Computability and Definability*

Trang Ha
Department of Mathematics
George Washington University
Washington, DC 20052
trangtha09@gmail.com

Valentina Harizanov
Department of Mathematics
George Washington University
Washington, DC 20052
harizanv@email.gwu.edu

Leah Marshall
Department of Mathematics
George Washington University
Washington, DC 20052
lbm@email.gwu.edu

Hakim Walker
Department of Mathematics
Harvard University
Cambridge, MA 02138
hjwalker@math.harvard.edu

August 7, 2019

Abstract

The connection between computability and definability is one of the main themes in computable model theory. A decidable theory has a *decidable model*, that is, a model with a decidable elementary diagram. On the other hand, in a *computable model* only the atomic diagram must be decidable. For some nonisomorphic structures that are elementarily equivalent, we can use computable (infinitary) sentences to describe different structures. In general, if two computable structures satisfy the same computable sentences, then they are isomorphic. Roughly speaking, computable formulas are $L_{\omega_1\omega}$-formulas with disjunctions and conjunctions over computably enumerable index sets. Let $\alpha > 0$ be a computable ordinal. A computable Σ_α (Π_α, resp.) formula is a computably enumerable disjunction (conjunction, resp.) of formulas $\exists\overline{u}\,\psi(\overline{x},\overline{u})$ ($\forall\overline{u}\,\psi(\overline{x},\overline{u})$, resp.) where ψ is computable Π_β (Σ_β, resp.) for some $\beta < \alpha$. A computable Σ_0 or Π_0 formula is a finitary quantifier-free formula. To show that our

*Harizanov was partially supported by the Simons FoundationCollaboration Grant and CCAS Dean's Research Chair award of the George Washington University.

descriptions of structures by computable infinitary formulas are optimal we often consider index sets and analyze their complexity. The index set of a structure $\mathcal{A}$ is the set of all Gödel indices for computable isomorphic copies of $\mathcal{A}$.

For a complexity class $\mathcal{P}$, a computable structure $\mathcal{A}$ is *$\mathcal{P}$-categorical* if for all computable $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism in $\mathcal{P}$. A computable structure $\mathcal{A}$ is *relatively $\mathcal{P}$-categorical* if for all $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism that is $\mathcal{P}$ relative to the atomic diagram of $\mathcal{B}$. There is a powerful syntactic condition that is equivalent to relative Δ^0_α-categoricity. The condition is that there is a computably enumerable Scott family of computable Σ_α formulas. For every computable α, there are Δ^0_α-categorical structures that do not have corresponding effective Scott families.

A relation R on a computable structure $\mathcal{A}$, which is not named in the language of $\mathcal{A}$, is called *intrinsically* $\mathcal{P}$ on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ onto another computable structure belongs to $\mathcal{P}$. A relation R is *relatively intrinsically* $\mathcal{P}$ on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to any structure $\mathcal{B}$ is $\mathcal{P}$ relative to the atomic diagram of $\mathcal{B}$. A relation R is relatively intrinsically Σ^0_α iff R is definable by a computable Σ_α formula with finitely many parameters. For every computable α, there are intrinsically Σ^0_α relations on computable structures, which are not definable by computable Σ_α formulas. On the other hand, intrinsically Δ^1_1 relations on computable structures coincide with relatively intrinsically Δ^1_1 relations, and are exactly the relations definable by computable infinitary formulas with finitely many parameters.

Contents

1 Introduction and preliminaries. Theories, diagrams, and models

In the 1970s, Metakides and Nerode, together with other researchers in the United States, initiated a systematic study of computability in mathematical structures and constructions by using modern computability-theoretic tools, such as the priority method and various coding techniques. At the same time and independently, computable model theory was developed in the Siberian school of constructive mathematics by Goncharov, Nurtazin and Peretyat'kin. While in classical mathematics we can replace some constructions by effective ones, for others such replacement is impossible in principle. For example, from the point of view of computability theory, isomorphic structures may have very different properties.

We will assume that all structures are at most countable and their languages are computable. A computable language is a countable language with algorithmically presented set of symbols and their arities. The universe A of an infinite countable structure $\mathcal{A}$ can be identified with ω. If L is the language of $\mathcal{A}$, then L_A is the language L expanded by adding a constant symbol for every $a \in A$, and $\mathcal{A}_A = (\mathcal{A}, a)_{a \in A}$ is the corresponding expansion of $\mathcal{A}$ to L_A. The *atomic* (*open*) *diagram* of a structure $\mathcal{A}$, $D(\mathcal{A})$, is the set of all quantifier-free sentences of L_A true in $\mathcal{A}_A$. A structure is *computable* if its atomic diagram is computable. The *Turing degree of* $\mathcal{A}$, $deg(\mathcal{A})$, is the Turing degree of the atomic diagram of $\mathcal{A}$. The *elementary* (complete, full) *diagram* of $\mathcal{A}$, denoted by $D^c(\mathcal{A})$, is the set of all sentences of L_A that are true in $\mathcal{A}_A$.

We will assume that our theories are countable and consistent. Henkin's construction of a model for a complete decidable theory is effective and produces a structure $\mathcal{A}$ with a computable domain such that the elementary diagram of $\mathcal{A}$ is computable. A structure $\mathcal{A}$ is called *decidable* if its elementary diagram $D^c(\mathcal{A})$ is computable. Thus, in the case of a computable structure, our starting point is semantic, while in the case of a decidable structure, the starting point is syntactic. It is easy to see that not every computable structure is decidable. For example, the *standard model of arithmetic*, $\mathcal{N} = (\omega, +, \cdot, S, 0)$, is computable but not decidable. On the other hand, Tennenbaum showed that if $\mathcal{A}$ is a nonstandard model of *Peano arithmetic* (PA), then $\mathcal{A}$ is not computable. Harrison-Trainor [87] has recently established that characterizing those computable structures that have a decidable copy is a Σ^1_1-complete problem. There are familiar structures $\mathcal{A}$ such that for all $\mathcal{B} \cong \mathcal{A}$, we have $D^c(\mathcal{B}) \equiv_T D(\mathcal{B})$. In particular, this is true for algebraically closed fields, and for other structures for which we have effective elimination of quantifiers.

In computable model theory, we investigate structures, their theories, fragments of diagrams, relations, and isomorphisms within various computability-theoretic hierarchies, such as arithmetic or, more generally, hyperarithmetical hierarchy, or within Turing degree and other degree-theoretic hierarchies. Computability-theoretic notation in this paper is standard and as in [151]. We review some basic notions and notation. For $X \subseteq \omega$, let $\varphi^X_0, \varphi^X_1, \varphi^X_2, \ldots$ be a fixed effective enumeration of all unary X-computable functions. For a structure $\mathcal{B}$, $\varphi^{\mathcal{B}}_e$ stands for $\varphi^{D(\mathcal{B})}_e$. If X is computable, we omit the superscript X. For $e \in \omega$, let $W^X_e = dom(\varphi^X_e)$. Hence $W_0, W_1, W_2, \ldots$ is an effective enumeration of all computably enumerable (c.e.) sets. By $X \leq_T Y$ ($X \equiv_T Y$, respectively) we denote that X is Turing reducible to Y (X is Turing equivalent to Y, respectively). By $X <_T Y$ we denote that $X \leq_T Y$ but $Y \not\leq_T X$. We write $\mathbf{x} = \deg(X)$ for the Turing degree of X. Thus, $\mathbf{0} = \deg(\emptyset)$. Let $n \geq 1$. Then $\mathbf{x}^{(n)} = \deg(X^{(n)})$, where $X^{(n)}$ is the n-th Turing jump of X. For a set X, we define the ω-*jump* of X by $X^{(\omega)} = \{\langle x, n\rangle : x \in X^{(n)}\}$, and let $\mathbf{x}^{(\omega)} = \deg(X^{(\omega)})$. The degree $\mathbf{0}^{(\omega)}$ is a natural upper bound for the sequence $(\mathbf{0}^{(n)})_{n\in\omega}$, although no ascending sequence of Turing degrees has a least upper bound. By $\oplus$ we denote the join of Turing degrees. A set $X \leq_T \emptyset'$ and its Turing degree $\mathbf{x}$ are called *low* if $\mathbf{x}' \leq \mathbf{0}'$, and *low*$_n$ if $\mathbf{x}^{(n)} \leq \mathbf{0}^{(n)}$.

An important question is when a theory has a decidable or a computable model, or a model of certain Turing degree. It is easy to see that the theory of a structure $\mathcal{A}$ is computable in $D^c(\mathcal{A})$, and that $D^c(\mathcal{A})$ is computable in $(D(\mathcal{A}))^{(\omega)}$. The *low basis theorem* of Jockusch and Soare can be used to obtain for a theory S, a model $\mathcal{A}$ with

$$(D^c(\mathcal{A}))' \leq_T S'.$$

The atomic diagram of a model of a theory may be of much lower Turing degree than the theory itself. For example, while the standard model of arithmetic $\mathcal{N}$ is computable, its theory, *true arithmetic*, $Th(\mathcal{N}) = TA$, is of Turing degree $\mathbf{0}^{(\omega)}$. Harrington and Knight (see [11]) proved that there is a nonstandard model $\mathcal{M}$ of PA such that $\mathcal{M}$ is *low* and $Th(\mathcal{M}) \equiv_T \emptyset^{(\omega)}$. Knight [105] proved that if $\mathcal{A}$ is a nonstandard model of PA, then there exists $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that $D(\mathcal{B}) <_T D(\mathcal{A})$.

A set is Σ^0_n if it is c.e. relative to $\mathbf{0}^{(n-1)}$. A set is Π^0_n if its negation is Σ^0_n, and a set is Δ^0_n if it is both Σ^0_n and Π^0_n. Let $\Delta^0_0 =_{def} \Delta^0_1$. A set X is *arithmetical* if $X \leq \emptyset^{(k)}$ for some $k \geq 0$. Gödel established that a set X, or a relation R, is arithmetical if and only if it is definable in the standard model of arithmetic $\mathcal{N}$. Moreover, R is Σ^0_n (Π^0_n, respectively) if and only if it is definable in $\mathcal{N}$ by a Σ^0_n (Π^0_n, respectively) formula. Σ^0_1 sets are exactly c.e. sets. One of the main results in computable mathematics, due to Matiyasevich, which implies the undecidability of the Hilbert Tenth Problem is that the Diophantine sets of natural numbers coincide with the c.e. sets. A set $X \subseteq \omega$ is *Diophantine* if there is a polynomial $p(y, x_1, ..., x_m)$ with integer coefficients such that for every natural number n, we have that $n \in X$ if and only if there are natural numbers $c_1, ..., c_m$ such that $p(n, c_1, ..., c_m) = 0$.

The set of all computable types of a complete decidable theory is a Π^0_2 set. Every principal type of such a theory is computable, and the set of all its principal types is Π^0_1. A countable structure $\mathcal{A}$ is *homogeneous* if for every two finite sequences $\overrightarrow{a}$ and $\overrightarrow{b}$ of the same length n, if $\overrightarrow{a}$ and $\overrightarrow{b}$ realize the same n-type in $\mathcal{A}$, then there is an automorphism of $\mathcal{A}$ taking $\overrightarrow{a}$ to $\overrightarrow{b}$. Prime models and countable saturated models are examples of homogeneous models. The study of the computable content of these models was initiated in the 1970s. A model $\mathcal{A}$ of a theory T is *prime* if for all models $\mathcal{B}$ of T, $\mathcal{A}$ elementarily embeds into $\mathcal{B}$. It is well known that all prime models of a given theory are isomorphic, and that every complete atomic theory has a prime model. Goncharov and Nurtazin [75], and independently Harrington [83] established that a complete decidable theory T with a prime model has a decidable prime model if and only if the set of all principal types of T is uniformly computable.

N. Khisamiev [101] showed that there is a complete theory of abelian groups with both a computable model and a prime model, but no computable prime model. More recently, Hirschfeldt [93] showed that there is a complete theory of linear orderings having a computable model and a prime model, but no computable prime model.

A countable model is *saturated* if it realizes every type of its language augmented by any finite tuple of constants for its elements. Morley [135] and T. Millar [129] independently proved that a complete decidable theory T has a decidable saturated model if and only if the set of all types of T is uniformly computable. For example, it was shown recently in [26] that the theory of differentially closed fields of characteristic 0, in symbols DCF_0, has a decidable saturated model. If the types are not uniformly computable, then the existence of a decidable saturated model is not guaranteed, as shown by counterexamples constructed independently by Goncharov and Nurtazin [75], Morley [135] and T. Millar [129].

For a structure $\mathcal{A}$, the *type spectrum* of $\mathcal{A}$ is the set of all types realized in $\mathcal{A}$. Goncharov [62], Peretyat'kin [140] and T. Millar [128] independently showed that there exists a complete decidable theory T having a homogeneous model $\mathcal{M}$ without a decidable copy, such that the type spectrum of $\mathcal{M}$ consists only of computable types and is computable. In fact, Goncharov [62] and Peretyat'kin [140] provided a criterion for a homogeneous model to be decidable. Their criterion can be stated in terms of the effective extension property. A computable set of computable types of a theory has the *effective extension property* if there is a partial computable function f that given a type Γ_n of arity k and a formula θ_i of arity $k+1$ (identified with their indices), outputs the index for a type containing Γ_n and θ_i, if there exists such a type.

A theory is called *Ehrenfeucht* if it has finitely many but at least two countable models, up to isomorphism. By Vaught's theorem, if a theory has two nonisomorphic models, then it has at least three nonisomorphic models. An example of a theory with exactly three countable models was given by Ehrenfeucht. His result can be easily generalized to obtain a theory with exactly n countable models, for any finite $n \geq 3$. Gavryushkin constructed examples of computable Ehrenfeucht models of arbitrarily high arithmetical and non-arithmetical complexity.

Theorem 1. *([58]) For every $n \geq 3$, there exists an Ehrenfeucht theory T of arbitrary arithmetical complexity such that it has n countable models, up to isomorphism, and it has a computable model among them. There also exists such a theory that is Turing equivalent to $\emptyset^{(\omega)}$.*

A theory is called *κ-categorical*, where κ is an infinite cardinal, if it has exactly one model of cardinality κ, up to isomorphism. The theories that are $\aleph_0$-categorical are also called *countably categorical.* Morley's categoricity theorem states that if a theory T is κ-categorical for some uncountable cardinal κ, then T is λ-categorical for all uncountable cardinals λ. Hence, theories categorical in an uncountable cardinal are also called *uncountably categorical.* A theory that is both countably and uncountably categorical is simply called *totally categorical*. For the case of countably categorical theories, Lerman and Schmerl [117] gave sufficient conditions for the existence of a computable model, which were later extended by Knight in [107]. Knight proved that if T is a countably categorical theory such that $T \cap \Sigma_{n+2}$ is Σ^0_{n+1} uniformly in n, then T has a computable model. The natural question is whether there exist such countably categorical theories of higher complexity. Fokina established the following result using the method of Marker's extension. (For Marker's extension see [71].)

Theorem 2. *([53]) There exists a countably categorical theory of arbitrary arithmetical complexity, which has a computable model.*

The problem of the existence of a countably categorical theory of non-arithmetical complexity was resolved by Khoussainov and Montalbán [103]. They showed that there exists a countably categorical theory S with a computable model such that $S \equiv_T \mathbf{0}^{(\omega)}$. The unique model of their theory in an infinite language is, up to isomorphism, a modification of the random graph. Andrews [3] later established the following result about truth-table (tt-) degrees. Truth-table reducibility, a version of strong Turing reducibility where the reduction presents a single list of questions to the oracle simultaneously (depending only on the input) and then after seeing the answers produces the output.

Theorem 3. *([3]) In every tt-degree that is $\leq 0^{(\omega)}$, there is a countably categorical theory in a finite language with a computable model.*

A complete theory T is *strongly minimal* if any definable (with parameters) subset of any model $\mathcal{M}$ of T is finite or cofinite. We call a structure strongly minimal if it has a strongly minimal theory. Andrews and Knight [5] showed that if T is a strongly minimal theory and for $n \geq 1$, $T \cap \Sigma_{n+2}$ is Δ^0_n, uniformly in n, then every model has a computable copy. This result relativized to $\emptyset'''$ gives the following corollary.

Theorem 4. *([5]) If a strongly minimal theory has a computable model, then every model has a Δ^0_4 copy.*

Andrews and J. Miller [7] defined the (*Turing degree*) *spectrum of a theory* T to be the set of Turing degrees of models of T. The idea behind this notion is to better understand the relationship between the model-theoretic properties of a theory and the computability-theoretic complexity of its models. On the other hand, the (*Turing*) *degree spectrum* of a structure $\mathcal{A}$ is

$$DgSp(\mathcal{A}) = \{\deg(D(\mathcal{B})) : \mathcal{B} \cong \mathcal{A}\}.$$

A structure $\mathcal{A}$ is called *automorphically trivial* if there exists a finite subset $\{a_1, \ldots, a_n\}$ of the domain of $\mathcal{A}$ such that every permutation f of the domain with $f(a_i) = a_i$ for $i \in \{1, \ldots, n\}$ is an automorphism of $\mathcal{A}$. These structures include all finite structures and also some infinite structures such as the complete graph on countably many vertices. A structure $\mathcal{A}$ in a finite language is automorphically trivial if and only if its spectrum is $\{\mathbf{0}\}$. The spectrum of an automorphically trivial structure always contains exactly one Turing degree, but if the language is infinite, that degree can be noncomputable. Knight proved the following fundamental result. If $\mathcal{A}$ is not automorphically trivial, then for any two Turing degrees $\mathbf{c} \leq \mathbf{d}$, if $\mathbf{c} \in DegSp(\mathcal{A})$, then also $\mathbf{d} \in DegSp(\mathcal{A})$. Moreover, we have the following result.

Theorem 5. *[82] (a) For every automorphically nontrivial structure $\mathcal{A}$, and every set $X \geq_T D^c(\mathcal{A})$, there exists $\mathcal{B} \cong \mathcal{A}$ such that*

$$D^c(\mathcal{B}) \equiv_T D(\mathcal{B}) \equiv_T X.$$

(b) For every automorphically trivial structure $\mathcal{A}$, we have $D^c(\mathcal{A}) \equiv_T D(\mathcal{A})$.

Theory spectra may coincide with degree spectra of structures, e.g., the cones above arbitrary Turing degrees are both theory spectra and degree spectra, as well as the set of all noncomputable degrees. On the other hand, there are examples of theory spectra that are not degree spectra for any structure, such as the degrees of complete extensions of Peano arithmetic, and the union of the cones above two incomparable Turing degrees [7]. On the other hand, by [76], there is a structure the degree spectrum of which consists of exactly the non-hyperarithmetical degrees, while as shown in [7], the set of non-hyperarithmetical degrees is not the spectrum of a theory. Further interesting examples of theory spectra can be found in [7], and for the case of atomic theories in [6].

A structure $\mathcal{A}$ is called *n-decidable* for $n \geq 1$ if the Σ_n-diagram of $\mathcal{A}$ is decidable. We will denote the Σ_n-diagram of $\mathcal{A}$ by $D_n(\mathcal{A})$. For sets X and Y, we say that Y is *c.e. in and above* (*c.e.a. in*) X if Y is c.e. relative to X, and $X \leq_T Y$. For any structure $\mathcal{A}$, $D_{n+1}(\mathcal{A})$ is c.e.a. in $D_n(\mathcal{A})$, uniformly in n, where $D_0(\mathcal{A}) = D(\mathcal{A})$. For all n, there are n-decidable structures that do not have $(n+1)$-decidable copies. that Chisholm and Moses [39] established that there is a linear ordering that is n-decidable for every $n \in \omega$, but has no decidable copy. Goncharov [64] obtained a similar result for Boolean algebras. Harrison-Trainor [87] has recently established that for $n \geq 1$, characterizing those computable structures that have an n-decidable copy is a Σ^1_1-complete problem.

Turing jump of a structure and different forms of the jump inversion for a structure have been studied independently by Baleva, Soskov, and A. Soskova in Bulgaria, by Morozov, Stukachev, and Puzarenko in Russia, and by Montalbán in the United States. We say that a structure $\mathcal{A}$ admits *strong jump inversion* if for every oracle X, if X' computes $D(\mathcal{C})'$ for some $\mathcal{C} \cong \mathcal{A}$, then X computes $D(\mathcal{B})$ for some $\mathcal{B} \cong \mathcal{A}$. Equivalently, for every oracle X, if $\mathcal{A}$ has a copy that is low over X, then it has a copy that is computable in X. Here, when we say that $\mathcal{C}$ is low over X, we mean that $D(\mathcal{C})' \leq_T X'$. For example, if $\mathcal{A}$ is an equivalence structure with infinitely many infinite classes, then $\mathcal{A}$ admits strong jump inversion. That is because if $\mathcal{A}$ is low over X, then the character of $\mathcal{A}$, consisting of pairs (n, k) such that there are at least k classes of size n, is Σ^0_2

relative to $\mathcal{A}$, so it is Σ_2^0 relative to X. Then $\mathcal{A}$ has an X-computable copy. Downey and Jockusch [44] established that every Boolean algebra admits strong jump inversion. More recently, D. Marker and R. Miller [121] have shown that all countable models of the theory of differentially closed fields of characteristic 0, in symbols DCF_0, admit strong jump inversion.

Not all countable structures admit strong jump inversion. Jockusch and Soare [98] proved that there are *low* linear orderings without computable copies. If T is a low completion of PA, then there is a model $\mathcal{A}$ such that its atomic diagram $D(\mathcal{A})$ is computable in T, hence $D(\mathcal{A})'$ is Δ_2^0. However, since $\mathcal{A}$ is necessarily non-standard, it does not have a computable copy.

The authors of [26] established a general result with sufficient conditions for a structure $\mathcal{A}$ to admit strong jump inversion. The conditions involve saturation properties and an enumeration of B_1-types, where these are made up of formulas that are Boolean combinations of existential formulas. The general result applies to some familiar kinds of structures, including some classes of linear orderings and trees, as well as DCF_0. It also applies to Boolean algebras with no 1-atom, with some extra information on the complexity of the isomorphism. Our general result gives the result of Marker and Miller. In order to apply our general result, we produce a computable enumeration of the types realized in models of DCF_0. This also yields the fact that the saturated model of DCF_0 has a decidable copy.

The general result applies to structures from familiar algebraic classes, including certain classes of linear orderings, abelian p-groups, equivalence structures, and trees. When a structure $\mathcal{A}$ admits strong jump inversion, and $\mathcal{A}$ is low relative to an oracle X, the authors of [26] also considered the complexity of the isomorphisms between $\mathcal{A}$ and its X-computable copies. In the case of an infinite Boolean algebra with no 1-atom, such an isomorphism can be chosen to be Δ_3^0 relative to X. This is interesting because Knight and Stob [111] established that any low Boolean algebra has a computable copy and a corresponding Δ_4^0 isomorphism, and this bound has been proven to be sharp.

2 Computable infinitary formulas. Scott rank

Several important notions of computability on effective structures have syntactic characterizations, which involve computable infinitary formulas introduced by Ash. Formulas of $L_{\omega_1\omega}$ are infinitary formulas with countable disjunctions and conjunctions, but only finite strings of quantifiers. If we restrict the disjunctions and conjunctions to c.e. sets, then we have the computable infinitary formulas. Ash defined computable Σ_α and Π_α formulas of $L_{\omega_1\omega}$, where α is a computable ordinal, recursively and simultaneously and together with their Gödel numbers.

An ordinal is *computable* if it is finite or is the order type of a computable well ordering on ω. The computable ordinals form a countable initial segment of the ordinals. *Kleene's* $\mathcal{O}$ is the set of notations for computable ordinals, together with a partial ordering $<_{\mathcal{O}}$ (see [146, 147]). The ordinal 0 gets notation 1. If a is a notation for α, then 2^a is a notation for $\alpha+1$. Then $a <_{\mathcal{O}} 2^a$, and also, if

$b <_{\mathcal{O}} a$, then $b <_{\mathcal{O}} 2^a$. Suppose α is a limit ordinal. If φ_e is a total function, giving notations for an increasing sequence of ordinals with limit α, then $3 \cdot 5^e$ is a notation for α. For all n, we have $\varphi_e(n) <_{\mathcal{O}} 3 \cdot 5^e$, and if $b <_{\mathcal{O}} \varphi_e(n)$, then $b <_{\mathcal{O}} 3 \cdot 5^e$. Let $|a|$ denote the ordinal with notation a. If $a \in \mathcal{O}$, then the restriction of $<_{\mathcal{O}}$ to the set $pred(a) = \{b \in \mathcal{O} : b <_{\mathcal{O}} a\}$ is a well ordering of type $|a|$. For $a \in \mathcal{O}$, $pred(a)$ is c.e., uniformly in a. The set $\mathcal{O}$ is Π^1_1-complete. A Π^1_1 subset of $\mathcal{O}$ is Δ^1_1 iff it is contained in a set of the form $\mathcal{O}_\alpha = \{b \in \mathcal{O} : |b| < \alpha\}$, where α is a computable ordinal.

For computable infinitary formulas, we cannot bring the quantifiers outside, but we can bring negations inside. We have a resemblance to normal form, and we can classify formulas according to the number of alternations of infinite disjunction/$\exists$ with infinite conjunction/$\forall$. The computable Σ_0 and Π_0 formulas are the finitary quantifier-free formulas. The computable $\Sigma_{\alpha+1}$ formulas are of the form

$$\bigvee_{n \in W_e} \exists \overrightarrow{y}_n \psi_n(\overrightarrow{x}, \overrightarrow{y}_n),$$

where for $n \in W_e$, ψ_n is a Π_α formula indexed by its Gödel number, and $\exists \overrightarrow{y}_n$ is a finite block of existential quantifiers. That is, $\Sigma_{\alpha+1}$ formulas are c.e. disjunctions of $\exists\Pi_\alpha$ formulas. Similarly, $\Pi_{\alpha+1}$ formulas are c.e. conjunctions of $\forall\Sigma_\alpha$ formulas. It can be shown that a computable Σ_1 formula is of the form

$$\bigvee_{n \in \omega} \exists \overrightarrow{y}_n \theta_n(\overrightarrow{x}, \overrightarrow{y}_n),$$

where $(\theta_n(\overrightarrow{x}, \overrightarrow{y}_n))_{n\in\omega}$ is a computable sequence of quantifier-free formulas. If α is a limit ordinal, then Σ_α (Π_α, respectively) formulas are of the form $\bigvee\limits_{n \in W_e} \psi_n$ ($\bigwedge\limits_{n \in W_e} \psi_n$, respectively), such that there is a sequence $(\alpha_n)_{n \in W_e}$ of ordinals having limit α, given by the ordinal notation for α, and every ψ_n is a Σ_{α_n} (Π_{α_n}, respectively) formula. For a more precise definition of computable Σ_α and Π_α formulas see [11].

The least noncomputable ordinal is denoted by ω_1^{CK}, where CK stands for Church-Kleene. To extend the arithmetical hierarchy, we define the representative sets in the hyperarithmetical hierarchy, H_a for $a \in \mathcal{O}$. The definition is recursive, and is based on iterating Turing jump: $H_1 = \emptyset$, $H_{2^a} = (H_a)'$, and $H_{3\cdot5^e} = \{2^x 3^n : x \in H_{\varphi_e(n)}\}$. Let β be an infinite computable ordinal. Then a set is Σ^0_β if it is c.e. relative to some H_a such that β is represented by notation a. A set is Π^0_β if its negation is Σ^0_β, and a set is Δ^0_β if it is both Σ^0_β and Π^0_β. A set is *hyperarithmetical* if it is Δ^0_α for some computable α. Hence, a set X is hyperarithmetical if $(\exists a \in \mathcal{O})[X \leq_T H_a]$. The hyperarithmetical sets coincide with Δ^1_1 sets.

The important property of these formulas is given in the following theorem due to Ash.

Theorem 6. *For a structure $\mathcal{A}$, if $\theta(\overrightarrow{x})$ is a computable Σ_α (Π_α) formula, then the set $\{\overrightarrow{a} : \mathcal{A}_A \models \theta(\overrightarrow{a})\}$ is Σ^0_α (Π^0_α) relative to the atomic diagram of $\mathcal{A}$.*

As an example, we will consider some definable properties of a countable reduced abelian p-group, where p is a prime number. Recall that a *p-group* is a group in which every element has order p^n for some n. Countable reduced Abelian p-groups are of particular interest because of their classification up to isomorphism by Ulm. We define a sequence of subgroups G_α, letting $G_0 = G$, $G_{\alpha+1} = pG_\alpha$, and for limit α, $G_\alpha = \cap_{\beta<\alpha} G_\beta$. There is a countable ordinal α such that $G_\alpha = G_{\alpha+1}$. The least such α is the *length* of G, denoted by $\lambda(G)$. The group is *reduced* if $G_{\lambda(G)} = \{0\}$. An element $x \neq 0$ has *height* β if $x \in G_\beta - G_{\beta+1}$. Let $P(G)$ be the set of element of G of order p. Let $P_\alpha = G_\alpha \cap P(G)$. For each $\beta < \lambda(G)$, $P_\beta/P_{\beta+1}$ is a vector space over $\mathbb{Z}_p$ of dimension $\leq \aleph_0$, and this dimension is denoted by $u_\beta(G)$. The *Ulm sequence* for G is the sequence $(u_\beta(G))_{\beta<\lambda(G)}$.

For any computable ordinal α, it is fairly straightforward to write a computable infinitary sentence stating that G is a reduced Abelian p-group of length at most α, and describing its Ulm invariants. In particular, Barker established the following results.

Proposition 1. *[16] Let G be a computable Abelian p-group.*

1. *$G_{\omega\cdot\alpha}$ is $\Pi^0_{2\alpha}$.*
2. *$G_{\omega\cdot\alpha+m}$ is $\Sigma^0_{2\alpha+1}$.*
3. *$P_{\omega\cdot\alpha}$ is $\Pi^0_{2\alpha}$.*
4. *$P_{\omega\cdot\alpha+m}$ is $\Sigma^0_{2\alpha+1}$.*

Proof. It is easy to see that 3 and 4 follow from 1 and 2, respectively. Toward 1 and 2, note the following:

$$\begin{aligned}
x \in G_m &\iff \exists y[p^m y = x];\\
x \in G_\omega &\iff \bigwedge\!\!\!\bigwedge_{m\in\omega} \exists y[p^m y = x];\\
x \in G_{\omega\cdot\alpha+m} &\iff \exists y[p^m y = x \;\&\; G_{\omega\cdot\alpha}(y)];\\
x \in G_{\omega\cdot\alpha+\omega} &\iff \bigwedge\!\!\!\bigwedge_{m\in\omega} \exists y[p^m y = x \;\&\; G_{\omega\cdot\alpha}(y)];\\
x \in G_{\omega\cdot\alpha} &\iff \bigwedge\!\!\!\bigwedge_{\gamma<\alpha} G_{\omega\cdot\gamma}(x) \text{ for limit } \alpha.
\end{aligned}$$

□

Using these results, it is easy to write, for any computable ordinal β, a computable $\Pi_{2\beta+1}$ sentence the models of which are exactly the reduced abelian p-groups of length $\omega\beta$.

Harizanov, Knight, and Morozov gave conditions for intrinsic collapse of the complete diagram to the n-diagram using infinitary formulas.

Theorem 7. *[82] For any structure $\mathcal{A}$ and any n, the following are equivalent.*

(i) For all $\mathcal{B} \cong \mathcal{A}$, $D^c(\mathcal{B}) \equiv_T D_n(\mathcal{B})$.

(ii) For some tuple $\overrightarrow{c}$, there is a computable function d taking each (finitary) formula $\theta(\overrightarrow{x})$ to a formula $d_\theta(\overline{c}, \overrightarrow{x})$, a c.e. disjunction of (finitary) Σ^0_{n+1} formulas with parameters $\overrightarrow{c}$, such that

$$\mathcal{A} \models \forall \overrightarrow{x}\, [\theta(\overrightarrow{x}) \Leftrightarrow d_\theta(\overrightarrow{c}, \overrightarrow{x})].$$

As an application of this theorem, let $\mathcal{B}$ be a linear ordering of type $\omega^n \cdot \eta$ for $n \geq 1$. Then

$$D^c(\mathcal{B}) \equiv_T D_{2n}(\mathcal{B})$$

and we obtain formulas d_θ as follows. Let $\mathcal{A}$ be an ordering of type $\omega^n \cdot \eta$ such that $D^c(\mathcal{A})$ is computable. For each tuple $\overrightarrow{a}$, we can find a (finitary) Σ^0_{2n+1} formula $\psi_{\overrightarrow{a}}(\overrightarrow{x})$ defining the orbit of $\overrightarrow{a}$. Then for each formula $\theta(\overrightarrow{x})$, $d_\theta(\overrightarrow{x})$ is the disjunction of these $\psi_{\overrightarrow{a}}(\overrightarrow{x})$ for $\overrightarrow{a}$ satisfying $\theta(\overrightarrow{x})$. It can be shown that there is a formula θ such that d_θ cannot be made finitary even for $n = 1$.

Ehrenfeucht gave an example of a complete theory with exactly three models, up to isomorphism. The language of the theory has a binary relation symbol $<$ and constants c_n for $n \in \omega$. The axioms say that $<$ is a dense linear ordering without endpoints, and the constants are strictly increasing. The theory T has the following three countable models, up to isomorphism. There is the *prime* model, in which there is no upper bound for the constants. There is the *saturated* model, in which the constants have an upper bound but no least upper bound. There is the *middle* model, in which there is a least upper bound for the constants. Let $\mathcal{A}^1$ be the prime model, let $\mathcal{A}^2$ be the middle model, and let $\mathcal{A}^3$ be the saturated model. The following examples of Scott sentences are due to S. Quinn (see [29]).

A computable Π_2 sentence characterizing the models of T such that

$$(\forall x) \bigvee\!\!\!\bigvee_{n\in\omega} x < c_n$$

is a Scott sentence for $\mathcal{A}^1$.

We have a computable Σ_3 Scott sentence for $\mathcal{A}^2$ describing a model of T such that

$$(\exists x)\, [\bigwedge\!\!\!\bigwedge_{n\in\omega} x > c_n \,\&\, (\forall y)\, [(\bigwedge\!\!\!\bigwedge_{n\in\omega} y > c_n) \rightarrow y \geq x]].$$

We have a computable Π_3 Scott sentence for $\mathcal{A}^3$, describing a model of T such that

$$(\exists x)\, [\bigwedge\!\!\!\bigwedge_{n\in\omega} x > c_n] \,\&\, (\forall y)\, [\bigwedge\!\!\!\bigwedge_{n\in\omega} y > c_n \Longrightarrow (\exists z)[\bigwedge\!\!\!\bigwedge_{n\in\omega} z > c_n \,\&\, z < y]].$$

A d-Σ^0_α *formula* is the conjunction of a Σ^0_α formula and a Σ^0_α formula. Knight and Saraph [110] proved that a finitely generated abelian group has a computable d-Σ^0_2 Scott sentence. The infinite dihedral group is given by the presentation $\langle a, b \mid a^2, b^2\rangle$. Knight and Saraph [110] also proved that a computable infinite dihedral group has a computable d-Σ^0_2 Scott sentence. Scott sentences for many other classes of groups were investigated. For example, Ho [97] proved that every computable polycyclic group has a computable d-Σ^0_2 Scott sentence.

We can measure the complexity of a countable structure by looking for a Scott sentence of minimal complexity. A. Miller [130], using a result of D. Miller [131], proved that for a countable ordinal $\gamma \geq 1$, if $\mathcal{A}$ has a $\Pi^0_{\gamma+1}$ Scott sentence and a $\Sigma^0_{\gamma+1}$ Scott sentence, then it must have a d-Σ^0_γ Scott sentence. Hence for $\alpha \geq 2$, if $\mathcal{A}$ has a Π^0_β Scott sentence and a Σ^0_β Scott sentence, then it must have a d-Σ^0_α Scott sentence for some $\alpha < \beta$. Therefore, the optimal Scott sentence for a given structure is Σ^0_α, Π^0_α, or d-Σ^0_α for some α. Alvir, Knight, and McCoy [2] established an effective version of A. Miller's result.

Theorem 8. *[2] Let $\alpha \geq 2$ be a computable ordinal. If a structure $\mathcal{A}$ has a computable Π^0_β Scott sentence and a computable Σ^0_β Scott sentence, then it must have a computable d-Σ^0_α Scott sentence for some $\alpha < \beta$.*

Knight and Saraph [110] showed that a finitely generated structure always has a Σ^0_3 Scott sentence. However, many finitely generated groups have a simpler description, which is d-Σ^0_2. Harrison-Trainor and Ho [88] gave a characterization of finitely generated structures for which the Σ^0_3 Scott sentence is optimal. A substructure $\mathcal{B}$ of a structure $\mathcal{A}$ is a Σ^0_1-elementary substructure of $\mathcal{M}$ if for every existential formula $\theta(\overrightarrow{x})$ and $\overrightarrow{b} \in B^{lh(\overrightarrow{x})}$, we have $\mathcal{A} \models \theta(\overrightarrow{b})$ iff $\mathcal{A}_B \models \theta(\overrightarrow{b})$.

Theorem 9. *[88, 2] For a finitely generated structure $\mathcal{A}$, the following are equivalent.*

(i) $\mathcal{A}$ has a d-Σ^0_2 Scott sentence.

(ii) $\mathcal{A}$ does not contain a proper Σ^0_1-elementary substructure isomorphic to itself.

(iii) For all (or some) generating tuples of $\mathcal{A}$, the orbit is defined by a Π^0_1 formula.

The equivalence of (i) and (iii) has been established by Alvir, Knight, and McCoy [2]. Since every finitely generated field satisfies condition (ii) of the previous theorem, it follows that a finitely generated field has a d-Σ^0_2 Scott sentence.

The compactness theorem of Kreisel and Barwise states that if Γ is a Π^1_1 set of computable infinitary sentences such that every Δ^1_1 subset of Γ has a model, then Γ has a model. As a corollary we obtain the following result (see [11]).

Theorem 10. *Let Γ be a Π^1_1 set of computable infinitary sentences. If every Δ^1_1 set $\Sigma \subseteq \Gamma$ has a computable model, then Γ has a computable model.*

The following result is a special case of a result of Ressayre in [144] (see [11]).

Theorem 11. *Let $\mathcal{A}$ be a hyperarithmetical structure. If $\overrightarrow{a}$ and $\overrightarrow{b}$ are tuples in $\mathcal{A}$ satisfying the same computable infinitary formulas, then there is an automorphism of $\mathcal{A}$ taking $\overrightarrow{a}$ to $\overrightarrow{b}$.*

Similarly, if $\mathcal{A}$ and $\mathcal{B}$ are hyperarithmetical structures satisfying the same computable sentences, then $\mathcal{A} \cong \mathcal{B}$ (see [72]).

The Scott isomorphism theorem says that for any countable structure $\mathcal{A}$ for a computable language, there is an $L_{\omega_1\omega}$ sentence σ such that the countable models of σ are exactly the isomorphic copies of $\mathcal{A}$. For a proof of the Scott isomorphism theorem see [11]. The proof leads to an assignment of ordinals to countable structures, which we call Scott rank. By a result of Nadel [137], for any hyperarithmetical structure, there is a computable Scott sentence iff the Scott rank is computable.

There are several different definitions of Scott rank. The one used by Sacks [148] involves a sequence of expansions of $\mathcal{A}$. Let $\mathcal{A}_0 = \mathcal{A}$, let $\mathcal{A}_{\alpha+1}$ be the result of adding to $\mathcal{A}_\alpha$ predicates for the types realized in $\mathcal{A}_\alpha$, and for limit α, let $\mathcal{A}_\alpha$ be the limit of the expansions $\mathcal{A}_\beta$, for $\beta < \alpha$. For some countable ordinal α, $\mathcal{A}_\alpha$ is atomic. The least such α is the *rank*. However, we will use the definition of the Scott rank given in [11] (also see [30]). First we define a family of equivalence relations on finite tuples $\overrightarrow{a}$ and $\overrightarrow{b}$ of elements in $\mathcal{A}$, of the same length.

1. We say that $\overrightarrow{a} \equiv^0 \overrightarrow{b}$ if $\overrightarrow{a}$ and $\overrightarrow{b}$ satisfy the same quantifier-free formulas.

2. For $\alpha > 0$, we say that $\overrightarrow{a} \equiv^\alpha \overrightarrow{b}$ if for all $\beta < \alpha$, for every $\overrightarrow{c}$, there exists $\overrightarrow{d}$, and for every $\overrightarrow{d}$, there exists $\overrightarrow{c}$, such that $\overrightarrow{a}, \overrightarrow{c} \equiv^\beta \overrightarrow{b}, \overrightarrow{d}$.

The *Scott rank of a tuple* $\overrightarrow{a}$ in $\mathcal{A}$ is the least β such that for all $\overrightarrow{b}$, the relation $\overrightarrow{a} \equiv^\beta \overrightarrow{b}$ implies $(\mathcal{A}, \overrightarrow{a}) \cong (\mathcal{A}, \overrightarrow{b})$. The *Scott rank of* $\mathcal{A}$, $SR(\mathcal{A})$, is the least ordinal α greater than the ranks of all tuples in $\mathcal{A}$. For example, if $\mathcal{L}$ is a linear order of type ω, then $SR(\mathcal{L}) = 2$. For a hyperarithmetical structure, the Scott rank is at most $\omega_1^{CK} + 1$. A *Harrison ordering* is a computable ordering of type $\omega_1^{CK}(1 + \eta)$, where η is the order type of the rationals. Here, for orderings $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_1\mathcal{L}_2$ is the result of replacing each element of $\mathcal{L}_2$ by a copy of $\mathcal{L}_1$. Harrison [85] showed the such an ordering exists. It can be shown that its Scott rank is $\omega_1^{CK} + 1$.

In general, it can be shown (see [11, 30]) that for a computable structure $\mathcal{A}$, we have the following.

1. $SR(\mathcal{A}) < \omega_1^{CK}$ if there is a computable ordinal β such that the orbits of all tuples are defined by computable Π_β formulas.

2. $SR(\mathcal{A}) = \omega_1^{CK}$ if the orbits of all tuples are defined by computable infinitary formulas, but there is no bound on the complexity of these formulas.

3. $SR(\mathcal{A}) = \omega_1^{CK} + 1$ if there is some tuple the orbit of which is not defined by any computable infinitary formula.

There are structures in natural classes, for example, abelian p-groups, where p is a prime number, with arbitrarily large computable ranks, and of rank $\omega_1^{CK} + 1$, but none of rank ω_1^{CK} (see [18]). Makkai was the first to prove the existence of a structure of Scott rank ω_1^{CK}.

Theorem 12. *[119] There is an arithmetical structure of Scott rank ω_1^{CK}.*

In [109], J. Millar and Knight showed that such structure can be made computable. Through the recent work of Calvert, Knight, and J. Millar [31], Calvert, Goncharov, and Knight [27], and Freer [56], we started to better understand the structures of Scott rank ω_1^{CK}.

Theorem 13. *[31, 27] There are computable structures of Scott rank ω_1^{CK} in the following classes: trees, undirected graphs, fields of any fixed characteristic, and linear orders.*

Sacks asked whether for known examples of computable structures of Scott rank ω_1^{CK}, the computable infinitary theories are $\aleph_0$-categorical. In [28], Calvert, Goncharov, J. Millar, and Knight gave an affirmative answer for known examples. In [126], J. Millar and Sacks introduced an innovative technique that produced a countable structure $\mathcal{A}$ of Scott rank ω_1^{CK} such that $\omega_1^{\mathcal{A}} = \omega_1^{CK}$ and the $L_{\omega_1^{CK},\omega}$-theory of $\mathcal{A}$ is *not* $\aleph_0$ -categorical. Finally, Harrison-Trainor, Igusa and Knight gave a negative answer to Sacks's question.

Theorem 14. *[?] There is a computable structure $\mathcal{M}$ of Scott rank ω_1^{CK} such that the computable infinitary theory of $\mathcal{M}$ is not $\aleph_0$-categorical.*

3 Index sets of structures and classes of structures

In order to measure computability-theoretic complexity of countable structures, one of the main goals is to find an optimal definition of the class of structures under investigation. This often requires the use of various internal properties of the structures in the class. After a reasonable definition is found, it is necessary to prove its sharpness. Usually, this is done by proving completeness in some complexity class.

We may state our goal as follows. Let K be a class of structures. We denote by K^c the set of computable structures in K. A *computable characterization* of K should separate computable structures in K from all other structures (those not in K, or noncomputable ones). A *computable classification* for K

up to an equivalence relation E (isomorphism, computable isomorphism, etc.) should determine each computable element, up to the equivalence E, in terms of relatively simple invariants. In [72], Goncharov and Knight presented three possible approaches to the study of computable characterizations of classes of structures.

Within the framework of the first approach, we say that K has a *computable characterization* if K^c is the set of computable models of a computable sentence. The class of linear orderings can be characterized by a single first-order sentence. The class of abelian p-groups is characterized by a single computable Π_2 sentence. The classes of well orderings and reduced abelian p-groups cannot be characterized by single computable sentences.

Furthermore, we say that there is a *computable classification* for K if there is a computable bound on the ranks of elements of K^c. By a *computable rank* $R^c(\mathcal{A})$ of a structure $\mathcal{A}$ we mean the least ordinal α such that for all tuples $\overrightarrow{a}$ and $\overrightarrow{b}$ in A, of the same length, if for all $\beta < \alpha$, all computable Π_β formulas that true of $\overrightarrow{a}$ are also true of $\overrightarrow{b}$, then there is an automorphism of $\mathcal{A}$ taking $\overrightarrow{a}$ to $\overrightarrow{b}$. For example, the computable rank of a vector space over $\mathbb{Q}$ is 1. There is no computable bound on computable ranks of linear orders and abelian p-groups. The computable rank is not the same as the Scott rank. However, for a hyperarithmetical structure, its computable rank is a computable ordinal just in case its Scott rank is computable (see [72]). If $\mathcal{A}$ is hyperarithmetical, then $R^c(\mathcal{A}) \leq \omega_1^{CK}$.

The second approach involves the notion of an index set. For a computable structure $\mathcal{A}$, an *index* is a number e such that $\varphi_e = \chi_{D(\mathcal{A})}$, where $(\varphi_e)_{e\in\omega}$ is a computable enumeration of all unary partial computable functions. We denote the structure with index e by $\mathcal{A}_e$. The *index set* for $\mathcal{A}$ is the set $I(\mathcal{A})$ of all indices for computable (isomorphic) copies of $\mathcal{A}$. For a class K of structures, closed under isomorphism, the *index set* is the set $I(K)$ of all indices for computable members of K. For an equivalence relation E on a class K, we define

$$I(E, K) = \{(m, n) : m, n \in I(K) \ \& \ \mathcal{A}_m E \mathcal{A}_n\}.$$

Within this approach, we say that K has a *computable characterization* if $I(K)$ is hyperarithmetical. The class K has a *computable classification* up to E if $I(E, K)$ is hyperarithmetical.

The first and the second approach are known to be equivalent [72]. In fact, we do not know a better way to estimate the complexity of an index set than by giving a description by a computable formula. (The third approach of Goncharov and Knight to computable characterization of classes of structures, equivalent to the other two approaches, involves the notion of an *enumeration*.)

Theorem 15. *([72]) For the following classes K, the index set $I(K)$ is Π^0_2:*

1. *linear orderings,*
2. *Boolean algebras,*

3. *abelian p-groups,*

4. *vector spaces over $\mathbb{Q}$.*

The results in the following theorem are well-known and can be attributed to Kleene and Spector.

Theorem 16. *For the following classes K, the index set $I(K)$ is not hyperarithmetical:*

1. *well-orderings,*

2. *superatomic Boolean algebras,*

3. *reduced abelian p-groups.*

In the next theorem, the complexity of index sets for classes of structures with important model-theoretic properties are given by White [158], Pavlovskii [139], and Fokina [52, 51]. By $\Sigma_3^0 - \Sigma_3^0$ we denote the difference of two Σ_3^0 sets. This difference is also denoted by d-Σ_3^0.

Theorem 17. (*a*) *([52]) The index set of structures with decidable countably categorical theories is an m-complete $\Sigma_3^0 - \Sigma_3^0$ set.*

(*b*) *([51]) The index set of decidable structures is Σ_3^0-complete.*

(*c*) *([158, 139]) The index set of computable prime models is an m-complete $\Pi_{\omega+2}^0$ set.*

(*d*) *([158]) The index set of computable homogeneous models is an m-complete $\Pi_{\omega+2}^0$ set.*

Calvert, Fokina, Goncharov, Knight, Kudinov, Morozov and Puzarenko [25] investigated index set complexity of structures of certain Scott ranks.

Theorem 18. *([25])*

(*a*) *The index set of computable structures with noncomputable Scott ranks is m-complete Σ_1^1.*

(*b*) *The index set of structures with the Scott rank ω_1^{CK} is m-complete Π_2^0 relative to Kleene's $\mathcal{O}$.*

(*c*) *The index set of structures with the Scott rank $\omega_1^{CK} + 1$ is m-complete Σ_2^0 relative to Kleene's $\mathcal{O}$.*

A computable structure $\mathcal{A}$ may not have a computable Scott sentence. If it does have a computable Scott sentence σ, then the complexity of the index set $I(\mathcal{A})$ is bounded by the complexity of σ. For many structures from familiar classes, it is often the case that the complexity of the index set matches that of an optimal Scott sentence. However, Knight and McCoy demonstrated in [108] that this is not always the case. Namely, they found a subgroup of $\mathbb{Q}$, which does not have a d-Σ_2^0 Scott sentence but its index set is d-Σ_2^0.

For some structures, we obtain more meaningful results by locating the given computable structure $\mathcal{A}$ within some natural class K closed under isomorphism.

Definition 1. A sentence σ is a *Scott sentence for $\mathcal{A}$ within K* if the countable models of σ in K are exactly the isomorphic copies of $\mathcal{A}$.

We say how to describe $\mathcal{A}$ within K, and also how to calculate the complexity of $I(\mathcal{A})$ within $I(K)$.

Definition 2. Let Γ be a complexity class.

1. $I(\mathcal{A})$ is Γ *within* K if $I(\mathcal{A}) = R \cap I(K)$ for some $R \in \Gamma$.
2. $I(\mathcal{A})$ is *m-complete* Γ *within* K if $I(\mathcal{A})$ is Γ within K and for any $S \in \Gamma$, there is a computable function $f : \omega \to I(K)$ such that there is a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ for which

$$n \in S \text{ iff } \mathcal{C}_n \cong \mathcal{A}.$$

That is,

$$n \in S \text{ iff } f(n) \in I(\mathcal{A}).$$

Recall Ehrenfeucht's example of a complete theory with exactly three countable models, up to isomorphism, in the language with a binary relation symbol $<$ and constants c_n for $n \in \omega$.

Theorem 19. *[29] Let K be the class of models of the original Ehrenfeucht theory T. Let $\mathcal{A}^1$ be the prime model, let $\mathcal{A}^2$ be the middle model, and let $\mathcal{A}^3$ be the saturated model.*
(a) $I(\mathcal{A}^1)$ is m-complete Π^0_2 within K.
(b) $I(\mathcal{A}^2)$ is m-complete Σ^0_3 within K.
(c) $I(\mathcal{A}^3)$ is m-complete Π^0_3 within K.

Finite structures are easy to describe. By $\Sigma^0_1 - \Sigma^0_1$ we denote the difference of two c.e. sets. Such a set is also called *d-c.e.* where d stands for difference.

Theorem 20. *Let L be a finite relational language. Let K be the class of finite L-structures, and let $\mathcal{A} \in K$.*
If $\mathcal{A}$ has size $n \geq 1$, then $I(\mathcal{A})$ is m-complete $\Sigma^0_1 - \Sigma^0_1$ within K.

Proof. We have a finitary existential sentence θ stating that there is a substructure isomorphic to $\mathcal{A}$, and another finitary existential sentence ψ stating that there are at least $n+1$ elements. Then $\theta \wedge \neg\psi$ is a Scott sentence for $\mathcal{A}$. It follows that $I(\mathcal{A})$ is d-c.e. within K. For completeness, let $S = S_1 - S_2$, where S_1 and S_2 are c.e. We have the usual finite approximations $S_{1,s}$, $S_{2,s}$.

Let $\mathcal{A}^-$ be a proper substructure of $\mathcal{A}$, and let $\mathcal{A}^+$ be a finite proper superstructure of $\mathcal{A}$. We will build a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that

$$\mathcal{A}_n \cong \begin{cases} \mathcal{A}^- & \text{if } n \notin S_1, \\ \mathcal{A} & \text{if } n \in S_1 - S_2, \\ \mathcal{A}^+ & \text{if } n \notin S_1 \cap S_2. \end{cases}$$

To accomplish this, let $D_0 = D(\mathcal{A}^-)$. At stage s, if $n \notin S_{1,s}$, we let D_s be the atomic diagram of $\mathcal{A}^-$. If $n \in S_{1,s} - S_{2,s}$, we let D_s be the atomic diagram of $\mathcal{A}$. If $n \in S_{1,s} \cap S_{2,s}$, we let D_s be the atomic diagram of $\mathcal{A}^+$. There is some s_0 such that for all $s \geq s_0$, $n \in S_1$ iff $n \in S_{1,s}$, and $n \in S_2$ iff $n \in S_{2,s}$. Let $\mathcal{A}_n$ be the structure with diagram D_s for $s \geq s_0$. It is clear that $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$. $\square$

The finite-dimensional vector spaces over a fixed infinite computable field are completely determined by a finite a basis. For concreteness, we assume that vector spaces are over $\mathbb{Q}$. Let K be the class of $\mathbb{Q}$-vector spaces, and let $\mathcal{A}$ be a 1-dimensional member of K. First, we show that $\mathcal{A}$ has a computable Π_2 Scott sentence. We have a computable Π_2 sentence characterizing the class K. We take the conjunction of this with the sentence saying

$$(\exists x)\ x \neq 0\ \&\ (\forall x)\,(\forall y) \bigvee\!\!\!\bigvee_{\lambda \in \Lambda} \lambda(x,y) = 0,$$

where Λ is the set of all nontrivial linear combinations $q_1 x + q_2 y$, for $q_i \in \mathbb{Q}$. Now, $I(\mathcal{A})$ is Π^0_2. We do not need to locate $\mathcal{A}$ within K, since the set of indices for members of K is Π^0_2. We can also show Π^0_2-hardness. Suppose $\mathcal{B}$ in K has dimension k, where $k > 1$. Then $\mathcal{B}$ has a d-Σ_2 Scott sentence. We take the conjunction of the axioms for $\mathbb{Q}$-vector spaces, and we add a sentence saying that there are at least k independent elements, and that there are not at least $k+1$. Then $I(\mathcal{B})$ is d-Σ^0_2. For $\mathcal{C}$, we have a computable Π_3 Scott sentence, obtained by taking the conjunction of the axioms for $\mathbb{Q}$-vector spaces and the conjunction over all $k \in \omega$ of computable Σ_2 sentences saying that the dimension is at least k. Therefore, $I(\mathcal{C})$ is Π^0_3. We can also establish the corresponding hardness results.

Theorem 21. *[29] Let K be the class of computable vector spaces over $\mathbb{Q}$, and let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$.*

(*a*) *If $\dim(\mathcal{A}) = 1$, then $I(\mathcal{A})$ is m-complete Π^0_2 within K.*

(*b*) *If $\dim(\mathcal{B}) > 1$, then $I(\mathcal{B})$ is m-complete d-Σ^0_2 within K.*

(*c*) *Let $\mathcal{C}$ be of infinite dimension. Then $I(\mathcal{A})$ is m-complete Π^0_3 within K.*

Archimedean ordered fields are isomorphic to subfields of the reals. They are determined by the Dedekind cuts that are filled. It follows that for any computable Archimedean ordered field $\mathcal{A}$, the index set $I(\mathcal{A})$ is Π^0_3. It is enough to show that $\mathcal{A}$ has a computable Π_3 Scott sentence. We have a computable Π_2 sentence σ_0 characterizing the Archimedean ordered fields. For each $a \in \mathcal{A}$, we have a computable Π_1 formula $c_a(x)$ saying that x is in the cut corresponding to a—we take the conjunction of a c.e. set of formulas saying $q < x < r$, for rationals q, r such that $\mathcal{A} \models q < a < r$. Let σ_1 be $\bigwedge\!\!\!\bigwedge_a (\exists x)\, c_a(x)$, and let σ_2 be $(\forall x) \bigvee\!\!\!\bigvee_a c_a(x)$. The conjunction of σ_0, σ_1, and σ_2 is a Scott sentence, which we may take to be computable Π_3.

Theorem 22. *[29] Let K be the class of Archimedean real closed ordered fields, and let $\mathcal{A}$ be a computable member of K.*

(a) *If the transcendence degree of $\mathcal{A}$ is* 0 (*i.e., $\mathcal{A}$ is isomorphic to the ordered field of algebraic reals*), *then $I(\mathcal{A})$ is m-complete Π^0_2 within K.*

(b) *If the transcendence degree of $\mathcal{A}$ is finite but not* 0*, then $I(\mathcal{A})$ is m-complete d-Σ^0_2 within K.*

(c) *If the transcendence degree of $\mathcal{A}$ is infinite, then $I(\mathcal{A})$ is m-complete Π^0_3 within K.*

For a computable member $\mathcal{A}$ of K, to show that $I(\mathcal{A})$ is Π^0_2, we show that there is a computable Π_2 Scott sentence. We take the conjunction of a sentence characterizing the real closed ordered fields, and a sentence saying that each element is a root of some polynomial.

For reduced abelian p-groups of length $< \omega^2$, we have the following result.

Theorem 23. *[29] Let K be the class of reduced Abelian p-groups of length $\omega M + N$ for some $M, N \in \omega$. Let $\mathcal{A} \in K$.*

(a) *If $\mathcal{A}_{\omega M}$ is minimal for the given length* (*of the form $\mathbb{Z}_{p^N}$*), *then $I(\mathcal{A})$ is m-complete Π^0_{2M+1} within K.*

(b) *If $\mathcal{A}_{\omega M}$ is finite but not minimal for the given length, then $I(\mathcal{A})$ is m-complete d-Σ^0_{2M+1} within K.*

(c) *If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{A}) = \infty$, and for all $m < k$, $u_{\omega M+m}(\mathcal{A}) = 0$, then $I(\mathcal{A})$ is m-complete Π^0_{2M+2} within K.*

(d) *If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{A}) = \infty$ and for some $m < k$ we have $0 < u_{\omega M+m}(\mathcal{A}) < \infty$, then $I(\mathcal{A})$ is m-complete d-Σ^0_{2M+2} within K.*

(e) *If there exist $m < k < N$ such that $u_{\omega M+m}(\mathcal{A}) = u_{\omega M+k}(\mathcal{A}) = \infty$, then $I(\mathcal{A})$ is m-complete Π^0_{2M+3} within K.*

The case when the length is ω was proved in [23].

Proof. Let K be the class of reduced Abelian p-groups of length ωM, and let $\mathcal{A} \in K$. Then $I(\mathcal{A})$ is m-complete Π^0_{2M+1} within K. Let $\mathcal{A} \in K$. First, we show that $\mathcal{A}$ has a computable Π_{2M+1} Scott sentence. There is a computable Π_2 sentence θ characterizing the Abelian p-groups. Next, there is a computable Π_{2M+1} sentence λ characterizing the groups which are reduced and have length at most ωM. For each $\alpha < \omega M$, we can find a computable Σ_{2M} sentence $\varphi_{\alpha,k}$ saying that $u_\alpha(\mathcal{A}) \geq k$. The set of these Σ_{2M} sentences true in $\mathcal{A}$ is Σ^0_{2M}. For each $\varphi_{\alpha,k}$, we can find a computable Π_{2M} sentence equivalent to the negation, and the set of these sentences true in $\mathcal{A}$ is Π^0_{2M}. We have a computable Π_{2M+1} sentence υ equivalent to the conjunction of the sentences $\pm\varphi_{\alpha,k}$ true in $\mathcal{A}$. Then we have a computable Π_{2M+1} Scott sentence equivalent to $\theta \wedge \lambda \wedge \upsilon$. It follows that $I(\mathcal{A})$ is Π^0_{2M+1}.

For completeness, let S be a Π^0_{2M+1} set. We will produce a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of elements of K, such that $n \in S$ if and only if $\mathcal{A}_n$ is isomorphic to $\mathcal{A}$. □

A group G is *free* if there is a set B of elements such that B generates G and there are no non-trivial relations on elements of B. We call B a *basis* for G. If B and U are two bases for a free group G, then B and U have the same cardinality.

For a free group G, the cardinality of a basis is called the *rank*. We write F_n for the free group of rank n, and F_∞ for the free group of rank $\aleph_0$. The groups F_n and F_∞ all have computable copies. Sela in a series of seven papers 2001–06 gave a positive solution to the problem of elementary equivalence of free groups of different finite ranks greater than 1, posed by Tarski in the 1940s. (Also, see work of Kharlampovich and Myasnikov [100].) That is, all non-abelian free groups with finitely many generators have the same elementary first-order theory. Inspired by this result, we investigated free groups in the context of computable model theory [32, 125].

Theorem 24. *[32] Within the class of free groups:*
(*a*) *$I(F_2)$ is m-complete Π^0_2;*
(*b*) *For $n > 2$, the set $I(F_n)$ is m-complete d-Σ^0_2;*
(*c*) *$I(F_\infty)$ is m-complete Π^0_3.*

Theorem 25. *[32, 125] Within the class of all groups:*
(*a*) *For $n \geq 1$, the set $I(F_n)$ is m-complete d-Σ^0_2;*
(*b*) *The set $I(F_\infty)$ is m-complete Π^0_4.*

We also define complexity of one class "within" a larger class. This definition allows us to analyze situations where, for instance, determining whether an index is in B is harder than Γ, but *once we know* that the index is in B, the problem of determining whether it is also in A not harder than Γ.

Definition 3. Let Γ be a complexity class and let $A \subseteq B$.

1. We say that A is Γ *within* B if there is some $C \in \Gamma$ such that $A = C \cap B$.

2. We say that A is Γ-*hard within* B if for any set S in Γ, there is a computable function $f : \omega \to B$ such that $f(n) \in A$ iff $n \in S$.

3. We say that A is *m-complete* Γ *within* B if A is Γ within B and A is Γ-hard within B.

Let $FinGen$ denote the class of all finitely generated groups.

Theorem 26. (*a*) *The set $I(FinGen)$ is m-complete Σ^0_3 within the class of free groups.*
(*b*) *The set $I(FinGen)$ is m-complete Σ^0_3 within the class of all groups.*
(*c*) *The set $I(LocFr)$ is m-complete Π^0_2 within the class of all groups.*

4 Relatively Δ^0_α-categorical structures

The complexity of isomorphisms between a computable structure and its isomorphic copies can be of various complexity. The main notion in this area of investigation is that of computable categoricity. A computable structure $\mathcal{M}$ is *computably categorical* if for every computable structure $\mathcal{A}$ isomorphic to $\mathcal{M}$, there exists a computable isomorphism from $\mathcal{M}$ onto $\mathcal{A}$. In [118], Mal'cev considered the notion of a recursively (computably) stable structure. A computable structure $\mathcal{M}$ is *computably stable* if every isomorphism from $\mathcal{M}$ to another computable structure is computable. In the same paper, Mal'cev investigated the notion of *autostability* of structures, which is equivalent to that of computably categoricity. Since then computable categoricity has been studied extensively. It has been extended to arbitrary levels of hyperarithmetical hierarchy, and more precisely to Turing degrees $\mathbf{d}$. Computable categoricity of a computable structure $\mathcal{M}$ can also be relativized to all (including noncomputable) structures $\mathcal{A}$ isomorphic to $\mathcal{M}$.

Definition 4. A computable structure $\mathcal{M}$ is $\mathbf{d}$-*computably categorical* if for every computable structure $\mathcal{A}$ isomorphic to $\mathcal{M}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{M}$ onto $\mathcal{A}$.

In the case when $\mathbf{d} = \mathbf{0}^{(n-1)}$, $n \geq 1$, we also say that $\mathcal{M}$ is Δ^0_n-*categorical*. Thus, computably categorical is the same as $\mathbf{0}$-computably categorical or Δ^0_1-categorical. We can similarly define Δ^0_α-categorical structures for any computable ordinal α.

Computably categorical structures tend to be very particular. For a structure in a typical algebraic class, being computably categorical is usually equivalent to having a finite basis or a finite generating set (such as in the case of a vector space), or to being highly homogeneous (such as in the case of a random graph). For example, Ershov established that a computable algebraically closed field is computably categorical if and only if it has a finite transcendence degree over its prime subfield. Goncharov, Lempp, and Solomon [73] proved that a computable, ordered, abelian group is computably categorical if and only if it has finite rank. Similarly, they showed that a computable, ordered, Archimedean group is computably categorical if and only if it has finite rank.

An *injection structure* $\mathcal{A} = (A,\ f)$ consists of a nonempty set A and an $1-1$ function $f : A \to A$. Given $a \in A$, the *orbit* $O_f(a)$ of a under f is $\{b \in A : (\exists n \in \mathbb{N})[f^n(a) = b \vee f^n(b) = a]\}$. An injection structure (A, f) may have two types of infinite orbits: $\mathbb{Z}$-orbits, which are isomorphic to $(\mathbb{Z}, S)$, and ω-orbits, which are isomorphic to (ω, S). Cenzer, Harizanov and Remmel [33] characterized computably categorical injection structures as those that have finitely many infinite orbits.

On the other hand, R. Miller and Schoutens [134] constructed a computable field that has *infinite* transcendence degree over the rationals, yet is computably categorical. Their idea uses a computable set of rational polynomials called Fermat polynomials to "tag" elements of a transcendence basis. Hence their field has an infinite computable transcendence basis that is computable in every isomorphic computable copy of the field, and with each single element effectively distinguishable from the others.

We can relativize the notion of Δ^0_α-categoricity by studying the complexity of isomorphisms from a computable structure to any countable isomorphic structure.

Definition 5. A computable structure $\mathcal{M}$ is *relatively* Δ^0_α*-categorical* if for every $\mathcal{A}$ isomorphic to $\mathcal{M}$, there is an isomorphism from $\mathcal{M}$ to $\mathcal{A}$, which is Δ^0_α relative to the atomic diagram of $\mathcal{A}$.

Clearly, a relatively Δ^0_α-categorical structure is Δ^0_α-categorical.

A remarkable feature of relative Δ^0_α-categoricity is that it admits a syntactic characterization. This characterization involves the existence of certain effective Scott families. Scott families come from the Scott Isomorphism Theorem. A *Scott family* for a structure $\mathcal{A}$ is a countable family Φ of $L_{\omega_1\omega}$-formulas with finitely many fixed parameters from A such that:

1. Each finite tuple in $\mathcal{A}$ satisfies some $\psi \in \Phi$;
2. If $\overrightarrow{a}$, $\overrightarrow{b}$ are tuples in $\mathcal{A}$, of the same length, satisfying the same formula in Φ, then there is an automorphism of $\mathcal{A}$, which maps $\overrightarrow{a}$ to $\overrightarrow{b}$.

If we strengthen condition (1) to require that the formulas in Φ define each tuple in $\mathcal{A}$, then Φ is called a *defining family* for $\mathcal{A}$. A *formally* Σ^0_α *Scott family* is a Σ^0_α Scott family consisting of computable Σ_α formulas. In particular, it follows that a formally c.e. Scott family is a c.e. Scott family consisting of finitary existential formulas. The following equivalence was established by Goncharov [65] for $\alpha = 1$, and by Ash, Knight, Manasse, and Slaman [12] and independently by Chisholm [37] for any computable ordinal α.

Theorem 27. *([12, 37]) The following are equivalent for a computable structure $\mathcal{A}$.*

(i) The structure $\mathcal{A}$ is relatively Δ^0_α-categorical.

(ii) The structure $\mathcal{A}$ has a formally Σ^0_α Scott family Φ with finitely many fixed parameters.

(iii) The structure $\mathcal{A}$ has a c.e. Scott family consisting of computable Σ_α formulas with finitely many fixed parameters.

For example, consider a computable equivalence structure $\mathcal{A}$, a computable set with a single equivalence relation. If $\mathcal{A}$ has a bound on the size of its finite equivalence classes, then we can show that $\mathcal{A}$ is relatively Δ^0_2-categorical. Let k be the maximum size of any finite equivalence class. Then $[a]$ is infinite if and only if $[a]$ contains at least $k+1$ elements, which is a Σ^0_1 condition. There is a Δ^0_2 formula that characterizes the elements a with a finite equivalence class of size m. Then a Scott formula for the tuple $(a_1, \ldots, a_m)$ includes a formula $\psi_i(x_i)$ for each a_i, giving the cardinality of $[a_i]$, together with formulas $\psi_{i,j}(x_i, x_j)$ for each i, j, which express whether $a_i E^{\mathcal{A}} a_j$ and whether $a_i = a_j$. Moreover, every computable equivalence structure is relatively Δ^0_3-categorical since every element with an infinite equivalence class has a Π_2 Scott formula, while the other elements even have Δ_2 Scott formulas. Thus, every finite tuple has a Σ_3 Scott formula.

A structure is *rigid* if it does not have nontrivial automorphisms. A computable structure is Δ^0_α*-stable* if every isomorphism from $\mathcal{A}$ onto a computable structure is Δ^0_α. If a computable structure is rigid and Δ^0_α-categorical, then it is Δ^0_α-stable. A *defining family* for a structure $\mathcal{A}$ is a set Φ of formulas with one free variable and a fixed finite tuple of parameters from A such that:

1. Every element of A satisfies some formula $\psi \in \Phi$;

2. No formula of Φ is satisfied by more than one element of A.

For a rigid computable structure $\mathcal{A}$, there is a formally Σ^0_α Scott family iff there is a formally Σ^0_α defining family.

Let us recall the definition of a Fraïssé limit. The *age* of a structure $\mathcal{M}$ is the class of all finitely generated structures that can be embedded in $\mathcal{M}$. Fraïssé showed that a (nonempty) finite or countable class $\mathbb{K}$ of finitely generated structures is the age of a finite or a countable structure if and only if $\mathbb{K}$ has the hereditary property and the joint embedding property. A class $\mathbb{K}$ has the *hereditary property* if whenever $\mathcal{C} \in \mathbb{K}$ and $\mathcal{S}$ is a finitely generated substructure of $\mathcal{C}$, then $\mathcal{S}$ is isomorphic to some structure in $\mathbb{K}$. A class $\mathbb{K}$ has the *joint embedding property* if for every $\mathcal{B}, \mathcal{C} \in \mathbb{K}$ there is $\mathcal{D} \in \mathbb{K}$ such that $\mathcal{B}$ and $\mathcal{C}$ embed into $\mathcal{D}$. A structure $\mathcal{U}$ is *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathcal{U}$ extends to an automorphism of $\mathcal{U}$. A structure $\mathcal{A}$ is a *Fraïssé limit* of a class of finitely generated structures $\mathbb{K}$ if $\mathcal{A}$ is countable, ultrahomogeneous, and has age $\mathbb{K}$. Fraïssé proved that the Fraïssé limit of a class of finitely generated structures is unique up to isomorphism. We say that a structure $\mathcal{A}$ is a Fraïssé limit if for some class $\mathbb{K}$, $\mathcal{A}$ is the Fraïssé limit of $\mathbb{K}$.

Theorem 28. *[1, 55] Let $\mathcal{A}$ be a computable structure, which is a Fraïssé limit. Then $\mathcal{A}$ is relatively Δ^0_2-categorical.*

Proof. Because of ultrahomogeneity, we can construct isomorphisms between $\mathcal{A}$ and an isomorphic structure $\mathcal{B}$ using a back-and-forth argument, as long as we can determine for every two sequences $\overrightarrow{a}$ and $\overrightarrow{b}$ of the same length of elements in $\mathcal{A}$ and $\mathcal{B}$, respectively, whether there is an isomorphism from the structure generated by $\overrightarrow{a}$ to the structure generated by $\overrightarrow{b}$, which maps $\overrightarrow{a}$ to $\overrightarrow{b}$ in order. This can be determined by $(D(\mathcal{B}))'$, since there is such an isomorphism precisely if there is no atomic formula ϕ with $\mathcal{A} \models \phi(\overrightarrow{a})$ and $\mathcal{B} \not\models \phi(\overrightarrow{b})$. This is a Π^0_1 condition relative to $\mathcal{A} \oplus \mathcal{B} \equiv_T \mathcal{B}$.

Therefore, we can use $(D(\mathcal{B}))'$ as an oracle to perform the back-and-forth construction of an isomorphism, and so there is an isomorphism that is Δ^0_2 relative to $\mathcal{B}$. $\square$

Adams and Cenzer [1] defined a structure $\mathcal{A}$ to be *weakly ultrahomogeneous* if there is a finite sequence of elements $\overrightarrow{a}$ from its domain such that $(\mathcal{A}, \overrightarrow{a})$ becomes ultrahomogeneous in the language extended by constants representing these elements. Adams and Cenzer [1] proved that every computable weakly ultrahomogeneous structure is relatively Δ^0_2-categorical. They also proved that every computable, relational, weakly ultrahomogeneous structure is relatively computably categorical. Hence every computable weakly ultrahomogeneous graph is computably categorical, but there are computably categorical graphs that are not weakly ultrahomogeneous.

Theorem 29. *[1] (a) A computable linear ordering is weakly homogeneous iff it is relatively computably categorical.*

(b) A computable equivalence structure is weakly homogeneous iff it is relatively computably categorical.

(c) For a computable injection structure, computable categoricity implies weak ultrahomogeneity, which implies relative Δ_2^0-categoricity, but neither implication can be reversed.

Goncharov and Dzgoev, and independently Remmel characterized computably categorical linear orderings in terms of the number of successor pairs (also called adjacencies). Similarly, they and also LaRoche (independently) characterized computably categorical Boolean algebras.

Theorem 30. *(a) ([66, 141]) A computable linear ordering is computably categorical if and only if it has only finitely many successor pairs.*

Every computably categorical linear ordering is relatively computably categorical.

(b) ([66, 141, 114]) A computable Boolean algebra is computably categorical if and only if it has finitely many atoms.

Every computably categorical Boolean algebra is relatively computably categorical.

Goncharov [61] and Smith [150] independently characterized computably categorical abelian p-groups. By $\mathbb{Z}(p^n)$ we denote the cyclic group of order p^n, and by $\mathbb{Z}(p^\infty)$ the quasicyclic (Prüfer) abelian p-group. Related to abelian p-groups are equivalence structures. Calvert, Cenzer, Harizanov, and Morozov [24] characterized computably categorical equivalence structures.

Theorem 31. *(a) ([61, 150]) An abelian p-group is computably categorical if and only if it can be written in one of the following forms: $(\mathbb{Z}(p^\infty))^l \oplus F$ for $l \in \omega \cup \{\infty\}$ and F is a finite group, or $(\mathbb{Z}(p^\infty))^n \oplus H \oplus (\mathbb{Z}(p^k))^\infty$, where $n, k \in \omega$ and H is a finite group.*

Every computably categorical abelian p-group is relatively computably categorical.

(b) [24] A computable equivalence structure $\mathcal{A}$ is computably categorical if and only if either $\mathcal{A}$ has finitely many finite equivalence classes, or $\mathcal{A}$ has finitely many infinite classes, upper bound on the size of finite classes, and exactly one finite k with infinitely many classes of size k.

Every computably categorical equivalence structure is relatively computably categorical.

Lempp, McCoy, R. Miller and Solomon [115] characterized computably categorical trees of finite height and showed that they are relatively computably categorical. R. Miller [132] previously established that no computable well-founded tree of infinite height is computably categorical. Equivalence structures can be generalized to allow for more than one equivalence relation on the universe. For finite $n \geq 2$, an n-equivalence structure is a structure $\mathcal{A} = (A, E_1, ..., E_n)$ where each E_i is an equivalence relation on A. An n-equivalence structure is nested if for $i < j \leq n$ we have $xE_jy \Rightarrow xE_iy$, i.e., $E_j \subseteq E_i$ as subsets of $A \times A$. For $a \in A$, we let $[a]_i$ denote the equivalence class of a under E_i. Thus for a nested equivalence structure, $i < j \leq n$ implies that $[a]_j \subseteq [a]_i$, so that the E_i classes are partitioned by E_j. There is also an equivalence relation $E_0 = A \times A$, so that $[a]_0 = A$ for all a.

In [122], Marshall described an effective correspondence between nested n-equivalence structures and certain trees of finite height where the branching of the tree reflects the containment of equivalence classes. This correspondence allows many effective properties to be transferred between nested n-equivalence structures and trees of finite height. More precisely, for any nested n-equivalence structure $\mathcal{A} = (A, E_1, \ldots, E_n)$, let E_{n+1} be the equality, and define the tree $T_{\mathcal{A}}$ as follows. The universe of $T_{\mathcal{A}}$ is the set $\{[a]_i : a \in A \wedge i = 1, \ldots, n\}$ and the partial ordering is inclusion. This means that for each a and $i \leq n$, $[a]_i$ is the predecessor of $[a]_{i+1}$. Marshall shows that a presentation of $T_{\mathcal{A}}$ can be computed from $\mathcal{A}$ so that the mapping from a to $[a]$ is also computable from $\mathcal{A}$.

Theorem 32. *[122] Let $\mathcal{A}$ be a computable nested n-equivalence structure and $T_{\mathcal{A}}$ its corresponding tree of finite height. Then the following are equivalent.*
(i) *$\mathcal{A}$ is computably categorical.*
(ii) *$\mathcal{A}$ is relatively computably categorical.*
(iii) *($T_{\mathcal{A}}$,$\prec$) is computably categorical.*
(iv) *($T_{\mathcal{A}}$,$\prec$) is relatively computably categorical.*

Now, let us denote the predecessor function in $T_{\mathcal{A}}$ by f.

Theorem 33. *[1] The following are equivalent.*
(i) *$\mathcal{A}$ is weakly ultrahomogeneous.*
(iii) *($T_{\mathcal{A}}$,f) is weakly ultrahomogeneous.*

Cenzer, Harizanov and Remmel [33] established that computably categorical injection structures are also relatively computably categorical. R. Miller and Shlapentokh [133] proved that a computable algebraic field F with a splitting algorithm is computably categorical if and only if it is decidable which pairs of elements of F belong to the same orbit under automorphisms. They also showed that this criterion is equivalent to relative computable categoricity of F.

Goncharov [63] was the first to show that computable categoricity of a computable structure does not imply relative computable categoricity. The main idea of his proof was to code a special kind of family of sets into a computable structure. Such families were constructed independently by Badaev [15] and Selivanov [149]. Hirschfeldt, Khoussainov, Shore, and Slinko [95] established a general result that implies that there are computably categorical but not relatively computably categorical structures in the following classes: partial orderings, lattices, 2-step nilpotent groups, commutative semigroups, and integral domains of arbitrary characteristic. Hirschfeldt, Kramer, R. Miller, and Shlapentokh [96] characterized relative computable categoricity for computable algebraic fields and used their characterization to construct a field with the following property.

Theorem 34. *([96]) There is a computably categorical algebraic field, which is not relatively computably categorical.*

Infinitary language is essential for Scott families. Cholak, Shore and Solomon [41] proved the existence of a computably categorical graph that does not have a Scott family of finitary formulas. It follows that this structure is not relatively computably categorical.

The result about the existence of computably categorical structures that are not relatively computably categorical was lifted to higher levels in the hyperarithmetical hierarchy by Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon for successor ordinals [69], and by Chisholm, Fokina, Goncharov, Harizanov, Knight and Quinn for limit ordinals [38]. It is an open question whether every Δ_1^1-categorical structure must be relatively Δ_1^1-categorical.

Theorem 35. *([69, 38]) For every computable ordinal α, there is a Δ_α^0-categorical but not relatively Δ_α^0-categorical structure.*

There is a complete description of higher levels of categoricity (in fact, stability) for well-orderings due to Ash [9]. Harris [84] has a description of Δ_n^0-categorical Boolean algebras for any $n < \omega$. However, not enough is known about Δ_n^0-categoricity for $n \geq 2$ for structures from many natural classes of algebraic structures. The study of higher level categoricity often leads to the study of algebraic properties of a family of relations specific for a given class (such as the independence relations or back-and-forth relations). Obtaining classification of categoricity is usually a difficult task. The reason is either the absence of algebraic invariants (such as for the linear orderings, and abelian and nilpotent groups), or the lack of suitable computability-theoretic notions that would capture the property of being Δ_n^0-categorical (such as in the case of Δ_2^0-categoricity of equivalence structures). Even for $n = 2$, the following problems remain open. Describe Δ_2^0-categorical linear orderings. Describe Δ_2^0-categorical equivalence relations. Describe Δ_2^0-categorical abelian p-groups. Describe Δ_2^0-categorical trees of finite height.

In [123], McCoy characterized relatively Δ_2^0-categorical linear orderings and Boolean algebras. In [124], McCoy gave a complete description of relatively Δ_3^0-categorical Boolean algebras. Frolov [57] found a Δ_3^0-categorical linear ordering that is not relatively Δ_3^0-categorical. In the following theorem we state McCoy's characterizations of relatively Δ_2^0-categorical linear orderings and Boolean algebras. As usual, by ω^* we denote the reverse order type of ω, and by η the order type of rationals.

Theorem 36. *[123] (a) A computable linear ordering is relatively Δ^0_2-categorical if and only if it is a sum of finitely many intervals, each of type $m, \omega, \omega^*, \mathbb{Z}$, or $n \cdot \eta$, so that each interval of type $n \cdot \eta$ has a supremum and infimum.*

(b) A computable Boolean algebra is relatively Δ^0_2-categorical if and only if it can be expressed as a finite direct sum $c_1 \vee \cdots \vee c_n$, where each c_i is either atomless, an atom, or a 1-atom.

Bazhenov [19] and Harris [84] independently showed that for Boolean algebras the notions of Δ^0_2-categoricity and relative Δ^0_2-categoricity coincide. It is not known whether every Δ^0_2-categorical linear ordering is relatively Δ^0_2-categorical.

Calvert, Cenzer, Harizanov and Morozov characterized relative Δ^0_2-categoricity for equivalence structures [24] and abelian p-groups [23]. Recall that the length of an abelian p-group G, $\lambda(G)$, is the least ordinal α such that $p^{\alpha+1}G = p^{\alpha}G$. The divisible part of G is $Div(G) = p^{\lambda(G)}G$ and is a direct summand of G. The group G is said to be reduced if $Div(G) = \{0\}$. For a group G, the *period* of G is $max\{order(g) : g \in G\}$ if this quantity is finite, and ∞ otherwise.

Theorem 37. *(a) ([24]) A computable equivalence structure is relatively Δ^0_2-categorical if and only if it either has finitely many infinite equivalence classes, or there is an upper bound on the size of its finite equivalence classes.*

(b) ([23]) A computable abelian p-group G is relatively Δ^0_2-categorical if and only if G is reduced and $\lambda(G) \leq \omega$, or G is isomorphic to $\bigoplus_{\alpha} \mathbb{Z}(p^{\infty}) \oplus H$, where $\alpha \leq \omega$ and H has finite period.

Kach and Turetsky [99] showed that there exists a Δ^0_2-categorical equivalence structure, which is not relatively Δ^0_2-categorical. Downey, Melnikov and Ng [48] built examples of abelian p-groups that show that the notions of Δ^0_2-categoricity and relative Δ^0_2-categoricity do not coincide for these groups. Every computable equivalence structure is relatively Δ^0_3-categorical. There is no such bound for a computable abelian p-group G. For example, it follows from the index set results in [29] that if $\lambda(G) = \omega \cdot n$ and $m \leq 2n-1$, or if $\lambda(G) > \omega \cdot n$ and $m \leq 2n-2$, then G is not Δ^0_m-categorical. Barker [16] proved that for every computable ordinal α, there are $\Delta^0_{2\alpha+2}$-categorical but not $\Delta^0_{2\alpha+1}$-categorical abelian p-groups.

Cenzer, Harizanov and Remmel [33] characterized relative Δ^0_2-categoricity for injection structures.

Theorem 38. *([33]) A computable injection structure is relatively Δ^0_2-categorical if and only if it has finitely many orbits of type ω, or finitely many orbits of type $\mathbb{Z}$. Every Δ^0_2-categorical injection structure is relatively Δ^0_2-categorical.*

Every computable injection structure is relatively Δ^0_3-categorical.

In [34], Cenzer, Harizanov and Remmel investigated computability-theoretic properties of a computable structure (A, f) with a single unary function f such that for every x in the pre-image, $f^{-1}(x)$ has exactly two elements, which is called a $2:1$ structure. Every computable $2:1$ structure is Δ^0_2-categorical.

Theorem 39. *[34] A $2:1$ structure is computably categorical if and only if it has finitely many $\mathbb{Z}$-chains.*

Structures for which $f^{-1}(x)$ has either exactly two or zero elements are called $(2,0):1$ structures. We can identify (A,f) with its directed graph $G(A,f)$, which has vertex set A and where the edge set consists of all pairs $(i, f(i))$ for $i \in A$. Given $a \in A$, we let the orbit $\mathcal{O}_A(a)$ consist of the set of all points in A, which lie in the connected component of $G(A,f)$ containing a. We let $tree_A(a) =_{def} \{y \in A : (\exists n)(f^n(y) = x)\}$. We say that a $(2,0):1$ structure (A,f) is *locally finite* if $tree_A(a)$ is finite for all $a \in A$. Every computable locally finite $(2,0):1$ structure is Δ^0_3-categorical. Cenzer, Harizanov and Remmel [34] proved that every computable locally finite $(2,0):1$ structure with finitely many ω-chains is Δ^0_2-categorical. Walker [156, 157] extended this investigation to $(2,1):1$ structures where for every x, the pre-image $f^{-1}(x)$ has either two or one element.

There is no known characterization of Δ^0_2-categoricity or of higher level categoricity for trees of finite height. Lempp, McCoy, R. Miller and Solomon [115] proved that for every $n \geq 1$, there is a computable tree of finite height, which is Δ^0_{n+1}-categorical but not Δ^0_n-categorical. Fokina, Harizanov and Turetsky established the following result, which also holds when a tree is presented as a directed graph.

Theorem 40. *[55] There is a Δ^0_2-categorical tree of finite height, which is not relatively Δ^0_2-categorical. There is also such a tree of infinite height.*

It follows from [32, 125] that every computable, free, nonabelian group is Δ^0_4-categorical, and the result cannot be improved to Δ^0_3. It was shown in [47] that every computable, free, abelian group is Δ^0_2-categorical, and the result cannot be improved to computable categoricity.

A *homogeneous, completely decomposable, abelian group* is a group of the form $\bigoplus_{i\in\kappa} H$, where H is a subgroup of the additive group of the rationals, $(\mathbb{Q}, +)$. Note that we have only a single H in the sum – any two summands are isomorphic. It is well known that such a group is computably categorical if and only if κ is finite; the proof is similar to the analogous result that a computable vector space is computably categorical if and only if it has finite dimension.

For P a set of primes, define $Q^{(P)}$ to be the subgroup of $(\mathbb{Q}, +)$ generated by $\{\frac{1}{p^k} : p \in P \wedge k \in \omega\}$. Downey and Melnikov [47] showed that a computable, homogeneous, completely decomposable, abelian group of infinite rank is Δ^0_2-categorical if and only if it is isomorphic to $\bigoplus_{\omega} Q^{(P)}$, where P is c.e. and the set (Primes $-$ P) is semi-low. Recall that a set $S \subseteq \omega$ is *semi-low* if the set $H_S = \{e : W_e \cap S \neq \emptyset\}$ is computable from $\emptyset'$. In [55], we proved that a computable, homogeneous, completely decomposable, abelian group of infinite rank is relatively Δ^0_2-categorical if and only if it is isomorphic to $\bigoplus_{\omega} Q^{(P)}$, where P is a computable set of primes. Since there exist co-c.e. sets that are semi-low and noncomputable, we obtained the following result.

Theorem 41. *There is a homogeneous, completely decomposable, abelian group, which is Δ_2^0-categorical but not relatively Δ_2^0-categorical.*

The notions of computable categoricity and relative computable categoricity coincide if we add more effectiveness requirements on the structure. Goncharov [65] proved that in the case of 2-decidable structures, computable categoricity and relative computable categoricity coincide. Kudinov showed that the assumption of 2-decidability cannot be weakened, by giving in [113] an example of 1-decidable and computably categorical structure, which is not relatively computably categorical. Recently, Fokina, Harizanov and Turetsky obtained such an example of a Fraïssé limit.

Theorem 42. *[55] There is a* 1*-decidable structure $\mathcal{F}$ that is a Fraïssé limit and computably categorical, but not relatively computably categorical. Moreover, the language for such $\mathcal{F}$ can be finite or it can be relational.*

Ash [8] established that for every computable ordinal α, under certain decidability conditions on $\mathcal{A}$, if $\mathcal{A}$ is Δ_α^0-categorical, then $\mathcal{A}$ is relatively Δ_α^0-categorical.

T. Millar [127] proved that if a structure $\mathcal{A}$ is 1-decidable, then any expansion of $\mathcal{A}$ by finitely many constants remains computably categorical. Cholak, Goncharov, Khoussainov and Shore showed that the assumption of 1-decidability is important. They showed that there is a computable structure, which is computably categorical, but ceases to be after naming any element of the structure. It follows that this structure is not relatively computably categorical since it cannot have a formally c.e. Scott family. Furthermore, Khoussainov and Shore [104] proved that there is a computably categorical structure $\mathcal{A}$ without a formally c.e. Scott family such that the expansion of $\mathcal{A}$ by any finite number of constants is computably categorical.

Downey, Kach, Lempp, and Turetsky have obtained the following result.

Theorem 43. *([46]) Any* 1*-decidable computably categorical structure is relatively Δ_2^0-categorical.*

Based on Theorem 43, we could conjecture that every computable structure that is computably categorical should be relatively Δ_3^0-categorical. However, this is not the case, as proved by Downey, Kach, Lempp, Lewis, Montalbán and Turetsky.

Theorem 44. *([45]) For every computable ordinal α, there is a computably categorical structure that is not relatively Δ_α^0-categorical.*

Thus, a natural question arises whether there is a computably categorical structure that is not relatively hyperarithmetically categorical.

Downey, Kach, Lempp, and Turetsky [46] established the following index set complexity result for relatively computably categorical structures.

Theorem 45. *([46]) The index set of relatively computably categorical structures is Σ_3^0-complete.*

On the other hand, in [45], Downey, Kach, Lempp, Lewis, Montalbán and Turetsky established that there is no simple syntactic characterization of computable categoricity, thus answering a long-standing open question.

Theorem 46. *([45]) The index set of computably categorical structures is* Π^1_1*-complete.*

Goncharov also investigated categoricity restricted to decidable structures (for example, see [59]).

Definition 6. A decidable structure $\mathcal{A}$ is called *decidably categorical* if every two decidable copies of $\mathcal{A}$ are computably isomorphic.

Nurtazin gave the following characterization of decidably categorical structures. Recall that for a complete theory T, a formula $\theta(\overrightarrow{x})$ is called *complete* if for every formula $\psi(\overrightarrow{x})$, either $T \vdash (\theta(\overrightarrow{x}) \Rightarrow \psi(\overrightarrow{x}))$ or $T \vdash (\theta(\overrightarrow{x}) \Rightarrow \neg\psi(\overrightarrow{x}))$.

Theorem 47. *([138]) Let $\mathcal{A}$ be a decidable structure. Then $\mathcal{A}$ is decidably categorical if and only if there is a finite tuple $\overrightarrow{c}$ of elements in A such that $(\mathcal{A}, \overrightarrow{c})$ is a prime model of the theory $Th(\mathcal{A}, \overrightarrow{c})$ and the set of complete formulas of this theory is computable.*

Moreover, Nurtazin proved that if there is no such $\overrightarrow{c}$, then there are infinitely many decidable copies of $\mathcal{A}$, no two of which are computably isomorphic.

Goncharov and Marchuk [74] showed that the index set of computable structures with decidably categorical copies is $\Sigma^0_{\omega+3}$-complete, while for decidably categorical structures the index set is Σ^0_3-complete. Index sets for decidably categorical structures with particular properties were further investigated by Bazhenov, Goncharov and Marchuk.

We say that a structure $\mathcal{A}$ is *categorical relative to n-decidable presentations* if any two n-decidable copies of $\mathcal{A}$ are computably isomorphic. For $n = 0$, we have a computably categorical structure. Fokina, Goncharov, Harizanov, Kudinov and Turetsky investigated for various $m, n \in \omega$, the index sets $I_{n,m}$ for n-decidable structures categorical relative to m-decidable presentations.

Theorem 48. *([54])*
(*a*) *In the case when $m \geq n \geq 0$, the index set $I_{n,m}$ is Π^1_1-complete.*
(*b*) *In the case when $m = n - 1 \geq 0$, the index set $I_{n,m}$ is Π^0_4-complete.*
(*c*) *In the case when $0 \leq m \leq n - 2$, the index set $I_{n,m}$ is Σ^0_3-complete.*

5 Definability and complexity of relations on structures

One of the important questions in computable model theory is how a specific property of a computable structure may change if the structure is isomorphically transformed so that it remains computable. A computable property of a computable structure $\mathcal{A}$, which Ash and Nerode [13] considered, is given by an additional (new) computable relation R on the domain A of $\mathcal{A}$. (That is, R is not named in the language of $\mathcal{A}$.) Ash and Nerode investigated syntactic conditions on $\mathcal{A}$ and R under which for every isomorphism f from $\mathcal{A}$ onto a computable structure $\mathcal{B}$, $f(R)$ is c.e. Such relations are called *intrinsically c.e.* on $\mathcal{A}$. In general, we have the following definition. Let $\mathcal{P}$ be a certain complexity class.

Definition 7. ([13]) An additional relation R on the domain of a computable structure $\mathcal{A}$ is called *intrinsically* $\mathcal{P}$ on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to a computable structure belongs to $\mathcal{P}$.

For example, the successor relation, and being an even number are not intrinsically computable relations on $(\omega, <)$. Clearly, if $\mathcal{A}$ is a computably stable structure, then every computable relation on its domain is intrinsically computable.

If R is definable in $\mathcal{A}$ by a computable Σ_1 formula with finitely many parameters, then R is intrinsically c.e. Ash and Nerode [13] proved that, under a certain extra decidability condition on $\mathcal{A}$ and R, the relation R is intrinsically c.e. on $\mathcal{A}$ iff R is definable by a computable Σ_1 formula with finitely many parameters. The *Ash-Nerode decidability condition* says that for an m-ary relation R, there is an algorithm that determines for every existential formula $\psi(x_0, \ldots, x_{m-1}, \overrightarrow{y})$ and every $\overrightarrow{c} \in A^{lh(\overrightarrow{y})}$, whether the following implication holds for every $\overrightarrow{a} \in A^m$:

$$(\mathcal{A} \vDash \psi(\overrightarrow{a}, \overrightarrow{c})) \Rightarrow R(\overrightarrow{a}).$$

Barker [17] lifted the Ash-Nerode theorem to arbitrary levels of the hyperarithmetical hierarchy. He proved that for a structure $\mathcal{A}$ and an additional relation R on $\mathcal{A}$, under some effectiveness conditions, R is definable by a computable Σ_α formula with finitely many parameters.

For the relative notions, the effectiveness conditions are not needed. Let $\mathcal{P}$ be a certain complexity class, which can be relativized, such as the class of all Σ^0_α sets.

Definition 8. An additional relation R on the domain of a computable structure $\mathcal{A}$ is called *relatively intrinsically* $\mathcal{P}$ on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to any structure $\mathcal{B}$ is $\mathcal{P}$ relative to the atomic diagram of $\mathcal{B}$.

The following equivalence is due to Ash, Knight, Manasse, and Slaman [12], and independently Chisholm [37].

Theorem 49. *([12, 37]) Let $\mathcal{A}$ be a computable structure. An additional relation R on $\mathcal{A}$ is relatively intrinsically Σ^0_α iff R is definable by a computable Σ_α formula with finitely many parameters.*

A relation R on a structure $\mathcal{A}$ that is definable by a computable Σ_α formula with finitely many parameters is also called *formally* Σ^0_α *on* $\mathcal{A}$.

Goncharov [63] and Manasse [120] gave examples of intrinsically c.e. relations on computable structures, which are not relatively intrinsically c.e. This result was lifted to higher levels of the hyperarithmetical hierarchy by Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon for successor ordinals [69], and by Chisholm, Fokina, Goncharov, Harizanov, Knight and Quinn for limit ordinals [38].

Theorem 50. *([69, 38]) For every computable ordinal α, there is a computable structure $\mathcal{A}$ with an intrinsically Σ^0_α relation R such that R is not definable by a computable Σ_α formula with finitely many parameters.*

We will assume that $\mathcal{A}$ is an infinite computable structure, and that R is an additional infinite co-infinite relation on $\mathcal{A}$. Without loss of generality, we assume that R is unary. We are interested in syntactic conditions under which there is a computable copy of $\mathcal{A}$ in which the image of R is simple. We may also ask when the image of $\neg R$ is only immune.

A subset of ω is called *immune* if it is infinite and contains no infinite c.e. subset. A set is *simple* if it is c.e. and its complement is immune. It is established in computability theory that a unary relation C on ω is *hyperimmune*, abbreviated by *h-immune*, iff it is infinite and no computable function majorizes its principal function p_C, where $p_C(n) =_{def} c_n$ provided that $C = \{c_0 < c_1 < c_2 < \cdots\}$. A set is called *hypersimple*, abbreviated by *h-simple*, if it is c.e. and its complement is hyperimmune. The following (canonical) indexing of finite sets is standard. Let $D_0 =_{def} \emptyset$. For $m > 0$, let $D_m = \{d_0, \dots, d_{k-1}\}$, where $d_0 < \cdots < d_{k-1}$ and $m = 2^{d_0} + \cdots + 2^{d_{k-1}}$. A sequence $(U_i)_{i\in\omega}$ of finite sets is a *strong array* if there is a unary computable function f such that for every $i \in \omega$, $U_i = D_{f(i)}$. A strong array is *disjoint* if its members are pairwise disjoint. Let $S \subseteq \omega$. The relation $\neg S$ is *h-immune* (on ω) if it is infinite and there is no disjoint strong array $(U_i)_{i\in\omega}$ such that for every $n \in \omega$, we have $U_i \cap \overline{S} \neq \emptyset$. We can similarly define h-immune relations on any computable set. Every h-immune set is *immune*, that is, infinite but without any infinite c.e. subset. Not every immune set is h-immune.

Results establishing various equivalences of syntactic and corresponding semantic conditions in computable copies of $\mathcal{A}$ usually involve additional effectiveness conditions, expressed in terms of $\mathcal{A}$ and R. To discover syntactic conditions governing the algorithmic properties of images of R in computable copies of $\mathcal{A}$, it is sometimes helpful to consider arbitrary copies of $\mathcal{A}$ and relative versions of the algorithmic properties. One advantage is that we may use the forcing method instead of the priority method—the latter is more complicated. In addition, the relative results should require no additional effectiveness conditions, which often mask the syntactic conditions.

A new relation on a countable structure $\mathcal{B}$ is *immune relative to* $\mathcal{B}$ if it is infinite and contains no infinite subset that is c.e. relative to $\mathcal{B}$. A new relation on a countable structure $\mathcal{B}$ is *simple relative to* $\mathcal{B}$ if it is c.e. relative to $\mathcal{B}$ and its complement is immune relative to $\mathcal{B}$. If we are to construct an isomorphic copy of $\mathcal{A}$ in which the image of $\neg R$ is relatively immune, there must be no infinite subset D of $\neg R$ definable in $\mathcal{A}$ by a computable Σ_1 formula $\varphi(\vec{c}, x)$ (with a finite tuple of parameters $\vec{c}$). This obvious necessary condition turns out to be sufficient.

Theorem 51. *([68]) . Let $\mathcal{A}$ be a computable L-structure, and let R be a unary infinite and co-infinite relation on A. Then the following are equivalent.*

(i) *For all copies $\mathcal{B}$ of $\mathcal{A}$ and all isomorphisms F from $\mathcal{A}$ onto $\mathcal{B}$, $\neg F(R)$ is not immune relative to $\mathcal{B}$.*

(ii) *There are an infinite set D and a finite tuple $\vec{c}$ such that $D \subseteq \neg R$ and D is definable in $\mathcal{A}$ by a computable Σ_1 formula $\varphi(\vec{c}, x)$.*

To prove $(i) \Rightarrow (ii)$ we build a "generic" copy $(\mathcal{B}, S)$ of $(\mathcal{A}, R)$. Under the assumption that $\neg S$, the image of $\neg R$, is not immune relative to $\mathcal{B}$, we produce the set D and a tuple $\vec{c}$ as in (ii). Let B be an infinite computable set, the universe of $\mathcal{B}$. The forcing conditions are the finite $1-1$ partial functions from B to A.

Let $\mathcal{A}$ be an L-structure, and $\mathbf{R}$ be an additional unary relation symbol. If we are interested in c.e. relations, computable Σ_1 formulas with positive occurrences of $\mathbf{R}$ in the expanded language $L \cup \{\mathbf{R}\}$ play an important role. Assume that there is an infinite set $D \subseteq \neg R$ such that D is definable in $(\mathcal{A}, R)$ by a computable Σ_1 formula with finitely many parameters and with only positive occurrences of $\mathbf{R}$. In any copy $\mathcal{B}$ of $\mathcal{A}$, if the image of R is c.e. relative to $\mathcal{B}$, then so is the image of D. Therefore, under this definability assumption, the image of R cannot be made simple relative to $\mathcal{B}$. It turns out that this is the only obstacle.

Theorem 52. *([68]). Let $\mathcal{A}$ be an infinite computable structure in a relational language L, and let R be a computable unary infinite and co-infinite relation on A. Then the following are equivalent.*

(i) *For all copies $\mathcal{B}$ of $\mathcal{A}$ and all isomorphisms F from $\mathcal{A}$ onto $\mathcal{B}$, $F(R)$ is not simple relative to $\mathcal{B}$.*

(ii) *There are an infinite set D and a finite tuple of parameters $\vec{c}$ such that $D \subseteq \neg R$, and D is definable in $(\mathcal{A}, R)$ by a computable Σ_1 formula $\varphi(\vec{c}, x)$ of $L \cup \{\mathbf{R}\}$ with only positive occurrences of $\mathbf{R}$.*

The following results give syntactic conditions that allow the existence of an isomorphism F from $\mathcal{A}$ onto a *computable* copy such that $\neg F(R)$ is immune (or simple). The results involve extra decidability conditions, which imply that both $\mathcal{A}$ and R are computable.

Theorem 53. *([68]). Let $\mathcal{A}$ be an infinite (computable) L-structure, and let R be a unary (computable) infinite and co-infinite relation on A. Assume that we have an effective procedure for deciding whether*

$$(\mathcal{A}_A, R) \models (\exists x \in \mathbf{R})\, \theta(\vec{c}, x),$$

where $\theta(\vec{c}, x)$ is a finitary existential formula of L with finitely many parameters. If there is no infinite set D such that $D \subseteq \neg R$ and D is definable in $\mathcal{A}$ by a *computable Σ_1 formula of L with finitely many parameters, then there is an isomorphism F from $\mathcal{A}$ onto a computable copy $\mathcal{B}$ such that the relation $\neg F(R)$ is immune.*

Proof uses the finite injury priority method.

Theorem 54. *([68]). Let $\mathcal{A}$ be an infinite (computable) L-structure, and let R be a unary (computable) infinite and co-infinite relation on A. Assume that we have an effective procedure for deciding whether*

$$(\mathcal{A}_A, R) \models (\exists x \in \mathbf{R})\, \varphi(\vec{c}, x),$$

where φ is a finitary existential formula in $L \cup \{\mathbf{R}\}$ with finitely many parameters and with positive occurrences of $\mathbf{R}$. If there is no infinite $D \subseteq \neg R$ definable by such a formula, then there is an isomorphism F from $\mathcal{A}$ onto a computable copy $\mathcal{B}$ such that $F(R)$ is simple.

For a finite sequence (tuple) of elements $\vec{c}$, we write $a \in \vec{c}$ to say that $a \in ran(\vec{c})$, and $\vec{c} \cap \vec{d} = \emptyset$ to denote that $ran(\vec{c}) \cap ran(\vec{d}) = \emptyset$.

Example 1. Let $\mathcal{A} = (\omega, <_\omega)$ and let R be the set of all even numbers. First, we show that no infinite subset of the odds is definable by a computable Σ_1 formula (in the language $\{<, \mathbf{R}\}$) with finitely many parameters $\vec{c}$ and positive occurrences of $\mathbf{R}$. Otherwise, we can assume, without loss of generality, that a disjunct of such a formula is a finitary formula $\exists \vec{u} \psi(\vec{c}, x, \vec{u})$ so that the following are true:

i) the formula $\psi(\vec{c}, x, \vec{u})$ is a conjunct which gives the complete ordering of $\vec{c}, x, \vec{u}$ and expresses that certain elements of $\vec{c}, \vec{u}$ are in R;

ii) there is a tuple $\vec{d}$, and an odd number a bigger than every element in $\vec{c}$ so that $(\mathcal{A}_A, R) \models \psi(\vec{c}, a, \vec{d})$.

Define a' and a tuple $\vec{d'}$ as follows:

i) $a' = a + 1$;

ii) if $d_i \in \vec{d}$ and d_i is less than a, set $d'_i =_{def} d_i$;

iii) if $d_i \in \vec{d}$ and d_i is greater than a , set $d'_i =_{def} d_i + 2$.

Clearly, $(\mathcal{A}_A, R) \models \psi(\vec{c}, a', \vec{d'})$. Hence $(\mathcal{A}_A, R) \models \exists \vec{u} \psi(\vec{c}, a', \vec{u})$, but a' is even, which is a contradiction.

Next, the structure $(\mathcal{A}, R)$ satisfies the decidability condition of Theorem 54. Therefore, there is a computable copy $\mathcal{B}$ of $\mathcal{A}$ and $F : \mathcal{A} \cong \mathcal{B}$ so that $F(R)$ is simple.

Example 2. Let $\mathcal{A}$ be an equivalence structure with infinitely many equivalence classes, all of size 2. Let R be a relation containing exactly one element from each class so that the pair $(\mathcal{A}, R)$ satisfies the decidability condition of Theorem

53. No infinite subset of $\neg R$ is definable by a computable Σ_1 formula (in the language $\{E\}$) with only finitely many parameters: if an element a and its equivalent are both outside the parameters, then any formula satisfied by a is also satisfied by its equivalent element. Therefore, there is a computable copy $\mathcal{B}$ and $F : \mathcal{A} \cong \mathcal{B}$ so that $\neg F(R)$ is immune.

However, $\neg R$ is definable by a computable Σ_1 formula $\varphi(x)$ in $\{E, \mathbf{R}\}$ with only positive occurrences of $\mathbf{R}$. Namely, $\varphi(x)$ is the following finitary formula: $\exists y(\mathbf{R}(y) \wedge yEx \wedge y \neq x)$. Therefore, in any copy $\mathcal{B}$ in which $F(R)$ is c.e. relative to $\mathcal{B}$, $F(R)$ is, in fact, computable relative to $\mathcal{B}$.

Example 3. Let $\mathcal{A}$ be an equivalence structure with infinitely many equivalence classes, all of size 2. Let R be a relation such that the following are satisfied:

i) there are infinitely many equivalence classes from which R contains exactly one element;

ii) there are no equivalence classes from which R contains both elements;

iii) there are infinitely many equivalence classes from which R contains neither element;

iv) the pair $(\mathcal{A}, R)$ satisfies the decidability condition of Theorem 53.

No infinite subset of $\neg R$ is definable by a computable Σ_1 formula (in the language $\{E\}$) with only finitely many parameters, so there is a computable copy $\mathcal{B}$ and $F : \mathcal{A} \cong \mathcal{B}$ in which $\neg F(R)$ is immune.

Furthermore, there is a computable copy $\mathcal{B}$ in which the image of R is c.e., but not computable. However, the formula $\varphi(x)$ in the language $\{E, \mathbf{R}\}$: $\exists y(\mathbf{R}(y) \wedge yEx \wedge y \neq x)$ defines an infinite subset of $\neg R$. Consequently, there is no $F : \mathcal{A} \cong \mathcal{B}$ such that $F(R)$ is simple relative to $\mathcal{B}$.

Example 4. Let $\mathcal{A}$ be the structure $(\mathcal{Q}, <_Q)$, and let R be the set of all rationals less than π. There is no computable formula (in the language $\{<\}$) with finitely many parameters which defines $\neg R$. However, the formula "$5 < x$" does define an infinite subset of $\neg R$. Consequently, there is no $F : \mathcal{A} \cong \mathcal{B}$ in which $\neg F(R)$ is immune relative to $\mathcal{B}$.

Example 5. Let $\mathcal{A}$ be an $\aleph_0$-dimensional vector space over a finite field, say over a field with 3 elements. Let R be the domain of a subspace of $\mathcal{A}$ of infinite dimension and infinite co-dimension. There is a computable copy of $\mathcal{A}$ in which the image of R is immune, since the only sets definable in $\mathcal{A}$ are finite and co-finite, and there is a copy also satisfying the effectiveness condition of Theorem 53.

For $a \notin R$, the formula $\varphi(a, x) = (\exists y)[x = a + y]$ defines an infinite subset of $\neg R$ that is c.e. (relative to $\mathcal{B}$) if the image of R is. It follows that the image of R can never be relatively simple.

The following definition of Hird introduces a syntactic property corresponding to h-immunity. We will term it "being formally h-immune on $\mathcal{A}$."

Definition 9. (Hird [91])

1. A *formal strong array on* $\mathcal{A}$ is a computable sequence of existential formulas in L with finitely many parameters $\vec{c}$, $(\psi_i(\vec{c},\vec{x}_i))_{i\in\omega}$, such that for every finite set $G \subseteq A$ there is $i \in \omega$ and a sequence $\vec{a}_i \in A^{lh(\vec{x_i})}$ with

$$(\mathcal{A}_A \models \psi_i(\vec{c},\vec{a}_i)) \wedge (\vec{a}_i \cap G = \emptyset).$$

2. We say that the relation $\neg R$ is *formally h-immune on* $\mathcal{A}$ if there is no formal strong array $(\psi_i(\vec{c},\vec{x}_i))_{i\in\omega}$ on $\mathcal{A}$ such that for every $i \in \omega$,

$$(\forall \vec{a}_i \in A^{lh(\vec{x}_i)})[(\mathcal{A}_A \models \psi_i(\vec{c},\vec{a}_i)) \Rightarrow (\vec{a}_i \cap \neg R \neq \emptyset)].$$

Being formally h-immune on $\mathcal{A}$ turns out to be a necessary condition for the existence of a computable copy of $\mathcal{A}$ such that the corresponding image of R is h-immune (see [91]). Assume that $\mathcal{B}$ is a computable copy of $\mathcal{A}$ and that F is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. The following result establishes that, under some extra decidability conditions for $(\mathcal{A}, R)$, the existence of a computable copy $\mathcal{B}$ of $\mathcal{A}$ such that the image of $\neg R$ is h-immune relative to $\mathcal{B}$ is equivalent to $\neg R$ being formally h-immune on $\mathcal{A}$.

Theorem 55. *([91])* (*a*) *Assume that* $\mathcal{B}$ *is a computable copy of* $\mathcal{A}$ *and that* F *is an isomorphism from* $\mathcal{A}$ *onto* $\mathcal{B}$*. If* $F(\neg R)$ *is h-immune on* B*, then* $\neg R$ *is formally h-immune on* $\mathcal{A}$*.*

(*b*) *Assume that there is an algorithm which decides for a given sequence* $\vec{c} \in A^{<\omega}$ *and an existential formula* $\psi(\vec{u},\vec{x})$ *in* L*,* $lh(\vec{u}) = lh(\vec{c})$*, whether*

$$(\forall \vec{a} \in A^{lh(\vec{x})})[(\mathcal{A}_A \models \psi(\vec{c},\vec{a})) \Rightarrow (\vec{a} \cap \neg R \neq \emptyset)].$$

If $\neg R$ *is formally h-immune on* $\mathcal{A}$*, then there is a computable structure* $\mathcal{B}$ *and an isomorphism* F *from* $\mathcal{A}$ *onto* $\mathcal{B}$ *such that the relation* $F(\neg R)$ *is h-immune on* B*.*

We now introduce a relative version of h-immunity.

Definition 10. Let S be an additional (unary) relation on the domain B of a countable structure $\mathcal{B}$.

1. A sequence $(U_i)_{i\in\omega}$ of finite sets is a *strong array relative to* $\mathcal{B}$ if there is a unary $\mathcal{B}$-computable function f such that for every $i \in \omega$, $U_i = D_{f(i)}$.

2. The relation $\neg S$ is *h-immune relative to* $\mathcal{B}$ if it is infinite and there is no disjoint strong array relative to $\mathcal{B}$, $(U_i)_{i\in\omega}$, such that for every $n \in \omega$, we have $U_i \cap \overline{S} \neq \emptyset$.

If there is an isomorphic copy of $\mathcal{A}$ on which the image of $\neg R$ is relatively h-immune, then $\neg R$ must be formally h-immune on $\mathcal{A}$. This necessary syntactic condition turns out to be sufficient.

Theorem 56. *([67]) Let $\mathcal{A}$ be a computable L-structure, and let R be a unary infinite and co-infinite relation on A. Then the following are equivalent.*

(i) For all copies $\mathcal{B}$ of $\mathcal{A}$ and all isomorphisms F from $\mathcal{A}$ onto $\mathcal{B}$, $\neg F(R)$ is not h-immune relative to $\mathcal{B}$.

(ii) The relation $\neg R$ is not formally h-immune on $\mathcal{A}$.

Hird [91] established that, under a suitable decidability condition, R is formally h-simple on $\mathcal{A}$ iff there is a computable copy $\mathcal{B}$ of $\mathcal{A}$ such that the image of R under an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ is h-simple on $\mathcal{B}$. In [67] we give a relative analogue of Hird's result for h-simple relations on computable copies. Harizanov [77] gave a sufficient general conditions for the existence of an h-simple relation on a computable copy of $\mathcal{A}$, in arbitrary nonzero c.e. Turing degree. For more on syntactic characterizations of relations having Post-type and similar properties on structures, or their degree-theoretic complexity see [90, 91, 14, 78, 68, 77, 67].

In addition to considering the complexity of relations on computable structures within hyperarithmetical hierarchy, we can also consider their degrees, such as Turing degrees or strong degrees. Harizanov introduced the following notion.

Definition 11. ([81]) The *Turing degree spectrum* of R on $\mathcal{A}$, in symbols $DgSp_{\mathcal{A}}(R)$, is the set of all Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ onto computable structures.

Let $\mathcal{L} = (\omega, \prec)$ be the following computable linear order of ordering type $\omega + \omega^*$:

$$0 \prec 2 \prec 4 \prec \cdots \prec 5 \prec 3 \prec 1.$$

We define a computable relation R to be the initial segment of type ω; that is, $R = 2\omega$. An early result, obtained independently by Tennenbaum and Denisov, is that there is an isomorphic computable copy of $\mathcal{L}$ such that its initial segment of type ω is not computable. It is easy to see that the relation R is intrinsically Δ^0_2 on $\mathcal{L}$, because of the following definability of R and $\neg R$:

$$\begin{aligned} x \quad &\in \quad R \Leftrightarrow \bigvee_{n\in\omega} \exists x_0 \dots \exists x_n [x_0 \prec x_1 \prec \cdots \prec x_n \wedge x = x_n \wedge \\ \forall y[\neg(y \quad &\prec \quad x_0) \wedge \neg(x_0 \prec y \prec x_1) \wedge \cdots \wedge \neg(x_{n-1} \prec y \prec x_n)], \end{aligned}$$

and

$$\begin{aligned} x \quad &\notin \quad R \Leftrightarrow \bigvee_{n\in\omega} \exists x_0 \dots \exists x_n [x_0 \succ x_1 \succ \cdots \succ x_n \wedge x = x_n \wedge \\ \forall y[\neg(y \quad &\succ \quad x_0) \wedge \neg(x_0 \succ y \succ x_1) \wedge \cdots \wedge \neg(x_{n-1} \succ y \succ x_n)]. \end{aligned}$$

It can further be shown that the degree spectrum $Dg_{\mathcal{L}}(R)$ consists of all Δ^0_2 degrees (see [79]).

Ershov classified Δ_2^0 sets as follows. Let α be a computable ordinal. A set $C \subseteq \omega$ is α-c.e. if there are a computable function $f : \omega^2 \to \{0,1\}$ and a computable function $o : \omega \times \omega \to \alpha + 1$ with the following properties:

$$\begin{aligned}(\forall x)[f(x,0) &= 0 \ \wedge \ \lim_{s\to\infty} f(x,s) = C(x)],\\ (\forall x)(\forall s)[o(x,0) &= \alpha \ \wedge \ o(x,s+1) \leq o(x,s)], \text{ and}\\ (\forall x)(\forall s)[f(x,s+1) &\neq f(x,s) \ \Rightarrow \ o(x,s+1) < o(x,s)].\end{aligned}$$

In particular, 1-c.e. sets are c.e. sets, and 2-c.e. sets are d.c.e. sets.

Let R be an additional unary computable relation on the domain of a computable structure $\mathcal{A}$. We are interested in syntactic conditions such that for an α-c.e. degree $\mathbf{c}$, there is an isomorphism f of degree $\mathbf{c}$ to a computable structure $\mathcal{B}$ for which $f(R)$ is also of degree $\mathbf{c}$. First we need the following definition.

The complement of R with respect to A is denoted $\overline{R}$. Let $\mathbf{R}$ be a symbol for R. If we are interested in the c.e. images of R, certain first-order formulas with positive occurrences of $\mathbf{R}$ in the expanded language $L(\mathcal{A}) \cup \{\mathbf{R}\}$ play a special role. A Σ_1 formula in $L(\mathcal{A}) \cup \{\mathbf{R}\}$ in which $\mathbf{R}$ occurs only positively is also called a Σ_1^Γ formula. This notation was introduced by Ash and Knight, who defined a hierarchy of infinitary formulas in a general setting in which Γ is a function assigning computable ordinals to relation symbols.

For an ordinal γ, define is a binary relation $\leq_\gamma$ on finite sequences of elements from A, of equal length, by: $\overrightarrow{l} \leq_\gamma \overrightarrow{r}$ iff every Π_γ formula true of $\overrightarrow{l}$ is also true of $\overrightarrow{r}$ (equivalently, every Σ_γ formula true of $\overrightarrow{r}$ is also true of $\overrightarrow{l}$).

Definition 12. Let $\overrightarrow{c} \in A^{<\omega}$ and $a \in A$.

1. (Harizanov) We say that a is *free over* $\overrightarrow{c}$ (also called 1-*free* over $\overrightarrow{c}$) if $a \in \overline{R}$ and for every finitary Σ_1^Γ formula $\psi(\overrightarrow{z}, x)$, $lh(\overrightarrow{z}) = lh(\overrightarrow{c})$, if

$$(\mathcal{A}_A, R) \models \psi(\overrightarrow{c}, a)$$

then $(\exists a' \in R)[(\mathcal{A}_A, R) \models \psi(\overrightarrow{c}, a')]$.

2. (Ash-Knight) Let β be a computable ordinal such that $\beta > 1$. The element a is β-*free over* $\overrightarrow{c}$ if $a \in \overline{R}$ and for every ordinal γ such that $1 \leq \gamma < \beta$,

$$(\forall \overrightarrow{u})(\exists a' \in R)(\exists \overrightarrow{v})[\overrightarrow{c}\,\hat{}\,a\,\hat{}\,\overrightarrow{u} \leq_\gamma \overrightarrow{c}\,\hat{}\,a'\,\hat{}\,\overrightarrow{v}].$$

Let the set of all free elements over $\overrightarrow{c}$ be denoted by $fr(\overrightarrow{c})$. Note that $fr(\overrightarrow{c}) \subseteq \overline{R}$. Clearly, if $a \in fr(\overrightarrow{c})$ and $\overrightarrow{d}$ is a subsequence of $\overrightarrow{c}$, then $a \in fr(\overrightarrow{d})$. Let

$$bd(\overrightarrow{c}) =_{def} \{a \in \overline{R}\text{: } a \text{ is not free over } \overrightarrow{c}\}.$$

Thus, if $a \in bd(\overrightarrow{c})$ and $\overrightarrow{c}$ is a subsequence of $\overrightarrow{d}$, then $a \in bd(\overrightarrow{d})$. A maximal relation on $\overline{R}$ (with respect to the set-theoretic inclusion) that is definable by a computable Σ_1^Γ formula with parameters $\overrightarrow{c}$ is of the form $bd(\overrightarrow{c})$. Conversely, if $bd(\overrightarrow{c})$ is definable by a computable Σ_1^Γ formula with parameters $\overrightarrow{c}$, then $bd(\overrightarrow{c})$ is a maximal relation on $\overline{R}$ definable by such a formula. For example, if $(\mathcal{A}, R) = (\mathcal{A}_0, R_0)$ then

$$a \in fr(\overrightarrow{c}) \Leftrightarrow a \prec c_{i_0},$$

where c_{i_0} is the $\prec$-least element in $\overline{R} \cap ran(\overrightarrow{c})$. If $\mathcal{A}$ is $(\omega, =)$ and R is a computable infinite co-infinite subset of ω, then

$$a \in fr(\overrightarrow{c}) \Leftrightarrow a \notin ran(\overrightarrow{c}).$$

Theorem 57. (*Harizanov [80]*) *Assume that there is an algorithm which for every $\overrightarrow{c} \in A^{<\omega}$ outputs an element $a \in A$ such that a is free over $\overrightarrow{c}$. Let $C \subseteq \omega$ be a c.e. set. Then there exists a computable structure $\mathcal{B}$ and an isomorphism $f : \mathcal{A} \rightarrow \mathcal{B}$ such that*

$$f(R) \equiv_T f \equiv_T C \& (f(R) \textit{ is c.e.}).$$

Proof. Let $\{C_s\}_{s\in\omega}$ be a computable enumeration of C such that at every stage s, C receives at most one new element, and that element is $\leq s$. Let $B = \omega$ and let $(\theta_e)_{e\in\omega}$ be an effective list of all atomic sentences in $L(\mathcal{A})_B$. For every e, either θ_e or $\neg\theta_e$ will be enumerated in the diagram of $\mathcal{B}$. At every stage s of the construction, we define a finite isomorphism f_s from $\mathcal{B}$ to $\mathcal{A}$. Let $X_s = f_s^{-1}(R)$ and

$$\overline{X_s} = \{d_0^s < d_1^s < d_2^s < \cdots\}.$$

We set $X =_{def} \bigcup_{s\in\omega} X_s$. During the construction, we define a partially computable function $h(n, s)$ such that for every s,

$$h(n, s) \downarrow \Leftrightarrow n \in \{0, \ldots, s + 1\}.$$

For every n, there exists $\lim_{s\to\infty} h(n, s)$, and for every c,

$$c \in C_s - C_{s-1} \Rightarrow d^{s-1}_{\gamma(c,s-1)} \in X_s.$$

□

Let α be a nonzero ordinal. By Cantor's normal form theorem, there is a unique representation

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \cdots + \omega^{\alpha_k} \cdot n_k,$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ and $0 < n_1, n_2, \ldots, n_k < \omega$. Let

$$c_n(\alpha) =_{def} \omega^{\alpha_1} \cdot (nn_1) + \omega^{\alpha_2} \cdot (nn_2) + \cdots + \omega^{\alpha_k} \cdot (nn_k),$$

and

$$c^*(\alpha) =_{def} \sup\{c_n(\alpha) : n \in \omega\}.$$

Hence, if β is the greatest ordinal such that $\omega^\beta \leq \alpha$, then $c^*(\alpha) = \omega^{\beta+1} \leq \alpha \cdot \omega$.

Theorem 58. (Ash-$Cholak$-$Knight$ [10]) *Let $C \subseteq \omega$ be an α-c.e. set where $\alpha \geq 2$. Assume that the relations $(\leq_\gamma)_{1 \leq \gamma < c^*(\alpha)}$ are uniformly c.e. Assume that for every $n \in \omega$, for every sequence $\overrightarrow{c} \in A^{<\omega}$, there is $a \in \overline{R}$ such that a is $c_n(\alpha)$-free over $\overrightarrow{c}$. Then there is a computable model $\mathcal{B}$ and an isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that*

$$f(R) \equiv_T C \equiv_T f.$$

Montalbán (see [86]) introduced the following definition of a degree spectrum of a relation on a cone. The intuition is that we have some fixed set of information we can access, and we must use the same information to view all copies of the structure.

Definition 13. Let R be an additional relation on a structure $\mathcal{A}$, and S an additional relation on a structure $\mathcal{B}$. We say that R and S have the *same Turing degree spectrum on a cone* if there is a Turing degree $\mathbf{d}$ such that for every Turing degree $\mathbf{c} \geq \mathbf{d}$, we have

$$\{\deg(R^{\mathcal{A}_1})\oplus\mathbf{c} : \mathcal{A}_1 \cong \mathcal{A} \,\&\, \deg(\mathcal{A}_1) \leq \mathbf{c}\} = \{\deg(S^{\mathcal{B}_1})\oplus\mathbf{c} : \mathcal{B}_1 \cong \mathcal{B} \,\&\, \deg(\mathcal{B}_1) \leq \mathbf{c}\}.$$

A set X and its Turing degree are called n-CEA for $n \in \omega$, if there is a sequence $X_0, X_1, \ldots, X_n$ such that $X = X_n$, X_0 is c.e., and X_{i+1} is c.e. in and above X_i for $0 \leq i \leq n-1$.

Theorem 59. *Let $\mathcal{A}$ be a structure and R an additional relation on $\mathcal{A}$.*

(a) *(Harizanov [80]) Relative to a cone, every degree spectrum of a relation is either the singleton consisting of a computable degree or contains all c.e. degrees.*

(b) *(Harrison-Trainor [86]) Relative to a cone, every degree spectrum of a relation is either intrinsically Δ^0_2 or contains all 2-CEA degrees.*

Harrison-Trainor also showed that there is a computable structure $\mathcal{A}$ with relatively intrinsically d-c.e. relations that have incomparable degree spectra relative to every oracle.

For some familiar relations on computable structures, their Turing degree spectra exhibit the dichotomy: either singletons or infinite. Harizanov [80] established that if for a non-intrinsically c.e. relation R on $\mathcal{A}$, the Ash-Nerode decidability condition holds, then $DgSp_{\mathcal{A}}(R)$ must be infinite. Moses [136] proved that a computable relation on a computable linear ordering is either definable by a quantifier-free formula with finitely many constants, or is not intrinsically computable. Hirschfeldt [92] gave a sufficient condition for a relation to have infinite degree spectrum. That is, if R is a computable relation on the domain of a computable structure $\mathcal{A}$ such that there is a Δ^0_2 function f such that $f(\mathcal{A})$ is a computable structure but $f(R)$ is not a computable relation, then $DgSp_{\mathcal{A}}(R)$ must be infinite. Applying this condition to linear orderings and using the proof of Moses's result, Hirschfeldt obtained the following result.

Theorem 60. *([92]) A computable relation on a computable linear ordering is either definable by a quantifier-free formula (in which case it is obviously intrinsically computable) or has an infinite Turing degree spectrum.*

Downey, Goncharov and Hirschfeldt [43] proved that a computable relation on a computable Boolean algebra is either intrinsically computable or has infinite Turing degree spectrum.

Theorem 61. *([43]) A computable relation on a computable Boolean algebra is either definable by a quantifier-free formula with finitely many constants (hence intrinsically computable) or has infinite degree spectrum.*

A similar question about Turing degree spectra dichotomy can be asked for computable relations on other classes of structures such as computable abelian groups. Another interesting question from [43] is whether the degree spectrum of an intrinsically Δ^0_2 relation on a computable linear order is always a singleton or infinite.

Csima, Harizanov, Miller and Montalbán studied computable Fraïssé limits and relations on their domains. A class K of structures is *computably locally finite* if there exists a computable function $g : \omega \to \omega$ such that every structure in K that is generated by n elements contains at most $g(n)$ elements.

Theorem 62. *[42] Let $\mathcal{A}$ be a* 1*-decidable structure for a finite language L, which is homogeneous and computably locally finite. Let R be a unary relation on $\mathcal{A}$. Then the following are equivalent.*

(i) *The degree spectrum of R on $\mathcal{A}$ is not upward closed under Turing reducibility.*

(ii) *The relation R is definable by a quantifier-free formula with parameters in $\mathcal{A}$.*

(iii) *The relation R is intrinsically computable.*

As a corollary of this theorem, we obtain that if K is a class of finite structures for a finite language L, such that $Th_L(K)$ is computably axiomatizable and locally finite, and with computable Fraïssé limit $\mathcal{A}$, then $\mathcal{A}$ is as in Theorem 62.

For more complicated relations on computable structures, Soskov [153] established the following definability result.

Theorem 63. *Let $\mathcal{A}$ be a computable structure and let R be a Δ^1_1 relation on its domain, which is invariant under automorphisms of $\mathcal{A}$. Then R is definable in $\mathcal{A}$ by a computable infinitary formula (without parameters).*

This led to the following characterization of intrinsically Δ^1_1 relations.

Theorem 64. *([153]) For a computable structure $\mathcal{A}$, and a relation R on $\mathcal{A}$, the following are equivalent:*

(i) *R is intrinsically Δ^1_1 on $\mathcal{A}$;*

(ii) *R is relatively intrinsically Δ^1_1 on $\mathcal{A}$;*

(iii) *R is definable in $\mathcal{A}$ by a computable infinitary formula with finitely many parameters.*

In the following theorem characterizing intrinsically Π^1_1 relations, Soskov [152] established the equivalence $(ii) \Leftrightarrow (iii)$, while $(i) \Leftrightarrow (ii)$ was established in [70].

Theorem 65. *([152, 70]) For a computable structure $\mathcal{A}$ and relation R on $\mathcal{A}$, the following are equivalent:*

(i) R is intrinsically Π^1_1 on $\mathcal{A}$;

(ii) R is relatively intrinsically Π^1_1 on $\mathcal{A}$;

(iii) R is definable in $\mathcal{A}$ by a Π^1_1 disjunction of computable infinitary formulas with finitely many parameters.

A relation R on $\mathcal{A}$ defined in $\mathcal{A}$ by a Π^1_1 disjunction of computable infinitary formulas with finitely many parameters is also called *formally* Π^1_1 *on* $\mathcal{A}$. In [70], we showed that if $\mathcal{A}$ is a computable structure and let R a relation on $\mathcal{A}$, which is Π^1_1 and invariant under automorphisms of $\mathcal{A}$, then R is definable in $\mathcal{A}$ by a Π^1_1 disjunction of computable infinitary formulas without parameters.

Here are some examples of computable structures with intrinsically Π^1_1 relations. Let $\mathcal{A}$ be a Harrison ordering, that is, a computable linear ordering of type $\omega_1^{CK}(1+\eta)$, where η is the order type of the rationals. Let R be the initial segment of type ω_1^{CK}. This set R is intrinsically Π^1_1, since it is defined by the disjunction of computable infinitary formulas saying that the interval to the left of x has order type β, for computable ordinals β.

For an ordering $\mathcal{L}$, the interval algebra $I(\mathcal{L})$ is the algebra generated, under finite union, by the half-open intervals $[a,b)$, $(-\infty,b)$, $[a,\infty)$, with endpoints in $\mathcal{L}$. A *Harrison Boolean algebra* is a computable Boolean algebra of type $I(\omega_1^{CK}(1+\eta))$. Let $\mathcal{A}$ be a Harrison Boolean algebra, and let R be the set of superatomic elements. This R is intrinsically Π^1_1, since it is defined by the disjunction of computable infinitary formulas saying that x is a finite join of α-atoms, for computable ordinals α.

A *Harrison p-group* is a computable abelian p-group $\mathcal{G}$ such that its length $\lambda(\mathcal{G}) = \omega_1^{CK}$, every element in its Ulm sequence $(u_\alpha(G))_{\alpha<\omega_1^{CK}}$ is ∞, and the divisible part has infinite dimension. A *Harrison group* is a Harrison p-group for some p. Recall that the Ulm subgroups G_α are defined by $G_\alpha = p^{\omega\alpha}G$, and $u_\alpha(G) =_{def} \dim_{\mathbb{Z}_p} P_\alpha(G)/P_{\alpha+1}(G)$, where $P_\alpha(G) = G_\alpha \cap \{x \in G : px = 0\}$. Let $\mathcal{A}$ be a Harrison group, and let R be the set of elements that have computable ordinal heights, that is, the complement of the divisible part. Then R is intrinsically Π^1_1 on $\mathcal{A}$, since it is defined by the disjunction of computable infinitary formulas saying that x has height α, for computable ordinals α. The divisible part of $\mathcal{G}$ has the same degree as its complement R.

Theorem 66. *[70] The following sets of Turing degrees are equal:*

(i) the set of Turing degrees of maximal well-ordered initial segments of Harrison orderings;

(ii) the set of Turing degrees of superatomic parts of Harrison Boolean algebras;

(iii) the set of Turing degrees of divisible parts of Harrison p-groups;

(iv) the set of Turing degrees of left-most paths of computable trees $\mathcal{T} \subseteq \omega^{<\omega}$ such that T *has a path, but no hyperarithmetical path;*

(v) the set of Turing degrees of Π^1_1 paths through Kleene's $\mathcal{O}$.

References

[1] F. Adams and D. Cenzer, Computability and categoricity of weakly untrahomogeneous structures, *Computability* 6 (2017), pp. 365–389.

[2] R. Alvir, J.F. Knight and C.F.D. McCoy, Complexity of Scott sentences, preprint.https://arxiv.org/abs/1807.02715

[3] U. Andrews, The degrees of categorical theories with recursive models, *Proceedings of the American Mathematical Society* 131 (2013), pp. 2501–2514.

[4] U. Andrews, New spectra of strongly minimal theories in finite languages, *Annals of Pure and Applied Logic* 162 (2011), pp. 367–372.

[5] U. Andrews and J.F. Knight, Strongly minimal theories with recursive models, *Journal of the European Mathematical Society* 20 (2018), pp. 1561–1594.

[6] U. Andrews and J.F. Knight, Spectra of atomic theories, *Journal of Symbolic Logic* 78 (2013), pp. 1189–1198.

[7] U. Andrews and J.S. Miller, Spectra of theories and structures, *Proceedings of the American Mathematical Society* 143 (2015), pp. 1283–1298.

[8] C.J. Ash, Categoricity in hyperarithmetical degrees, *Annals of Pure and Applied Logic* 34 (1987), pp. 1–14.

[9] C.J. Ash, Recursive labeling systems and stability of recursive structures in hyperarithmetical degrees, *Transactions of the American Mathematical Society* 298 (1986), pp. 497–514.

[10] C.J. Ash, P. Cholak and J.F. Knight, Permitting, forcing, and copying of a given recursive relation, *Annals of Pure and Applied Logic* 86 (1997), pp. 219–236.

[11] C. Ash and J. Knight, *Computable Structures and the Hyperarithmetical Hierarchy*, Elsevier, Amsterdam, 2000.

[12] C. Ash, J. Knight, M. Manasse, and T. Slaman, Generic copies of countable structures, *Annals of Pure and Applied Logic* 42 (1989), pp. 195–205.

[13] C.J. Ash and A. Nerode, Intrinsically recursive relations, in: J.N. Crossley, editor, *Aspects of Effective Algebra* (U.D.A. Book Co., Steel's Creek, Australia, 1981), pp. 26–41.

[14] C.J. Ash, J.F. Knight, and J.B. Remmel, Quasi-simple relations in copies of a given recursive structure, *Annals of Pure and Applied Logic* 86 (1997), 203–218.

[15] S.A. Badaev, Computable enumerations of families of general recursive functions, *Algebra and Logic* 16 (1977), pp. 129–148 (Russian); (1978) pp. 83–98 (English translation).

[16] E. Barker, Back and forth relations for reduced abelian p-groups, *Annals of Pure and Applied Logic* 75 (1995), pp. 223–249.

[17] E. Barker, Intrinsically Σ^0_α relations, *Annals of Pure and Applied Logic* 39 (1988), pp. 105–130.

[18] J. Barwise, Infinitary logic and admissible sets, *Journal of Symbolic Logic* 34 (1969), pp. 226–252.

[19] N.A. Bazhenov, Δ^0_2-categoricity of Boolean algebras, *Journal of Mathematical Sciences* 203 (2014), pp. 444–454.

[20] W. Calvert, *Algebraic Structure and Computable Structure*, PhD dissertation, University of Notre Dame, 2005.

[21] W. Calvert, The isomorphism problem for computable abelian p-groups of bounded length, *Journal of Symbolic Logic* 70 (2005), pp. 331–345.

[22] W. Calvert, The isomorphism problem for classes of computable fields, *Archive for Mathematical Logic* 43 (2004), pp. 327–336.

[23] W. Calvert, D. Cenzer, V. Harizanov, and A. Morozov, Effective categoricity of Abelian p-groups," *Annals of Pure and Applied Logic* 159 (2009), pp. 187–197.

[24] W. Calvert, D. Cenzer, V. Harizanov, and A. Morozov, Effective categoricity of equivalence structures, *Annals of Pure and Applied Logic* 141 (2006), pp. 61–78.

[25] W. Calvert, E. Fokina, S. Goncharov, J. Knight, O. Kudinov, A. Morozov, and V. Puzarenko, Index sets for classes of high rank structures, *Journal of Symbolic Logic* 72 (2007), pp. 1418–1432.

[26] W. Calvert, A. Frolov, V. Harizanov, J. Knight, C. McCoy, A. Soskova, and S. Vatev, Strong jump inversion, *Journal of Logic and Computation* 7 (2018), pp. 1499–1522.

[27] W. Calvert, S.S. Goncharov and J.F. Knight, Computable structures of Scott rank ω_1^{CK} in familiar classes, in: S. Gao, S. Jackson, and Y. Zhang, eds., Advances in logic, *Contemporary Mathematics* 425 (American Mathematical Society, Providence, RI, 2007), pp. 49–66.

[28] W. Calvert, S. Goncharov, J. Millar, and J. Knight, Categoricity of computable infinitary theories, *Archive for Mathematical Logic* 48 (2009), pp. 25–38.

[29] W. Calvert, V.S. Harizanov, J.F. Knight, and S. Miller, Index sets of computable structures, *Algebra and Logic* 45 (2006), pp. 306–325.

[30] W. Calvert and J.F. Knight, Classification from a computable point of view, *Bulletin of Symbolic Logic* 12 (2006), pp. 191–218.

[31] W. Calvert, J.F. Knight, and J. Millar, Computable trees of Scott rank ω_1^{CK}, and computable approximability, *Journal of Symbolic Logic* 71 (2006), pp. 283–298.

[32] J. Carson, V. Harizanov, J. Knight, K. Lange, C. Safranski, C. McCoy, A. Morozov, S. Quinn, and J. Wallbaum, Describing free groups, *Transactions of the American Mathematical Society* 364 (2012), pp. 5715–5728.

[33] D. Cenzer, V. Harizanov, and J. Remmel, Effective categoricity of injection structures, *Algebra and Logic* 53 (2014), pp. 39–69 (English translation).

[34] D. Cenzer, V. Harizanov, and J. Remmel, Two-to-one structures, *Journal of Logic and Computation* 23 (2013), pp. 1195–1223.

[35] D. Cenzer and J.B. Remmel, Complexity-theoretic model theory and algebra, in: Yu.L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, eds., *Handbook of Recursive Mathematics*, vol. 1, Studies in Logic and the Foundations of Mathematics 139 (North-Holland, Amsterdam, 1998), pp. 381–513.

[36] C.C. Chang and H.J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

[37] J. Chisholm, Effective model theory vs. recursive model theory, *Journal of Symbolic Logic* 55 (1990), pp. 1168–1191.

[38] J. Chisholm, E. Fokina, S. Goncharov, V. Harizanov, J. Knight, and S. Quinn, Intrinsic bounds on complexity and definability at limit levels, *Journal of Symbolic Logic* 74 (2009), pp. 1047–1060.

[39] J. Chisholm and M. Moses, An undecidable linear order that is n-decidable for all n, *Notre Dame Journal of Formal Logic* 39 (1998), pp. 519–526.

[40] P. Cholak, S. Goncharov, B. Khoussainov, and R.A. Shore, Computably categorical structures and expansions by constants, *Journal of Symbolic Logic* 64 (1999), pp. 13–37.

[41] P. Cholak, R.A. Shore, and R. Solomon, A computably stable structure with no Scott family of finitary formulas. *Archive for Mathematical Logic* 45 (2006), pp. 519–538.

[42] B. Csima, V. Harizanov, R. Miller and A. Montalbán, Computability of Fraïssé limits, *Journal of Symbolic Logic* 76 (2011), pp. 66–93.

[43] R.G. Downey, S.S. Goncharov, and D.R. Hirschfeldt, Degree spectra of relations on Boolean algebras, *Algebra and Logic* 42 (2003), pp. 105–111.

[44] R. Downey and C.G. Jockusch, Every low Boolean algebra is isomorphic to a recursive one, *Proceedings of the American Mathematical Society* 122 (1994), pp. 871–880.

[45] R.G. Downey, A.M. Kach, S. Lempp, A.E.M. Lewis-Pye, A. Montalbán, and D.D. Turetsky, The complexity of computable categoricity, *Advances in Mathematics* 268 (2015), pp. 423–466.

[46] R.G. Downey, A.M. Kach, S. Lempp, and D.D. Turetsky, Computable categoricity versus relative computable categoricity, *Fundamenta Mathematicae* 221 (2013), pp. 129–159.

[47] R.G. Downey, and A.G. Melnikov, Effectively categorical abelian groups, *Journal of Algebra* 373 (2013), pp. 223–248.

[48] R.G. Downey, A.G. Melnikov, and K.M. Ng, Abelian p-groups and the halting problem, *Annals of Pure and Applied Logic* 167 (2016), 1123–1138.

[49] Yu.L. Ershov, *Decidability Problems and Constructive Models*, Nauka, Moscow, 1980 (Russian).

[50] Yu.L. Ershov and S.S. Goncharov, *Constructive Models*, Siberian School of Algebra and Logic, Kluwer Academic/Plenum Publishers, 2000 (English translation).

[51] E.B. Fokina, Index sets for some classes of structures, *Annals of Pure and Applied Logic* 157 (2009), pp. 139–147.

[52] E.B. Fokina, Index sets of decidable models, *Siberian Mathematical Journal* 48 (2007), pp. 939–948 (English translation).

[53] E. Fokina, On complexity of categorical theories with computable models, *Vestnik NGU* 5 (2005), pp. 78–86 (Russian).

[54] E. Fokina, S. Goncharov, V. Harizanov, O. Kudinov, and D. Turetsky, Index sets for n-decidable structures categorical relative to m-decidable presentations, *Algebra and Logic* 54 (2015), pp. 336–341 (English translation).

[55] E. Fokina, V. Harizanov and D. Turetsky, Computability-theoretic categoricity and Scott families, *Annals of Pure and Applied Logic* 170 (2019), pp. 699–717.

[56] C. Freer, *Models with High Scott Rank*, PhD dissertation, Harvard University, 2008.

[57] A.N. Frolov, Effective categoricity of computable linear orderings, *Algebra and Logic* 54 (2015), pp.415–417 (English translation).

[58] A.N. Gavryushkin, Spectra of computable models for Ehrenfeucht theories, *Algebra and Logic* 46 (2007), pp. 149–157 (English translation).

[59] S.S. Goncharov, Autostability of prime models with respect to strong constructivizations, *Algebra and Logic* 48 (2009), pp. 410–417 (English translation).

[60] S.S. Goncharov, Autostable models and algorithmic dimensions, in: Yu.L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, eds. *Handbook of Recursive Mathematics*, vol. 1, (North-Holland, Amsterdam, 1998), pp. 261–287.

[61] S.S. Goncharov, Autostability of models and abelian groups, *Algebra and Logic* 19 (1980), pp. 13–27 (English translation).

[62] S.S. Goncharov, Strong constructivizability of homogeneous models, *Algebra and Logic* 17 (1978), pp. 247–263 (English translation).

[63] S.S. Goncharov, The quantity of nonautoequivalent constructivizations, *Algebra and Logic* 16 (1977), pp. 169–185 (English translation).

[64] S.S. Goncharov, Restricted theories of constructive Boolean algebras, *Siberian Mathematical Journal* 17 (1976), pp. 601–611 (English translation).

[65] S.S. Goncharov, Selfstability and computable families of constructivizations, *Algebra and Logic* 14 (1975), pp. 647–680 (Russian).

[66] S.S. Goncharov and V.D. Dzgoev, Autostability of models, *Algebra and Logic* 19 (1980), pp. 28–37 (English translation).

[67] S.S. Goncharov, V.S. Harizanov, J.F. Knight, and C.F.D. McCoy, Relatively hyperimmune relations on structures, *Algebra and Logic* 43 (2004), pp. 94–101 (English translation).

[68] S.S. Goncharov, V.S. Harizanov, J.F. Knight, and C.F.D. McCoy, Simple and immune relations on countable structures, *Archive for Mathematical Logic* 42 (2003), pp. 279–291.

[69] S. Goncharov, V. Harizanov, J. Knight, C. McCoy, R. Miller, and R. Solomon, Enumerations in computable structure theory, *Annals of Pure and Applied Logic* 136 (2005), pp. 219–246.

[70] S.S. Goncharov, V.S. Harizanov, J.F. Knight, and R.A. Shore, Π^1_1 relations and paths through $\mathcal{O}$, *Journal of Symbolic Logic* 69 (2004), pp. 585–611.

[71] S.S. Goncharov and B. Khoussainov, Complexity of theories of computable categorical models, *Algebra and Logic* 43 (2004), pp. 365–373 (English translation).

[72] S.S. Goncharov and J.F. Knight, Computable structure and non-structure theorems, *Algebra and Logic* 41 (2002), pp. 351–373.

[73] S. Goncharov, S. Lempp, and R. Solomon, The computable dimension of ordered abelian groups, *Advances in Mathematics* 175 (2003), pp. 102–143.

[74] S.S. Goncharov and M.I. Marchuk, Index sets of constructive models that are autostable under strong constructivizations, *Journal of Mathematical Sciences (New York)* 205 (2015), pp. 368–388.

[75] S.S. Goncharov and A.T. Nurtazin, Constructive models of complete decidable theories, *Algebra and Logic* 12 (1974) pp. 67–77 (English translation).

[76] N. Greenberg, A. Montalbán, and T.A. Slaman, Relative to any non-hyperarithmetic set, *Journal of Mathematical Logic* 13 (2013), pp. 1–26.

[77] V. Harizanov, Turing degrees of hypersimple relations on computable structures, *Annals of Pure and Applied Logic* 121 (2003), pp. 209–226.

[78] V. Harizanov, Effectively nowhere simple relations on computable structures, in: M.M. Arslanov and S. Lempp, eds., *Recursion Theory and Complexity* (Walter de Gruyter, Berlin, 1999), pp. 59–70.

[79] V.S. Harizanov, Turing degrees of certain isomorphic images of computable relations, *Annals of Pure and Applied Logic* 93 (1998), pp. 103–113.

[80] V.S. Harizanov, Some effects of Ash-Nerode and other decidability conditions on degree spectra, *Annals of Pure and Applied Logic* 55 (1991), pp. 51–65.

[81] V.S. Harizanov, *Degree Spectrum of a Recursive Relation on a Recursive Structure*, PhD dissertation, University of Wisconsin, Madison, 1987.

[82] V.S. Harizanov, J.F. Knight and A.S. Morozov, Sequences of n-diagrams, *Journal of Symbolic Logic* 67 (2002), pp. 1227–1247.

[83] L. Harrington, Recursively presentable prime models, *Journal of Symbolic Logic* 39 (1974), pp. 305–309.

[84] K. Harris, Δ^0_2-categorical Boolean algebras, preprint.

[85] J. Harrison, Recursive pseudo-well-orderings, *Transactions of the American Mathematical Society* 131 (1968), pp. 526–543.

[86] M. Harrison-Trainor, *Degree Spectra of Relations on a Cone*, Memoirs of the American Mathematical Society, vol. 253, 2018.

[87] M. Harrison-Trainor, There is no classification of the decidably presentable structures, *Journal of Mathematical Logic* 18 (2018), 1850010, 41pp.

[88] M. Harrison-Trainor and M-C. Ho, On optimal sentences of finitely generated algebraic structures, *Proceedings of the American Mathematical Society* 146 (2018), pp. 4473–4485.

[89] M. Harrison-Trainor, G. Igusa, and J.F. Knight, Some new computable structures of high rank, *Proceedings of the American Mathematical Society* 146 (2018), pp. 3097–3109.

[90] G. Hird, Recursive properties of intervals of recursive linear orders, in: J. N. Crossley, J. B. Remmel, R. A. Shore, and M. E. Sweedler, eds., *Logical Methods* (Birkhäuser, Boston, 1993), pp. 422–437.

[91] G.R. Hird, Recursive properties of relations on models, *Annals of Pure and Applied Logic* 63 (1993), pp. 241–269.

[92] D.R. Hirschfeldt, Degree spectra of relations on computable structures in the presence of Δ^0_2 isomorphisms, *Journal of Symbolic Logic* 67 (2002), pp. 697–720.

[93] D.R. Hirschfeldt, Prime models of theories of computable linear orderings, *Proceedings of the American Mathematical Society* 129 (2001), pp. 3079–3083.

[94] D.R. Hirschfeldt, B. Khoussainov, and R.A. Shore, A computably categorical structure whose expansion by a constant has infinite computable dimension, *Journal of Symbolic Logic* 68 (2003), pp. 1199–1241.

[95] D.R. Hirschfeldt, B. Khoussainov, R.A. Shore, and A.M. Slinko, Degree spectra and computable dimensions in algebraic structures, *Annals of Pure and Applied Logic* 115 (2002), pp. 71–113.

[96] D. Hirschfeldt, K. Kramer, R. Miller, and A. Shlapentokh, Categoricity properties for computable algebraic fields, *Transactions of the American Mathematical Society* 367 (2015), pp. 3955–3980.

[97] M-C. Ho, Describing groups, *Proceedings of the American Mathematical Society* 145 (2017), pp. 2233–2239.

[98] C.G. Jockusch and R.I. Soare, Degrees of orderings not isomorphic to recursive linear orderings, *Annals of Pure and Applied Logic* 52(1991), pp. 39–64.

[99] A.M. Kach and D. Turetsky, Δ^0_2-categoricity of equivalence structures, *New Zealand Journal of Mathemtics* 39 (2009), pp. 143–149.

[100] O. Kharlampovich and A. Myasnikov, Elementary theory of free non-abelian groups, *Journal of Algebra* 302 (2006), pp. 215–242.

[101] N.G. Khisamiev, Theory of abelian groups with constructive models, *Siberian Mathematical Journal* 27 (1986), pp. 572–585 (English translation).

[102] N.G. Khisamiev, Strongly constructive models of a decidable theory, *Izvestiya Akademii Nauk Kazakhskoj SSR, Seriya Fiziko-Matematicheskaya* 1 (1974), pp. 83–84 (Russian).

[103] B. Khoussainov and A. Montalbán, A computable $\aleph_0$-categorical structure whose theory computes true arithmetic, *Journal of Symbolic Logic* 75 (2010), pp. 728–740.

[104] B. Khoussainov and R.A. Shore, Computable isomorphisms, degree spectra of relations and Scott families, *Annals of Pure and Applied Logic* 93 (1998), pp. 153–193.

[105] J.F. Knight, Minimality and completions of PA, *Journal of Symbolic Logic* 66 (2001), pp. 1447–1457.

[106] J.F. Knight, Degrees coded in jumps of orderings, *Journal of Symbolic Logic* 51 (1986), pp. 1034–1042.

[107] J. Knight, Nonarithmetical $\aleph_0$-categorical theories with recursive models, *Journal of Symbolic Logic* 59 (1994), pp. 106–112.

[108] J.F. Knight and C. McCoy, Index sets and Scott sentences, *Archive for Mathematical Logic* 53 (2014), pp. 519–524.

[109] J.F. Knight and J. Millar, Computable structures of Scott rank ω_1^{CK}, *Journal of Mathematical Logic* 10 (2010), pp. 31–43.

[110] J.F. Knight and V. Saraph, Scott sentences for certain groups, *Archive for Mathematical Logic* 57 (2017), pp. 453–473.

[111] J.F. Knight and M. Stob, Computable Boolean algebras, *Journal of Symbolic Logic* 65 (2000), pp. 1605–1623.

[112] K.Zh. Kudaibergenov, Effectively homogenous models, *Siberian Mathematical Journal* 27 (1986), pp. 180–182 (Russian).

[113] O. Kudinov, An autostable 1-decidable model without a computable Scott family of $\exists$-formulas, *Algebra and Logic* 35 (1996), pp. 458–467.

[114] P. LaRoche, Recursively presented Boolean algebras, *Notices AMS* 24 (1977), A552–A553.

[115] S. Lempp, C. McCoy, R. Miller and R. Solomon, Computable categoricity of trees of finite height, *Journal of Symbolic Logic* 70 (2005), pp. 151–215.

[116] S. Lempp, C.F.D. McCoy, A.S. Morozov, and R. Solomon, Group theoretic properties of the group of computable automorphisms of a countable dense linear order, *Order* 19 (2002), pp. 343–364 .

[117] M. Lerman and J. Schmerl, Theories with recursive models, *Journal of Symbolic Logic* 44 (1979), pp. 59–76.

[118] A.I. Mal'cev, On recursive Abelian groups, *Soviet Mathematics. Doklady* 3 (1962), pp. 1431–1434 (English translation).

[119] M. Makkai, An example concerning Scott heights, *Journal of Symbolic Logic* 46 (1981), pp. 301–318.

[120] M. Manasse, *Techniques and Counterexamples in Almost Categorical Recursive Model Theory*, PhD dissertation, University of Wisconsin, Madison, 1982.

[121] D. Marker and R. Miller, Turing degree spectra of differentially closed fields, *Journal of Symbolic Logic* 82 (2017), pp. 1–25.

[122] L.B. Marshall, *Computability-Theoretic Properties of Partial Injections, Trees, and Nested Equivalences,* PhD dissertation, George Washington University, 2015.

[123] C.F.D. McCoy, Δ^0_2-categoricity in Boolean algebras and linear orderings, *Annals of Pure and Applied Logic* 119 (2003), pp. 85–120.

[124] C.F.D. McCoy, On Δ^0_3-categoricity for linear orders and Boolean algebras, *Algebra and Logic* 41 (2002), pp. 295–305 (English translation).

[125] C. McCoy and J. Wallbaum, Describing free groups, part II: Π^0_4-hardness and no Σ^0_2 basis, *Transactions of the American Mathematical Society* 364 (2012), pp. 5729–5734.

[126] J. Millar and G.E. Sacks, Atomic models higher up, *Annals of Pure and Applied Logic* 155 (2008), pp. 225–241.

[127] T. Millar, Recursive categoricity and persistence, *Journal of Symbolic Logic* 51 (1986), pp. 430–434.

[128] T.S. Millar, Homogeneous models and decidability, *Pacific Journal of Mathematics* 91 (1980), pp. 407–418.

[129] T.S. Millar, Foundations of recursive model theory, *Annals of Mathematical Logic* 13 (1978), pp. 45–72.

[130] A. Miller, The Borel classification of the isomorphism class of a countable model, *Notre Dame Journal of Formal Logic* 24 (1983), pp. 22–34.

[131] D.E. Miller, The invariant $\mathbf{\Pi}^0_\alpha$ separation principle, *Transactions of the American Mathematical Society* 242 (1978), pp.185–204.

[132] R. Miller, The computable dimension of trees of infinite height, *Journal of Symbolic Logic* 70 (2005), pp. 111–141.

[133] R. Miller and A. Shlapentokh, Computable categoricity for algebraic fields with splitting algorithms, *Transactions of the American Mathematical Society* 367 (2015), pp. 3955–3980.

[134] R. Miller and H. Schoutens, Computably categorical fields *via* Fermat's Last Theorem, *Computability* 2 (2013), pp. 51–65.

[135] M. Morley, Decidable models, *Israel Journal of Mathematics* 25 (1976), pp. 233–240.

[136] M. Moses, Relations intrinsically recursive in linear orders, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 32 (1986), pp. 467–472.

[137] M. Nadel, Scott sentences and admissible sets, *Annals of Mathematical Logic* 7 (1974), pp. 267–294.

[138] A.T. Nurtazin, Strong and weak constructivizations and computable families, *Algebra and Logic* 13 (1974), pp. 177–184 (English translation).

[139] E.N. Pavlovskii, An estimate for the algorithmic complexity of classes of computable models, *Siberian Mathematical Journal* 49 (2008), pp. 512–523 (English translation).

[140] M.G. Peretyat'kin, Criterion for strong constructivizability of a homogeneous model, *Algebra and Logic* 17 (1978), pp. 290–301 (English translation).

[141] J.B. Remmel, Recursively categorical linear orderings, *Proceedings of the American Mathematical Society* 83 (1981), pp. 387–391.

[142] J.B. Remmel, Recursive isomorphism types of recursive Boolean algebras, *Journal of Symbolic Logic* 46 (1981), pp. 572–594.

[143] J.B. Remmel, Recursively enumerable Boolean algebras, *Annals of Mathematical Logic* 15 (1978), pp. 75–107.

[144] J.-P. Ressayre, Models with compactness properties relative to an admissible language, *Annals of Mathematical Logic* 11(1977), pp. 31–55.

[145] J.-P. Ressayre, Boolean models and infinitary first order languages, *Annals of Mathematical Logic* 6 (1973), pp. 41–92.

[146] H. Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.

[147] G.E. Sacks, *Higher Recursion Theory*, Springer, Berlin, 1990.

[148] G.E. Sacks, On the number of countable models, in: *Southeast Asian Conference on Logic* (Singapore, 1981), C.T. Chong and M.J. Wicks, eds., pp. 185–195, North-Holland, 1983.

[149] V.L. Selivanov, Enumerations of families of general recursive functions, *Algebra and Logic* 15 (1976), pp. 128–141 (English translation).

[150] R.L. Smith, Two theorems on autostability in p-groups, in *Logic Year 1979–80*, Univ. Connecticut, Storrs, Lecture Notes in Mathematics 859, (Springer, Berlin, 1981), pp. 302–311.

[151] R.I. Soare, *Turing Computability* (*Theory and Applications*) (Springer, Berlin, 2016).

[152] I.N. Soskov, Intrinsically Π^1_1 relations, *Mathematical Logic Quarterly* 42 (1996), pp. 109–126.

[153] I.N. Soskov, Intrinsically hyperarithmetical sets, *Mathematical Logic Quarterly* 42 (1996), pp. 469–480.

[154] A.I. Stukachev, A jump inversion theorem for the semilattices of Σ-degrees, *Sibirskie Èlektronnye Matematicheskie Izvestiya* 6 (2009), pp. 182–190 (Russian). (English translation in: *Siberian Advances in Mathematics* 20 (2010), pp. 68–74.)

[155] R.L. Vaught, Sentences true in all constructive models, *Journal of Symbolic Logic* 25 (1960), pp. 39–58.

[156] H. Walker, *Computable Isomorphisms of Directed Graphs and Trees*, PhD dissertation, GWU, 2017.

[157] H. Walker, Computable isomorphisms for certain classes of infinite graphs, *Journal of Knot Theory and Ramifications* 27 (2018), pp. 1841012 (1–17).

[158] W. White, *Characterization for Computable Structures*, PhD dissertation, Cornell University, 2000.