment of $\mathcal{L}$. Let P be the Π^0_1 class of all initial segments of $\mathcal{L}$. By Corollary 4.8, P has a perfect Π^0_1 subclass Q. Let T be the set of all strings in $2^{<\omega}$ that can be extended to elements of Q. Then T is a perfect tree and clearly $T \leq_{\mathrm{T}} \emptyset'$. So the proof of Lemma 4.2 yields $S \leq_{\mathrm{T}} \emptyset'$ such that $S \subseteq |\mathcal{L}|$ and $(S, <_{\mathcal{L}}) \cong (\mathbb{Q}, <)$. □

The following result is a modified converse to Corollary 4.9.

Proposition 4.10. *Suppose that a computable linear ordering $\mathcal{L}$ has a computable subset S such that $(S, <_{\mathcal{L}}) \cong (\mathbb{Q}, <)$. Then for every set $D \subseteq \omega$ there is an initial segment I of $\mathcal{L}$ such that $I \equiv_{tt} D$.*

Proof. We can assume that the universe of $\mathcal{L}$ is ω. For numbers $l <_{\mathcal{L}} r$, let (l, r) denote the set of all z such that $l <_{\mathcal{L}} z <_{\mathcal{L}} r$. When we say that a triple x, y, z of natural numbers is the least one with a certain property, we mean that it is the triple for which $\langle x, y, z\rangle$ is least among those with the given property.

We build I by stages. At each stage n, we will choose elements l_n, x_n, r_n of S with $l_n <_{\mathcal{L}} x_n <_{\mathcal{L}} r_n$ in such a way that $n \notin (l_n, r_n)$. At the beginning of stage $n+1$, the value of $I(z)$ will have been decided iff $z \notin (l_n, r_n)$. We will then decide the value of $I(x_n)$ depending on whether $n \in D$.

Stage 0. Let l_0, x_0, r_0 with $l_0 <_{\mathcal{L}} x_0 <_{\mathcal{L}} r_0$ be the least triple of elements of S such that $0 \notin (l_0, r_0)$. Declare that every $z \leq_{\mathcal{L}} l_0$ is in I, and that every $z \geq_{\mathcal{L}} r_0$ is not in I.

Stage $n+1$. Given $l_n <_{\mathcal{L}} x_n <_{\mathcal{L}} r_n$, we have the following cases.

(1) Suppose $n \in D$. Let $l_{n+1}, x_{n+1}, r_{n+1}$ with $l_{n+1} <_{\mathcal{L}} x_{n+1} <_{\mathcal{L}} r_{n+1}$ be the least triple of elements of $S \cap (l_n, r_n)$ such that $n+1 \notin (l_{n+1}, r_{n+1})$ and $x_n <_{\mathcal{L}} l_{n+1}$.
(2) Suppose $n \notin D$. Let $l_{n+1}, x_{n+1}, r_{n+1}$ with $l_{n+1} <_{\mathcal{L}} x_{n+1} <_{\mathcal{L}} r_{n+1}$ be the least triple of elements of $S \cap (l_n, r_n)$ such that $n+1 \notin (l_{n+1}, r_{n+1})$ and $r_{n+1} <_{\mathcal{L}} x_n$.

In either case, declare that every $z \leq_{\mathcal{L}} l_{n+1}$ is in I, and that every $z \geq_{\mathcal{L}} r_{n+1}$ is not in I. Note that this action ensures that $I(n+1)$ is decided at this stage, and that $x_n \in I$ iff $n \in D$.

This completes the construction.

Since the value of $I(n+1)$ is decided at stage $n+1$, we have a functional Ψ such that $I = \Psi^D$. Furthermore, Ψ is total, so $I \leq_{\mathrm{tt}} D$.

In the other direction, consider the functional Ψ defined as follows on oracle Y. First, let $\Psi^Y(0) = Y(x_0)$. Given $\Psi^Y(0), \ldots, \Psi^Y(k)$, run the above construction up to stage $k+1$, with Ψ^Y in place of D. (That is, replace the question of whether $n \in D$ by the question of whether $\Psi^Y(n) = 1$.) This process yields a value for x_{k+1}. Now let $\Psi^Y(k+1) = Y(k+1)$. It is easy to check that $\Psi^I = D$. Since Ψ is total, $D \leq_{\mathrm{tt}} I$. $\square$

It is natural to ask whether there is a low (or superlow) c.e. set C such that C cannot be wtt-reduced to any ranked set. The methods of this section do not suffice to answer this question because they depend on making C complex. If C is complex and c.e., then, by Theorem 4.5, there is a DNC function $f \leq_{\mathrm{wtt}} C$, and it follows from the proof of the Arslanov Completeness Theorem (see [37], Theorem 5.1 on p. 88) that C is wtt-complete. We answer the above question (for lowness) in the next section by using a refinement of the method of this section.

5. Degrees of c.e. sets not wtt-reducible to ranked sets

In this section we show that there is a low c.e. set D that is not wtt-reducible to any initial segment of any scattered computable linear ordering. As we have seen, the initial segments of a computable linear ordering form a Π^0_1 class, so we can prove this result using computable trees. We will use the concepts given in the next definition.

Definition 5.1. We shall say that a (possibly finite) sequence $A = \langle \sigma_0, \sigma_1, \ldots \rangle \subset 2^{<\omega}$ is a *transversal* of a given tree $T \subset 2^{<\omega}$ if every $f \in [T]$ extends some σ_i.

Given $p \in \omega$, a transversal A of T is an (i,p)-*converging transversal* of T if $|\sigma_k| \geq \varphi_i(\langle p, k \rangle)$ for every $\sigma_k \in A$.

The next theorem is the key fact about (i,p)-converging transversals that we will need. In its proof, we will use the following known sufficient condition for a set X be be complex (as usual, we use C to denote Kolmogorov complexity): If there is a computable function f such that

$$\forall k \; C(X \upharpoonright f(k)) \geq k,$$

then X is complex (see [30] for a proof of this fact).

Theorem 5.2. *Let T be a computable tree such that $[T]$ has no perfect Π^0_1 subclass, and let φ_i be total. Then for every p, the tree T has a finite (i,p)-converging transversal.*

Proof. Fix p. Define the (noneffective) sequence $\sigma_0, \sigma_1, \ldots$ as follows. Split ω into blocks $I_0 < I_1 < \cdots$ with $|I_k| = 2^k$. Let $f(k) = \max_{n \in I_k} \varphi_i(\langle p, n\rangle)$. For each k and each $n \in I_k$ in numerical order, choose σ_n to be the string of length $f(k)$ with least Kolmogorov complexity among the strings not yet chosen, breaking ties by taking the leftmost string. Note that, by the definition of f, for every n we have $|\sigma_n| \geq \varphi_i(\langle p, n\rangle)$.

Since there are at most $2^k - 1$ strings of Kolmogorov complexity $< k$, every such string of length $f(k)$ is σ_n for some $n \in I_k$. Thus, if $C(X \upharpoonright f(k)) < k$, then X extends some σ_n. So if X does not extend any σ_n then it is complex, and hence not in $[T]$, by Theorem 4.7. Thus $\sigma_0, \sigma_1, \ldots$ is a transversal of T, which must, therefore, by compactness, have a finite initial subsequence $\sigma_0, \ldots, \sigma_m$ that is also a transversal. This sequence is an (i,p)-converging transversal. $\square$

Before proving the main result of this section, we reprove part (1) of Corollary 4.8 using transversals. More precisely, we show that there exists a c.e. set D that is not wtt-reducible to any ranked real. Later we will discuss how to make D low.

Theorem 5.3. *There exists a c.e. set D that is not wtt-reducible to any ranked real. That is, the wtt-cone above D is disjoint from every countable Π^0_1 class. In fact, every Π^0_1 class that contains an element $\geq_{wtt} D$ has a perfect Π^0_1 subclass.*

The following corollary is an immediate consequence of Theorem 5.3 and Proposition 4.3.

Corollary 5.4. *There exists a c.e. set D that is not wtt-reducible to any initial segment of any computable scattered linear ordering.*

For the following discussion, let $\{T_j\}_{j\in\omega}$ be a fixed effective enumeration of all c.e. subtrees of $2^{<\omega}$.

Proof of Theorem 5.3. The desired set D will be the union of uniformly c.e. sets $D_{eij} \subset \{\langle e,i,j,k\rangle\}_{k\in\omega}$ constructed below. To assist in constructing D_{eij}, we will attempt to find a finite transversal A_{eij} of T_j. Given the sth approximation D^s_{eij} to D_{eij}, we produce D^{s+1}_{eij} as follows.

Construction of D.

Step 1. Define the transversals.

If A_{eij} has not been defined by stage s, determine whether there exists some $A = \langle \sigma_0, \sigma_1, \dots, \sigma_M \rangle \in (2^{\leq s})^{<\omega}$ with $M \leq s$ known to be an $(i, \langle e, i, j \rangle)$-converging transversal of T_j. That is,

(1) for every $\sigma_k \in A$, the computation $\varphi_{i,s}^{\sigma_k}(\langle e, i, j, k \rangle)$ converges to a value $\leq |\sigma_k|$; and
(2) there exists $L \leq s$ such that T_j^s converges on every $\rho \in 2^L$, and every $\rho \in \left(2^L \cap T_j\right)$ extends some $\sigma_k \in A$.

If there is such an A, let A_{eij} be the least such. Otherwise, A_{eij} remains undefined, so $D_{eij}^{s+1} = D_{eij}^s = \emptyset$.

Step 2. Enumerate elements into D_{eij}.

Given that $A_{eij} = \langle \sigma_0, \sigma_1, \dots \rangle$ has been defined by stage s, enumerate $\langle e, i, j, k \rangle$ into D_{eij} if and only if $\varphi_{e,s}^{\sigma_k}(\langle e, i, j, k \rangle) = 0$.

This ends the construction. From Theorem 5.2 we may readily derive the conclusion of Theorem 5.3 as follows: Fix j such that T_j is total and does not have a perfect Π_1^0 subclass, and let β be an infinite path of T. Assume that $D \leq_{\text{wtt}} \beta$ via φ_e with use bounded by φ_i. Note that φ_i is total. Applying Theorem 5.2 with $p = \langle e, i, j \rangle$, it follows that A_{eij} will eventually be defined. As A_{eij} is a transversal of T_j, some $\sigma_k \in A_{eij}$ must lie on β. Let $n = \langle e, i, j, k \rangle$. Observe that

$$\varphi_e^\beta(n) = \varphi_e^{\beta \upharpoonright \varphi_i(n)}(n) = \varphi_e^{\sigma_k}(n).$$

Therefore,

$$\varphi_e^\beta(\,\langle e, i, j, k \rangle\,) = 1 \iff \varphi_e^{\sigma_k}(\,\langle e, i, j, k \rangle\,) = 1,$$

but, by construction,

$$\langle e, i, j, k \rangle \in D \iff \langle e, i, j, k \rangle \in D_{eij} \iff \varphi_e^{\sigma_k}(\,\langle e, i, j, k \rangle\,) = 0.$$

We conclude that $D \not\leq_{\text{wtt}} \beta$ via φ_e and φ_i. □

Next, we will show that the set D of Theorem 5.3 can be chosen to be low. One might attempt to do this by adding permitting to the proof of Theorem 5.3 to show that D can be chosen to be Turing reducible to any given non-computable c.e. set. However, problems arise because multiple permissions are required to meet a given requirement. In fact, the following proposition shows that it is impossible to strengthen Theorem 5.3 in this fashion.

Proposition 5.5. *There is a non-computable c.e. set A such that every set $D \leq_T A$ is wtt-reducible to a ranked set, and, in fact, to the ω-part of a computable linear ordering of order type $\omega + \omega^*$.*

Proof. Let A be a non-computable c.e. set of *strongly contiguous degree*, i.e., the Turing degree of A consists of a single wtt-degree. Such a set exists by [9], $(2.1)'$. Every c.e. Turing degree contains a c.e. set that is the ω-part of a computable linear ordering of order type $\omega + \omega^*$ (Example 2.4 in [17], or by Theorem 3.4 in this article). Thus we may assume, without loss of generality, that A has this property. Since the Turing degree of A is strongly contiguous, it follows easily that every set Turing reducible to A is wtt-reducible to A, and hence to a set that is the ω-part of a computable linear ordering of order type $\omega + \omega^*$. □

Although the proof of Theorem 5.3 does not mix with ordinary permitting, it would be routine to add steps to ensure that D meets the usual lowness requirements. Instead of writing out the details of this argument, we define a class $\mathcal{A}$ of degrees that includes some low c.e. degrees such that all (c.e.) elements of $\mathcal{A}$ compute a (c.e.) set D that satisfies the conclusion of Theorem 5.3. A natural candidate for $\mathcal{A}$ is the set of array non-computable degrees (see [13, 14]), since these are often well-suited to arguments that require multiple permitting. However, it seems unlikely that they provide sufficient permissions to carry out the proof of Theorem 5.3. Instead, we consider a uniform version of array non-computability, and degrees with this property will indeed provide sufficient permissions.

In working with the definition below, it will be useful to notice the well-known fact that when $f : \omega \to \omega$ is a (total) function, $f \leq_{\mathrm{wtt}} \emptyset'$ iff f is ω-c.e., that is, there are computable functions $h(\cdot,\cdot)$ and $p(\cdot)$ such that $f(n) = \lim_s h(n,s)$ for all n, and

$$|\{s : h(n,s) \neq h(n,s+1)\}| \leq p(n).$$

Additionally, there exists a uniformly Δ^0_2 enumeration $\{f_e\}_{e\in\omega}$ of all functions that are wtt-reducible to $\emptyset'$.

Definition 5.6. (i) A degree $\mathbf{d}$ is *array non-computable* (or *ANC*) if for each $f \leq_{\mathrm{wtt}} \emptyset'$ there is a $\mathbf{d}$-computable function g such that f does not dominate g. (That is, $g(n) > f(n)$ for infinitely many n.)

If a degree is not *ANC*, then it is called *array computable.*

(ii) A degree $\mathbf{d}$ is *uniformly ANC* if there is a fixed $\mathbf{d}$-computable function g that is not dominated by any $f \leq_{\mathrm{wtt}} \emptyset'$.

Schaeffer [36], Proposition 6.3, showed that every superlow degree is array computable. It follows from this fact and Corollary 4.8 that there is a set D of array computable degree $\mathbf{d}$ such that D is not wtt-reducible to any ranked set.

Downey, Greenberg, and Weber [12] introduced a closely related class of c.e. degrees, called the totally ω-c.e. degrees, which are naturally definable in the c.e. degrees. This class is meant to capture a certain notion of "multiple permitting" involved in constructions such as that of a critical triple of c.e. degrees. (See [12] for further discussion of such constructions.) A degree $\mathbf{d}$ is *totally ω-c.e.* if every $\mathbf{d}$-computable function is wtt-computable in $\emptyset'$. Downey (personal communication) pointed out that, for c.e. degrees, the notions of being totally ω-c.e. and being not uniformly ANC coincide. The following proof was provided to us by Downey and Greenberg (personal communication).

Theorem 5.7. *Let $\mathbf{d}$ be a c.e. degree. Then $\mathbf{d}$ is totally ω-c.e. iff $\mathbf{d}$ is not uniformly ANC.*

Proof. Let $\mathbf{d}$ be totally ω-c.e. Then every $\mathbf{d}$-computable function f is wtt-computable in $\emptyset'$, and hence dominated by a function that is wtt-computable in $\emptyset'$. So $\mathbf{d}$ is not uniformly ANC. (This direction does not require $\mathbf{d}$ to be c.e.)

Now let $\mathbf{d}$ be c.e. and not uniformly ANC. Let g be $\mathbf{d}$-computable. Let $D \in \mathbf{d}$ be c.e. and let Φ be a Turing functional such that $\Phi^D = g$. Let $h(n)$ be the least stage s such that $\Phi^D(n)[s]\downarrow$ with a D-correct computation. Since h is $\mathbf{d}$-computable, it is dominated by some $f \leq_{\mathrm{wtt}} \emptyset'$. For each n, we have $g(n) = \Phi^D(n)[f(n)]$, from which it follows that $g \leq_{\mathrm{wtt}} f \leq_{\mathrm{wtt}} \emptyset'$. □

Of course, every uniformly ANC degree is ANC. On the other hand, it was shown in [12] that the array computable c.e. degrees are properly contained in the totally ω-c.e. degrees. Thus, by Lemma 5.7, there are c.e. ANC degrees that are not uniformly ANC. Still, many results about uniformly ANC degrees are similar to those about ANC degrees.

Recall that a degree $\mathbf{d}$ is in $\overline{\mathrm{GL}_2}$ if $\mathbf{d}'' > (\mathbf{d} \cup \mathbf{0}')'$. It is remarked in Proposition 1.2 of [14] that every degree in $\overline{\mathrm{GL}_2}$ is ANC, and this result extends easily to the analogous result for uniformly ANC degrees.

Proposition 5.8. *Let $\mathbf{d} \in \overline{GL_2}$. Then $\mathbf{d}$ is uniformly ANC. In particular, every degree $\mathbf{d} \leq \mathbf{0}'$ that is not low$_2$ is uniformly ANC.*

Proof. Recall that, since $\mathbf{d} \in \overline{\mathrm{GL}_2}$, there is no $\mathbf{0}'$-computable function that dominates every $\mathbf{d}$-computable function. Let $\{f_e\}_{e\in\omega}$ be a Δ^0_2 enumeration of all functions that are wtt-reducible to $\emptyset'$, and let $F(n) = \max_{e\leq n} f_e(n)$. As F is $\mathbf{0}'$-computable, there exists some $\mathbf{d}$-computable function g such that $g(n) > F(n)$ for infinitely many n. Fixing a function f_e that is wtt-reducible to $\emptyset'$, observe that $F(n) \geq f_e(n)$ for all $n \geq e$, and hence $g(n) > f_e(n)$ for infinitely many n. □

Proposition 5.9. *There exists a low c.e. uniformly ANC degree.*

Proof. Let F be as in the proof of the previous proposition. Use the finite injury method to construct a c.e. set A that meets the usual lowness requirements and also satisfies $a_n > F(n)$ for infinitely many n, where $a_0, a_1, \dots$ lists $\overline{A}$ in increasing order. We omit the routine details. $\square$

Theorem 5.10.

(1) *For every uniformly ANC degree* $\mathbf{d}$*, there exists a* $\mathbf{d}$*-computable set* D *such that* D *is not wtt-reducible to any ranked set, and, in fact, every* Π^0_1 *class that contains an element* $\geq_{wtt} D$ *has a perfect* Π^0_1 *subclass.*
(2) *Furthermore, when* $\mathbf{d}$ *in* (1) *is a c.e. degree, we can take the set* D *to be c.e.*

Proof. Our construction proceeds in much the same way as in the proof of Theorem 5.3. However, rather than using just one transversal A_{eij} to define the set D_{eij}, we will seek to produce infinitely many such transversals A^r_{eij}, where A^r_{eij} is the first finite $(i, \langle e, i, j, r\rangle)$-converging transversal found within T_j, if it exists, and A^r_{eij} is undefined if no such transversal exists. In addition, to make $\deg(D) \leq \mathbf{d}$, an element will enter D only when it is allowed to do so by a "permitting function" *Deadline*(r), where *Deadline* is a $\mathbf{d}$-computable function that is not dominated by any function $f \leq_{\mathrm{wtt}} \emptyset'$.

To prove (1), define $D = \bigcup_s D_s$, where the sets D_s are defined as follows.

Given that $\langle e, i, j, r, k\rangle \notin D_s$, then $\langle e, i, j, r, k\rangle \in D_{s+1}$ iff

(1) $s \leq \mathit{Deadline}(r)$,
(2) A^r_{eij} has been defined by stage s, and
(3) $\varphi^{\sigma_k}_e(\langle e, i, j, r, k\rangle) = 0$, where σ_k denotes the kth element of A^r_{eij}.

Clearly $D \leq_{\mathrm{T}} \mathit{Deadline}$, and hence D is $\mathbf{d}$-computable.

Lemma 5.11. *Assume that* A^r_{eij} *exists for every* r*. Then there exists some* r *such that* $\langle e, i, j, r, k\rangle \in D$ *iff* $\varphi^{\sigma_k}_{e,s}(\langle e, i, j, r, k\rangle) = 0$*, for all* $k < |A^r_{eij}|$*.*

Proof. Assume otherwise. For all r and all $k < |A^r_{eij}|$, let

$$s_{rk} = \begin{cases} 0 & \text{if } \varphi^{\sigma_k}_{e,s}(\langle e,i,j,r,k\rangle)\uparrow\,; \\ \left(\begin{array}{c}\text{least } s \text{ such that} \\ \varphi^{\sigma_k}_{e,s}(\langle e,i,j,r,k\rangle)\downarrow\end{array}\right) & \text{otherwise.}\end{cases}$$

Let $f(r) = \sup_k s_{rk}$. Note that $f \leq_{\mathrm{wtt}} \emptyset'$. That is, letting $\{f_s\}_{s\in\omega}$ be the obvious sequence of computable approximations of f, we have that $f_s(r) \neq f_{s-1}(r)$ for no more than $|A^r_{eij}|$ many values of s. Therefore, *Deadline*$(r) > f(r)$ for some r, implying by construction that $\langle e,i,j,r,k\rangle \in D$ iff $\varphi_e^{\sigma_k}(\langle e,i,j,r,k\rangle) = 0$. □

To complete the proof of part (1) of Theorem 5.10, fix $e, i, j \in \omega$ such that φ_i is total and increasing and T_j has no perfect Π^0_1 subclass. Then, by Theorem 5.2, we know A^r_{eij} exists for every r. Therefore, the conclusion of Lemma 5.11 holds, and thus the same argument as in the previous cases shows that for every branch β of T_j, the set D is not wtt-reducible to β via φ_e and φ_i.

To prove (2), observe that when $\mathbf{d}$ is a c.e. degree, we may assume that the function *Deadline* is the limit of uniformly computable approximations *Deadline*$_s$, where *Deadline*$_s(r) \leq$ *Deadline*$_{s+1}(r)$ for all r, s. To see this, fix a c.e. set $X \in \mathbf{d}$ and e_0 such that the function $g = \varphi^X_{e_0}$ is not dominated by any $f \leq_{\text{wtt}} \emptyset'$. Let *Deadline*$_s(r) = \max_{t\leq s} \varphi^{X_t}_{e_0,t}(r)$. Then, clearly, $g(r) \leq$ *Deadline*$(r) = \lim_s$ *Deadline*$_s(r)$, and hence *Deadline* is not dominated by any $f \leq_{\text{wtt}} \emptyset'$. Furthermore, the fact that X is c.e. implies that *Deadline* $\leq_T \varphi^X_{e_0} \leq_T X$.

Then, given that $\langle e,i,j,r,k\rangle \notin D_s$, we have that $\langle e,i,j,r,k\rangle \in D_{s+1}$ iff

(1) $s \leq$ *Deadline*$_s(r)$,
(2) A^r_{eij} has been defined by stage s, and
(3) $\varphi^{\sigma_k}_{e,s}(\langle e,i,j,r,k\rangle) = 0$, where σ_k denotes the kth element of A^r_{eij}.

That is, the previous ($\mathbf{d}$-computable) first condition "$s \leq$ *Deadline*(r)" has been replaced by the *computable* condition "$s \leq$ *Deadline*$_s(r)$".

Then D is clearly c.e. since each of the conditions (1)–(3) is computable. Furthermore, it can easily be seen that the same argument used to prove Lemma 5.11 still works. □

The following corollary follows at once from Proposition 5.9 and Theorem 5.10.

Corollary 5.12. *The set D of Theorem 5.3 and Corollary 5.4 can be taken to have low Turing degree. That is, there exists a low c.e. set D such that the wtt-cone above D is disjoint from every countable Π^0_1 class; in fact, every Π^0_1 class that contains an element $\geq_{wtt} D$ has a perfect Π^0_1 subclass. Furthermore, D is not wtt-reducible to any initial segment of any computable scattered linear ordering.*

Downey and Greenberg (personal communication) have recently announced independent proofs that every c.e., totally ω-c.e. set is wtt-reducible to a c.e. set that is the ω-part of a computable linear ordering of type $\omega + \omega^*$. Thus, for c.e. sets, part 2 of Theorem 5.10 cannot be extended beyond the uniformly ANC degrees. In other words, a c.e. degree **d** contains a c.e. set that is not wtt-reducible to any ranked set if and only if **d** is uniformly ANC. In particular, every superlow c.e. set is wtt-reducible to a ranked set (indeed, to the ω-part of a linear ordering of type $\omega+\omega^*$), and hence Corollary 5.12 cannot be extended from lowness to superlowness.

The following result is implicit in the work of Cenzer and Smith [8]. It can also be proved by combining the method of Theorem 5.3 with permitting.

Theorem 5.13. *For every non-computable c.e. set C there exists a c.e. set $D \leq_{wtt} C$ such that the tt-cone above D is disjoint from every countable Π^0_1 class.*

Proof. It is shown in [8], Theorem 6.1 that for every non-computable c.e. set C, there is a c.e. set $D \equiv_{\mathrm{T}} C$ such that D is unranked. It follows from the proof of this result that $D \equiv_{\mathrm{wtt}} C$. Furthermore, by [8], Lemma 1.2(b), every set A with $D \leq_{\mathrm{tt}} A$ is unranked because D is unranked. □

References

[1] C. J. Ash, P. Cholak, and J. F. Knight, Permitting, forcing, and copies of a given recursive relation, *Annals of Pure and Applied Logic* 86 (1997) 219–236.

[2] C. J. Ash and J. F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy* (Elsevier, Amsterdam, 2000).

[3] C. J. Ash and A. Nerode, Intrinsically recursive relations, in: *Aspects of Effective Algebra* (Proc. Conf. Monash Univ., Clayton, Australia, Aug. 1–4, 1979), J. N. Crossley (ed.), (U.D.A. Book Co., Yarra Glen, Victoria, Australia, 1981), 26–41.

[4] G. Barmpalias, Hypersimplicity and semicomputability in the weak truth table degrees, *Archive for Mathematical Logic* 44 (2005), 1045–1065.

[5] E. Barker, Intrinsically Σ^0_α relations, *Annals of Pure and Applied Logic* 39 (1988), 105–130.

[6] M. Bickford and C. F. Mills, Lowness properties of r.e. sets, unpublished preprint.

[7] D. Cenzer, P. Clote, R. Smith, and S. Wainer, Members of countable Π^0_1 classes, *Annals of Pure and Applied Logic* 31 (1986), 145–163.

[8] D. Cenzer and R. Smith, The ranked members of a Π^0_1 set, *Journal of Symbolic Logic* 54 (1989), 975–991.

[9] R. Downey, Δ^0_2 degrees and transfer theorems, *Illinois Journal of Mathematics* 31 (1987), 419–427.

[10] R. Downey, On Π^0_1 classes and their ranked points, *Notre Dame Journal of Formal Logic* 32 (1991), 499–512.

[11] R. G. Downey, S. S. Goncharov, and D. R. Hirschfeldt, Degree spectra of relations on Boolean algebras, *Algebra and Logic* 42 (2003), 105–111.

[12] R. Downey, N. Greenberg, and R. Weber, Totally ω-computably enumerable degrees I: bounding critical triples, submitted.

[13] R. G. Downey, C. Jockusch, Jr., and M. Stob, Array nonrecursive sets and multiple permitting arguments, in: *Recursion Theory Week Proceedings,* 1989, J. Ambos-Spies, G. H. Müller, and G. E. Sacks (eds.), (Lecture Notes in Mathematics 1432, Springer-Verlag, New York, 1990), 141–173.

[14] R. G. Downey, C. Jockusch, Jr., and M. Stob, Array nonrecursive sets and genericity, in: *Computability, Enumerability, Unsolvability: Directions in Recursion Theory*, S. B. Cooper, T. A. Slaman, and S. S. Wainer (eds.), (London Mathematical Society Lecture Notes Series 224, Cambridge University Press, 1996), 93–104.

[15] S. S. Goncharov and B. Khoussainov, On the spectrum of degrees of decidable relations, *Doklady Mathematics* 55 (1997), 55–57.

[16] V. S. Harizanov, Uncountable degree spectra, *Annals of Pure and Applied Logic* 54 (1991), 255–263.

[17] V. S. Harizanov, Some effects of Ash-Nerode and other decidability conditions on degree spectra, *Annals of Pure and Applied Logic* 55 (1991), 51–65.

[18] V. S. Harizanov, Turing degrees of certain isomorphic images of recursive relations, *Annals of Pure and Applied Logic* 93 (1998), 103–113.

[19] V. S. Harizanov, Relations on computable structures, in: *Contemporary Mathematics*, N. Bokan (ed.), (University of Belgrade, 2000), 65–81.

[20] D. R. Hirschfeldt, Degree spectra of relations on computable structures, *Bulletin of Symbolic Logic* 6 (2000), 197–212.

[21] D. R. Hirschfeldt, Degree spectra of intrinsically c.e. relations, *Journal of Symbolic Logic* 66 (2001), 441–469.

[22] D. R. Hirschfeldt, Degree spectra of relations on computable structures in the presence of Δ^0_2 isomorphisms, *Journal of Symbolic Logic* 67 (2002), 697–720.

[23] D. R. Hirschfeldt and W. M. White, Realizing levels of the hyperarithmetic hierarchy as degree spectra of relations on computable structures, *Notre Dame Journal of Formal Logic* 43 (2002), 51–64.

[24] C. G. Jockusch, Jr, Semirecursive sets and positive reducibility, *Transactions of the American Mathematical Society* 131 (1968), 420–536.

[25] C. G. Jockusch, Jr. and T. McLaughlin, Countable retracing function and Π^0_2 predicates, *Pacific Journal of Mathematics* 30 (1969), 67–93.

[26] C. G. Jockusch, Jr. and R. Shore, Pseudo-jump operators II: transfinite iterations hierarchies and minimal covers, *Journal of Symbolic Logic* 49 (1984), 1205–1236.

[27] C. G. Jockusch, Jr. and R. I. Soare, Π^0_1 classes and degrees of theories, *Transactions of the American Mathematical Society* 173 (1972), 33–56.

[28] B. Khoussainov and R. A. Shore, Solutions of the Goncharov-Millar and degree spectra problems in the theory of computable models, *Doklady Mathematics* 61 (2000), 178–179.

[29] G. Kreisel, Analysis of the Cantor-Bendixson Theorem by means of the analytic hierarchy, *Bulletin de l'Académie Polonaise des Sciences* 7 (1959), 621–626.

[30] B. Kjos-Hanssen, W. Merkle, and F. Stephan, Kolmogorov complexity and the Recursion Theorem, STACS 2006: *Twenty-Third Annual Symposium on Theoretical Aspects of Computer Science* (Marseille, France, February 23–25, 2006, Proceedings, Springer LNCS 3884), 149–161.

[31] M. Li and P. Vitányi, *An Introduction to Kolmogorov Complexity and Its Applications*, 2nd edition, Springer-Verlag, 1997.

[32] T. McNicholl, Intrinsic reducibilities, *Mathematical Logic Quarterly* 46 (2000), 393–407.

[33] J. Mohrherr, *Index Sets and Truth-Table Degrees in Recursion Theory*, PhD Dissertation, University of Illinois at Chicago, 1982.

[34] J. Mohrherr, A refinement of low_n and high_n for the r.e. degrees, *Zeitschrift für matematische Logik und Grundlagen der Mathematik* 32 (1986), 5–12.

[35] P. Odifreddi, *Classical Recursion Theory* (North-Holland, Amsterdam, 1989).

[36] B. Schaeffer, Dynamic notions of genericity and array noncomputability, *Annals of Pure and Applied Logic* 95 (1998), 37–69.

[37] R. I. Soare, *Recursively Enumerable Sets and Degrees.* A Study of Computable Functions and Computably Generated Sets (Springer-Verlag, Berlin, 1987).

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