RELATIVELY HYPERIMMUNE RELATIONS ON STRUCTURES

S. S. GONCHAROV, V. S. HARIZANOV, J. F. KNIGHT, AND C. F. D. MCCOY

ABSTRACT. Let R be a relation on the domain of a computable structure $\mathcal{A}$. We establish that the existence of an isomorphic copy $\mathcal{B}$ of $\mathcal{A}$ such that the image of R ($\neg R$, resp.) is h-simple (h-immune, resp.) relative to $\mathcal{B}$ is equivalent to a syntactic condition, termed R is formally h-simple (formally h-immune, resp.) on $\mathcal{A}$.

1. INTRODUCTION

We consider only countable structures for computable relational languages, and investigate relatively hyperimmune and relatively hypersimple relations on these structures. We can identify the universe A of $\mathcal{A}$ with a subset of ω, and think of it as a set of constants. The language L_A is the extension of L by a constant symbol a for every $a \in A$, and the corresponding expanded structure is $\mathcal{A}_A$. The *atomic diagram* of $\mathcal{A}$, $D(\mathcal{A})$, is the set of all atomic and negated atomic sentences of L_A, which are true in $\mathcal{A}_A$. A structure is *computable* if its atomic diagram is computable.

Here, we consider relative versions of classical computability-theoretic notions of hyperimmunity and hypersimplicity of relations on countable structures. A relation on ω is called *hypersimple*, abbreviated by *h-simple*, if it is computably enumerable (c.e.) and its complement is hyperimmune. It is established in computability theory that a unary relation C on ω is *hyperimmune*, abbreviated by *h-immune*, iff it is infinite and no computable function majorizes its principal function p_C, where $p_C(n) =_{def} c_n$ provided that $C = \{c_0 < c_1 < c_2 < \dots\}$.

Let $\mathcal{A}$ be an infinite computable structure for language L, and let R be an infinite and co-infinite relation on A. We consider the following problems.

Problem 1. *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$ such that $\neg F(R)$ is h-immune relative to $\mathcal{B}$?*

The authors gratefully acknowledge support of the NSF Binational Grant DMS-0075899.

Problem 2. *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$ such that $F(R)$ is h-simple relative to $\mathcal{B}$?*

In [8], Hird introduced the notions of *formal h-immunity* and *formal h-simplicity* for relations on computable structures. These are syntactic analogues of h-immunity and h-simplicity. It is not hard to see that if there is an isomorphic copy of $\mathcal{A}$ on which the image of $\neg R$ is relatively h-immune, then $\neg R$ must be formally h-immune on $\mathcal{A}$. This necessary syntactic condition turns out to be sufficient. Thus, our main result on Problem 1 is the following equivalence of a semantic and a corresponding syntactic notion.

Theorem 2.1. *The following are equivalent for $(\mathcal{A}, R)$.*
(i) There is a copy $\mathcal{B}$ of $\mathcal{A}$ and an isomorphism F from $\mathcal{A}$ onto $\mathcal{B}$ such that $\neg F(R)$ is h-immune relative to $\mathcal{B}$.
(ii) The relation $\neg R$ is formally h-immune on $\mathcal{A}$.

Similarly, we obtain our main result on Problem 2.

Theorem 3.1. *The following are equivalent for $(\mathcal{A}, R)$.*
(i) There is a copy $\mathcal{B}$ of $\mathcal{A}$ and an isomorphism F from $\mathcal{A}$ onto $\mathcal{B}$ such that $F(R)$ is h-simple relative to $\mathcal{B}$.
(ii) The relation R is formally h-simple on $\mathcal{A}$.

Unlike analogous results for computable copies $\mathcal{B}$ of $\mathcal{A}$, these relative results do not involve extra decidability conditions on $(\mathcal{A}, R)$. Examples of such relative results are also presented in [1], [2], [3], and [5]. In particular, in [5] we investigated relative immunity and relative simplicity of relations on countable structures, and established equivalences of syntactic and corresponding semantic conditions.

In the remainder of this section, we review some computability-theoretic notation and concepts (see [13]). In Section 2, we review the notions of h-immunity and formal h-immunity, define the notion of relative h-immunity and prove our main theorem on relatively h-immune relations. In Section 3, we review the notions of h-simplicity and formal h-simplicity, define the notion of relative h-immunity and prove our main theorem on relatively h-simple relations.

Let $\varphi_0, \varphi_1, \varphi_2, \dots$ be a fixed effective enumeration of all unary partial computable functions. Let $X \subseteq \omega$. Then $\varphi_0^X, \varphi_1^X, \varphi_2^X, \dots$ is a fixed effective enumeration of all unary X-partial computable functions. For a structure $\mathcal{B}$, $\varphi_e^{\mathcal{B}}$ stands for $\varphi_e^{D(\mathcal{B})}$. The following (canonical) indexing of finite sets is standard. Let $D_0 =_{def} \emptyset$. For $m > 0$, let $D_m = \{d_0, \dots, d_{k-1}\}$, where $d_0 < \dots < d_{k-1}$ and $m = 2^{d_0} + \dots + 2^{d_{k-1}}$. A sequence $(U_i)_{i\in\omega}$ of finite sets is a *strong array* if there is a unary computable function f such that for every $i \in \omega$, $U_i = D_{f(i)}$. A strong array is *disjoint* if its members are pairwise disjoint.

Throughout the paper, we will assume that $\mathcal{A}$ is an infinite computable structure for language L, and that R is an infinite co-infinite relation on its domain. Without loss of generality, we assume that R is unary. By $\vec{a}$ we denote a finite sequence (tuple) of elements. We often write $a \in \vec{a}$ instead of $a \in ran(\vec{a})$, and $\vec{a} \cap G = \emptyset$ instead of $ran(\vec{a}) \cap G = \emptyset$.

2. Relatively h-immune relations

The following definition of an h-immune relation on natural numbers is a usual one.

Definition 2.1. *Let $S \subseteq \omega$. The relation $\neg S$ is* h-immune *(on ω) if it is infinite and there is no disjoint strong array $(U_i)_{i\in\omega}$ such that for every $n \in \omega$, we have $U_i \cap \overline{S} \neq \emptyset$.*

We can similarly define h-immune relations on any computable set. Every h-immune set is *immune*, that is, infinite but without any infinite c.e. subset. Not every immune set is h-immune. However, Jockusch [9] proved that every immune initial segment of a computable linear order is h-immune. Hird [7] studied co-c.e. intervals of a computable linear order with h-immune (equivalently, immune) images on some computable copy of the linear order. Remmel [12] proved that if $\mathcal{A}$ is a computable Boolean algebra, whose atoms form an infinite computable set, then there is a computable copy $\mathcal{B}$ of $\mathcal{A}$ such that the set of all atoms of $\mathcal{B}$ is h-immune and of an arbitrary nonzero c.e. Turing degree.

The following definition of Hird introduces a syntactic property corresponding to h-immunity. We will term it "being formally h-immune on $\mathcal{A}$."

Definition 2.2. (*Hird* [8]) (i) *A* formal strong array on $\mathcal{A}$ *is a computable sequence of existential formulas in L with finitely many parameters $\vec{c}$, $(\psi_i(\vec{c}, \vec{x}_i))_{i\in\omega}$, such that for every finite set $G \subseteq A$ there is $i \in \omega$ and a sequence $\vec{a}_i \in A^{lh(\vec{x}_i)}$ with*

$$[\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)] \wedge [\vec{a}_i \cap G = \emptyset].$$

(ii) *We say that the relation $\neg R$ is* formally h-immune on $\mathcal{A}$ *if there is no formal strong array $(\psi_i(\vec{c}, \vec{x}_i))_{i\in\omega}$ on $\mathcal{A}$ such that for every $i \in \omega$,*

$$(\forall \vec{a}_i \in A^{lh(\vec{x}_i)})[(\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)) \Rightarrow (\vec{a}_i \cap \neg R \neq \emptyset)].$$

Being formally h-immune on $\mathcal{A}$ turns out to be a necessary condition for the existence of a computable copy of $\mathcal{A}$ such that the corresponding image of R is h-immune (see [8]). Assume that $\mathcal{B}$ is a computable copy of $\mathcal{A}$ and that F is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. It is easy to show that if $(\psi_i(\vec{c}, \vec{x}_i))_{i\in\omega}$ is a formal strong array on $\mathcal{A}$, then $(\psi_i(F(\vec{c}), \vec{x}_i))_{i\in\omega}$ is a formal strong array on $\mathcal{B}$ (see [6]). The following result establishes that, under some extra decidability conditions for $(\mathcal{A}, R)$, the existence of a computable copy $\mathcal{B}$ of $\mathcal{A}$ such that the image of $\neg R$ is h-immune relative to $\mathcal{B}$ is equivalent to $\neg R$ being formally h-immune on $\mathcal{A}$.

Theorem 2.1. (*Hird* [8]) (i) *Assume that $\mathcal{B}$ is a computable copy of $\mathcal{A}$ and that F is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. If $F(\neg R)$ is h-immune on B, then $\neg R$ is formally h-immune on $\mathcal{A}$.*

(ii) *Assume that there is an algorithm which decides for a given sequence $\vec{c} \in A^{<\omega}$ and an existential formula $\psi(\vec{u}, \vec{x})$ in L, $lh(\vec{u}) = lh(\vec{c})$, whether*

$$(\forall \vec{a} \in A^{lh(\vec{x})})[(\mathcal{A}_A \models \psi(\vec{c}, \vec{a})) \Rightarrow (\vec{a} \cap \neg R \neq \emptyset)].$$

If $\neg R$ is formally h-immune on $\mathcal{A}$, then there is a computable structure $\mathcal{B}$ and an isomorphism F from $\mathcal{A}$ onto $\mathcal{B}$ such that the relation $F(\neg R)$ is h-immune on B.

An infinitary Σ_1 formula is an $L_{\omega\omega_1}$ formula of the form

$$\bigvee_{i\in I} \exists \vec{u}_i \psi_i(\vec{x}, \vec{u}_i),$$

where for every $i \in I$, $\psi_i(\vec{x}, \vec{u}_i)$ is a finitary quantifier-free formula. We assume that the finitary quantifier-free formulas are coded by some effective Gödel numbering, and ψ_i is the i^{th} formula under this numbering. If the index set I is c.e., then we have a *computable* Σ_1 formula. We can continue recursively with the definition of computable infinitary formulas, by defining Σ_α and Π_α formulas for every computable ordinal α (see [1]).

Definition 2.3. (i) *An* infinitary formal strong array on $\mathcal{A}$ *is a computable sequence of computable Σ_1 formulas, $(\psi_i(\vec{c}, \vec{x}_i))_{i\in\omega}$, such that for every finite set $G \subseteq A$ there is $i \in \omega$ and $\vec{a}_i \in A^{lh(\vec{x_i})}$ with*

$$[\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)] \wedge [\vec{a}_i \cap G = \emptyset].$$

(ii) *The relation $\neg R$ is* infinitary formally h-immune *on $\mathcal{A}$ if there is no infinitary formal strong array $(\psi_i(\vec{c}, \vec{x}_i))_{i\in\omega}$ on $\mathcal{A}$ such that for every $i \in \omega$,*

$$(\forall \vec{a}_i \in A^{lh(\vec{x}_i)})[(\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)) \Rightarrow (\vec{a}_i \cap \overline{R} \neq \emptyset)].$$

The following proposition follows immediately.

Proposition 2.2. *The relation $\neg R$ is infinitary formally h-immune on $\mathcal{A}$ if and only it is formally h-immune on $\mathcal{A}$.*

We now introduce a relative version of h-immunity.

Definition 2.4. *Let S be an additional (unary) relation on the domain B of a countable structure $\mathcal{B}$.*

(i) A sequence $(U_i)_{i\in\omega}$ of finite sets is a strong array relative to $\mathcal{B}$ *if there is a unary $\mathcal{B}$-computable function f such that for every $i \in \omega$, $U_i = D_{f(i)}$.*

*(ii) The relation $\neg S$ is h-*immune relative to $\mathcal{B}$ *if it is infinite and there is no disjoint strong array relative to $\mathcal{B}$, $(U_i)_{i\in\omega}$, such that for every $n \in \omega$, we have $U_i \cap \overline{S} \neq \emptyset$.*

If there is an isomorphic copy of $\mathcal{A}$ on which the image of $\neg R$ is relatively h-immune, then $\neg R$ must be formally h-immune on $\mathcal{A}$. This necessary syntactic condition turns out to be sufficient.

Theorem 2.1. *Let $\mathcal{A}$ be a computable L-structure, and let R be a unary infinite and co-infinite relation on A. Then the following are equivalent:*

(i) For all copies $\mathcal{B}$ of $\mathcal{A}$ and all isomorphisms F from $\mathcal{A}$ onto $\mathcal{B}$, $\neg F(R)$ is not h-immune relative to $\mathcal{B}$.

(ii) The relation $\neg R$ is not formally h-immune on $\mathcal{A}$.

Proof. $(ii) \Rightarrow (i)$ Assume that $\neg R$ is not formally h-immune on $\mathcal{A}$. Let $(\psi_i(\vec{c}, \vec{x}_i))_{i\in\omega}$ be a corresponding formal strong array on $\mathcal{A}$. Let $\mathcal{B}$ be a copy of $\mathcal{A}$ and F an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Hence, $(\psi_i(F(\vec{c}), \vec{x}_i))_{i\in\omega}$ is a formal strong array on $\mathcal{B}$. Since for every $\vec{a}_i \in A^{<\omega}$, we have

$$(\vec{a}_i \cap \overline{R} \neq \emptyset) \Rightarrow (F(\vec{a}_i) \cap F(R) \neq \emptyset),$$

it follows that

$$(\forall \vec{b}_i \in B^{lh(\vec{x}_i)})[(\mathcal{B}_B \models \psi_i(F(\vec{c}), \vec{b}_i)) \Rightarrow (\vec{b}_i \cap F(R) \neq \emptyset)].$$

We now show that $F(R)$ is not h-immune relative to $\mathcal{B}$, by enumerating a corresponding strong array relative to $\mathcal{B}$. We simultaneously enumerate finite sequences of elements in B, satisfying formulas in $(\psi_i(F(\vec{c}), \vec{x}_i))_{i\in\omega}$, such that none of these sequences intersects any of the previously enumerated ones. This is possible by the main property of a formal strong array. For every such sequence $\vec{b}_i$ with $\mathcal{B}_B \models \psi_i(F(\vec{c}), \vec{b}_i))$, it follows that $\vec{b}_i \cap F(R) \neq \emptyset$, as required. The ranges of these sequences form a strong array relative to $\mathcal{B}$.

$(i) \Rightarrow (ii)$ We build a generic copy $(\mathcal{B}, S)$ of $(\mathcal{A}, R)$. Assuming that $\neg S$ is not h-immune relative to $\mathcal{B}$, we will produce an infinitary formal strong array on $\mathcal{A}$, $(\psi_i)_{i\in\omega}$, which will witness that $\neg R$ is not infinitary formally h-immune on $\mathcal{A}$. Let $B = \{b_0, b_1, b_2, \ldots\}$ be an infinite computable set, the universe of $\mathcal{B}$. Without loss of generality, we can assume that $(\forall i)[b_i = i]$. The set $\mathcal{F}$ of *forcing conditions* consists of all finite 1-1 partial functions from B to A. We use letters $p, q, r, \ldots$ to denote elements of $\mathcal{F}$. Let $\mathbf{R}$ be a unary relation symbol that is not in L. As a forcing language, we take a propositional language P^* in which the propositional variables are the atomic sentences in the expanded language $(L \cup \{\mathbf{R}\})_B$. Let P be the sublanguage of P^* consisting of atomic sentences in L_B. Let $\mathcal{S}^*$ be the set of computable infinitary sentences in the language P^*, and let $\mathcal{S}$ be the set of computable infinitary sentences in P. We consider only computable infinitary formulas in their *normal* form. Hence the negations can occur only in finitary open subformulas. For a sentence ψ, we write $neg(\psi)$ for the computable infinitary sentence (in the normal form) that is dual to ψ, i.e. equivalent to the negation of ψ. A sentence ψ in forcing language P^* is propositional, but we may also think of it as a predicate sentence in the language $(L \cup \{\mathbf{R}\})_B$. The constants of ψ are the constants appearing in the propositional variables in ψ.

We are especially interested in sentences in $\mathcal{S}$ or $\mathcal{S}^*$ expressing the following facts in $(\mathcal{B}, S)$ for some fixed $e \in \omega$:

(i) $\varphi_e^{\mathcal{B}}$ is total (expressed in $\mathcal{S}$);

(ii) $D_{\varphi_e^{\mathcal{B}}(n)}$ and $D_{\varphi_e^{\mathcal{B}}(n')}$ are disjoint for $n \neq n'$ (expressed in $\mathcal{S}$);

(iii) $D_{\varphi_e^{\mathcal{B}}}(n) \cap \overline{S} \neq \emptyset$ for every n (expressed in $\mathcal{S}^*$).

We express the sentence in (i) by

$$\bigwedge_{n\in\omega} \bigvee_{m\in\omega} \bigvee_{\{\sigma:\varphi_e^{\sigma}(n)=m\}} \theta_\sigma.$$

Here, $\varphi_e^\sigma(n) = m$ is the halting computation of an oracle Turing machine with Gődel index e on input n with output m in $\leq lh(\sigma)$ steps using the finite oracle $\sigma \in 2^{lh(\sigma)}$, where $\theta_\sigma = \theta_{\sigma_{n,m}}(\vec{b}_{\sigma_{n,m}})$ is an open sentence in L_B, expressing information about $D(\mathcal{B})$, contained in σ.

We express the sentence in (ii) by

$$\bigwedge_{n\neq n'} \bigvee_{D_m \cap D_{m'}=\emptyset} \bigvee \{\theta_\sigma \wedge \theta_{\sigma'} : \varphi_e^\sigma(n) = m \wedge \varphi_e^{\sigma'}(n') = m'\}$$

where, again, θ_σ and $\theta_{\sigma'}$ (resp.) are open sentences in L_B, expressing suitable oracle information about $D(\mathcal{B})$, needed for the computations $\varphi_e^\sigma(n) = m$ and $\varphi_e^{\sigma'}(n') = m'$ (resp.).

We express the sentence in (iii) by

$$\bigwedge_{n,m} [\bigvee\{\theta_\sigma : \varphi_e^\sigma(n) = m\} \Rightarrow \bigvee_{\{j:j\in D_m\}} \neg\mathbf{R}(b_j)].$$

Let $p \in \mathcal{F}$ and a sentence ψ in $\mathcal{S}^*$, we define a forcing relation, $p \Vdash \psi$, as follows:

(1) If ψ is a finitary sentence, then $p \Vdash \psi$ iff the constants of ψ are all in $dom(p)$, and p makes ψ true in $(\mathcal{A}_A, R)$;
(2) $p \Vdash \bigvee\limits_{i\in I} \psi_i$ iff $(\exists i \in I)[p \Vdash \psi_i]$;
(3) $p \Vdash \bigwedge_{i\in I} \psi_i$ iff $(\forall q \supseteq p)(\forall i \in I)(\exists r \supseteq q)[r \Vdash \psi_i]$.

We have the usual forcing lemmas for every $p, q \in \mathcal{F}$ and every sentence ψ in $\mathcal{S}^*$.

Lemma 2.2. *$[p \Vdash \psi \wedge q \supseteq p] \Rightarrow q \Vdash \psi$.*

Lemma 2.3. *$\neg[(p \Vdash \psi) \wedge (p \Vdash neg(\psi))]$.*

Lemma 2.4. *$(\exists r \supseteq p)[(r \Vdash \psi) \vee (r \Vdash neg(\psi))]$.*

We say that r *decides* ψ iff $[(r \Vdash \psi) \vee (r \Vdash neg(\psi))]$. A *complete forcing sequence* is a chain $(p_n)_{n\in\omega}$ of forcing conditions such that for every sentence ψ in $\mathcal{S}^*$, there is n such that p_n decides ψ; for every $a \in A$, there is n such that $a \in ran(p_n)$; and for every $b \in B$, there is n such that $b \in dom(p_n)$. The existence of a complete forcing sequence $(p_n)_{n\in\omega}$ follows from Lemma 2.4. It is easy to see that $\bigcup\limits_{n\in\omega} p_n$ a 1-1 function from B onto A. Let $F =_{def} (\bigcup\limits_{n\in\omega} p_n)^{-1}$. Then F induces on B a copy $(\mathcal{B}, S)$ of $(\mathcal{A}, R)$, where $S = F(R)$.

We have the following *Truth-and-Forcing Lemma*.

Lemma 2.5. *For any $\psi \in \mathcal{S}^*$, we have $(\mathcal{B}_B, F(R)) \models \psi$ iff there is $n \in \omega$ such that $p_n \Vdash \psi$.*

To complete the proof of Theorem 2.1, we need the following lemma.

Lemma 2.6. *The relation $F(\neg R)$ is h-immune relative to $\mathcal{B}$.*

Proof of Lemma 2.6. Suppose otherwise. Then there is $e \in \omega$ such that $(D_{\varphi_e^{\mathcal{B}}(n)})_{n\in\omega}$ is a strong array on B, such that for every $n \in \omega$, $D_{\varphi_e^{\mathcal{B}}(n)} \cap \overline{S} = \emptyset$. By the *Truth-and-Forcing Lemma*, there is $p \in \mathcal{F}$ ($p = p_n$ for some n) such that p forces the statements expressing this fact. Let p map $\vec{d}$ onto $\vec{c}$.

We will define a computable sequence of computable Σ_1 formulas in L, with parameters $\vec{c}$. Assume that $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$ is an effective enumeration of all finite sequences of variables of L. For $i \in \omega$, let $\psi_i(\vec{c}, \vec{x}_i)$ be the following formula

$$\bigvee_{q \supseteq p} \bigvee \{q \models \text{“}\varphi_e^{\mathcal{B}}(n) = m\text{”} : n, m \in \omega \wedge D_m = \vec{b}_i \wedge q(\vec{b}_i) = \vec{x}_i\}$$

Let us prove that $(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$ is an infinitary formal strong array on $\mathcal{A}$. Let $G \subseteq A$ be a finite set. Let

$$M =_{def} \{n \in \omega : D_{\varphi_e^{\mathcal{B}}(n)} \cap F(G) \neq \emptyset\}.$$

Clearly, M is a finite set. If there is $p' \supseteq p$, such that for every $a \in G$, there is $n \in \omega$ such that $p' \Vdash$ “$F(a) \in D_{\varphi_e^{\mathcal{B}}(n)}$J, then choose such p'. Otherwise, let $p' \supseteq p$ be such that no $p'' \supseteq p$ is such that p'' forces any new elements of $F(G)$ (that is, the ones that p' does not force) into $\bigcup\limits_{n \in \omega} D_{\varphi_e^{\mathcal{B}}(n)}$. Choose $n \notin M$ such that for some $q \supseteq p'$, for some m, we have $q \Vdash$ “$\varphi_e^{\mathcal{B}}(n) = m$”, where $D_m \subseteq dom(q)$. Let $\vec{a}_i =_{def} q(D_m)$. Then $\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)$.

Now assume that for some $\vec{a}_i \in A^{lh(\vec{x}_i)}$, we have $\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)$. Then for some $q \supseteq p$, there are $\vec{b}_i, n, m$ such that $q(\vec{b}_i) = \vec{a}_i$, $D_m = \vec{b}_i$, and $q \Vdash$ “$\varphi_e^{\mathcal{B}}(n) = m$”. Then there is $q' \supseteq q$ such that for some $j \in D_m$, we have $q' \Vdash \neg\mathbf{R}(b_j)$. Hence $a_j \in \overline{R}$. □

3. Relatively h-simple relations

The following definition of an h-simple relation on natural numbers is standard.

Definition 3.1. *Let $S \subseteq \omega$. The relation S is h-*simple *(on ω) if S is c.e. and $\neg S$ is h-immune.*

We can similarly define h-simple relations on any computable set. The set of all h-simple sets is a proper subset of the set of all *simple* sets, that is, the c.e. sets whose complements are *immune*. Dekker [4] showed that the deficiency set of a computable 1-1 enumeration of a noncomputable c.e. set is h-simple. Hence every nonzero c.e. Turing degree contains an h-simple set.

In [8], Hird introduced a syntactic analogue of h-simplicity for relations on computable structures, which we will term formal h-simplicity. Since we are considering c.e. relations, computable Σ_1 formulae with positive occurrences of $\mathbf{R}$ in the expanded language $L \cup \{\mathbf{R}\}$ play an important role.

Definition 3.2. (i) *An* infinitary $\mathbf{R}^+$-formal strong array *on $(\mathcal{A}, R)$ is a computable sequence of computable Σ_1 formulas in $L \cup \{\mathbf{R}\}$ with only positive occurrences of $\mathbf{R}$ and with finitely many parameters $\vec{c}$,*

$(\psi_i(\vec{c}, \vec{x}_i))_{i\in\omega}$, *such that for every finite set* $G \subseteq A$ *there is* $i \in \omega$ *and a sequence* $\vec{a}_i \in A^{lh(\vec{x}_i)}$ *such that*

$$[(\mathcal{A}_A, R) \models \psi_i(\vec{c}, \vec{a}_i)] \wedge [\vec{a}_i \cap G = \emptyset].$$

If computable Σ_1 *formulas are just finitary existential ones, then we have an* $\mathbf{R}^+$-formal strong array.

(ii) *We say that the relation* R *is* formally h-simple on $\mathcal{A}$ *if* R *is c.e. and there is no* $\mathbf{R}^+$*-formal strong array* $(\psi_i(\vec{c}, \vec{x}_i))_{i\in\omega}$ *on* $(\mathcal{A}, R)$ *such that for every* $i \in \omega$,

$$(\forall \vec{a}_i \in A^{lh(\vec{x}_i)})[((\mathcal{A}_A, R) \models \psi_i(\vec{c}, \vec{a}_i)) \Rightarrow (\vec{a}_i \cap \overline{R} = \emptyset)].$$

Hird [8] established that, under a suitable decidability condition, R is formally h-simple on $\mathcal{A}$ iff there is a computable copy $\mathcal{B}$ of $\mathcal{A}$ such that the image of R under an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ is h-simple on $\mathcal{B}$. Harizanov [6] gave sufficient general conditions for the existence of an h-simple relation on a computable copy of $\mathcal{A}$, in arbitrary nonzero c.e. Turing degree.

Next, we introduce a relative version of h-simplicity.

Definition 3.3. *Let* S *be an additional (unary) relation on the domain* B *of a countable structure* $\mathcal{B}$*. The relation* S *is* h-simple relative to $\mathcal{B}$ *if* S *is c.e. relative to* $\mathcal{B}$ *and* $\neg S$ *is* h*-immune relative to* $\mathcal{B}$.

In the next theorem we give a relative analogue of Hird's result for h-simple relations on computable copies.

Theorem 3.1. *Let* $\mathcal{A}$ *be an infinite computable structure in a relational language* L*, and let* R *be a computable unary infinite and co-infinite relation on* A*. Then the following are equivalent:*

(i) *For all copies* $\mathcal{B}$ *of* $\mathcal{A}$ *and all isomorphisms* F *from* $\mathcal{A}$ *onto* $\mathcal{B}$, $F(R)$ *is not* h*-simple relative to* $\mathcal{B}$.

(ii) *The relation* R *is not formally* h*-simple on* $\mathcal{A}$.

Proof. We can prove that $(ii) \Rightarrow (i)$ as in Theorem 2.1. We now prove by contrapositive that $(i) \Rightarrow (ii)$. Assume that R is formally h-simple on $\mathcal{A}$. If R is definable in $\mathcal{A}$ by a computable Σ_1 formula with finitely many parameters, then in any copy $\mathcal{B}$ of $\mathcal{A}$, the image of R is c.e. relative to $\mathcal{B}$. Hence, if $\mathcal{B}$ is a copy in which the image of $\neg R$ is relatively h-immune, then the image of R is relatively h-simple.

Assume that R is not definable by a computable Σ_1 formula with finitely many parameters. Then in a generic copy $\mathcal{B}$ of $\mathcal{A}$, the image of R is not c.e. relative to $\mathcal{B}$ (as established in [2] and [3]). Therefore, as in [5], we introduce an expanded language, $L^* = L \cup \{\mathbf{R}'\} \cup \{\mathbf{Q}\}$, where $\mathbf{R}'$ is a new unary relation symbol and $\mathbf{Q}$ is a new binary relation symbol. We extend the L-structure $\mathcal{A}$ into an L^*-structure $\mathcal{A}^*$. The structure $\mathcal{A}^*$ is obtained by extending the universe A by another infinite computable set R', and expanding $\mathcal{A}$ to include the unary relation R' and a binary relation Q, which is a 1-1 mapping from R' onto R. In $\mathcal{A}^*$, the open L^*-formula $\neg\mathbf{R}'(x)$ defines A, and the existential L^*-formula $\exists\, y\mathbf{Q}(y,x)$ defines R. The following lemma follows from a proof in [5].

Lemma 3.2. *Let $D \subseteq A$. If the set D is definable in $\mathcal{A}^*$ by a computable Σ_1 formula $\psi^*(\vec{c}^*, x)$ of L^*, then it is definable in $(\mathcal{A}, R)$ by some computable Σ_1 formula $\psi(\vec{c}, x)$ in $L \cup \{\mathbf{R}\}$, with finitely many parameters $\vec{c}$ and only positive occurrences of $\mathbf{R}$. Moreover, we can obtain $\psi(\vec{c}, x)$ effectively from $\psi^*(\vec{c}^*, x)$, uniformly in the Gődel code of ψ^*.*

Hence, it follows that the relation $\neg(R \cup R')$ is not formally h-immune on $\mathcal{A}^*$. If we apply Theorem 2.1 to the relation $R \cup R'$ on structure $\mathcal{A}^*$, we get a copy $\mathcal{B}^*$ of $\mathcal{A}^*$ and an isomorphism F from $\mathcal{A}^*$ onto $\mathcal{B}^*$, with $\mathcal{B}$ corresponding to $\mathcal{A}$ under F, such that the following holds:

(a) the structure $\mathcal{B}$ is computable in structure $\mathcal{B}^*$;

(b) the relation $F(R)$ is c.e. relative to $\mathcal{B}^*$;

(c) the relation $(B^* - F(R \cup R'))$ is h-immune relative to $\mathcal{B}^*$.

Thus, by (a), any function computable relative to $\mathcal{B}$ is computable relative to $\mathcal{B}^*$. Clearly, $B - F(R) = B^* - F(R \cup R')$. Consequently, $B - F(R)$ is h-immune relative to $\mathcal{B}$. However, we still cannot conclude that $F(R)$ is h-simple relative to $\mathcal{B}$, because $F(R)$ is not necessarily c.e. relative to $\mathcal{B}$.

Now, using Knight's construction in [10], we can prove, as in [5], that either there is an automorphism H of $\mathcal{A}$ for which $H(R)$ is h-simple, or there is an isomorphism H from $\mathcal{B}$ onto a structure $\mathcal{C}$ such that $H(F(R))$ is h-simple relative to $\mathcal{C}$. Hence, the structure $\mathcal{C}$ is an isomorphic copy of $\mathcal{A}$, on which the corresponding image of R is relatively h-simple. □

References

[1] C. J. Ash and J. F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy*, Elsevier, Amsterdam, 2000.

[2] C. Ash, J. Knight, M. Manasse and T. Slaman, Generic copies of countable structures, *Annals of Pure and Applied Logic* 42 (1989), 195-205.

[3] J. Chisholm, Effective model theory vs. recursive model theory, *Journal of Symbolic Logic* 55 (1990), 1168-1191.

[4] J. C. E. Dekker, A theorem on hypersimple sets, *Proceedings of the American Mathematical Society* 5 (1954), 791-796.

[5] S. S. Goncharov, V. S. Harizanov, J. F. Knight, and C. McCoy, Simple and immune relations on countable structures, submitted to *Archive for Mathematical Logic.*

[6] V. S. Harizanov, Turing degrees of hypersimple relations on computable Structures, submitted to *Annals of Pure and Applied Logic*.

[7] G. Hird, Recursive properties of intervals of recursive linear orders, in: J. N. Crossley, J. B. Remmel, R. A. Shore, and M. E. Sweedler, eds., *Logical Methods* (Birkhäuser, Boston, 1993), 422-437.

[8] G. R. Hird, Recursive properties of relations on models, *Annals of Pure and Applied Logic* 63 (1993), 241-269.

[9] C. G. Jockusch, Jr., Semirecursive sets and positive reducibility, *Transactions of the American Mathematical Society* 131 (1968), 420-436.

[10] J. F. Knight, Degrees coded in jumps of orderings, *Journal of Symbolic Logic* 51 (1986), 1034-1042.

[11] A. Nerode and J. B. Remmel, A survey of lattices of r.e. substructures, in: A. Nerode and R. Shore, eds., *Recursion Theory,* Proceedings of Symposia in Pure Mathematics of the American Mathematical Society 42 (American Mathematical Society, Providence, 1985), 323-375.

[12] J. B. Remmel, Recursive isomorphism types of recursive Boolean algebras, *Journal for Symbolic Logic* 46 (1981), 572-594.

[13] R. I. Soare, *Recursively Enumerable sets and Degrees. A Study of Computable Functions and Computably Generated Sets*, Springer-Verlag, Berlin, 1987.

Siberian Branch of the Russian Academy of Sciences, Sobolev Institute of Mathematics, 630090 Novosibirsk, Russia, gonchar@math.nsc.ru

Department of Mathematics, The George Washington University, Washington, D.C. 20052, USA, harizanv@gwu.edu

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA, Knight.1@nd.edu

Department of Mathematics, University of Wisconsin, Madison, Madison, WI 53706, USA, mccoy@math.wisc.edu