Computable Model Theory[*][†]

Ekaterina B. Fokina
Kurt Gödel Research Center
University of Vienna
Vienna, Austria
efokina@logic.univie.ac.at

Valentina Harizanov
Department of Mathematics
George Washington University
Washington, DC 20052
harizanv@gwu.edu

Alexander G. Melnikov
Division of Mathematical Sciences
Nanyang Technological University
Singapore 639798
AMelnikov@ntu.edu.sg

October 23, 2014

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[*]Fokina was supported by the Austrian Science Fund (FWF) through the Elise Richter project V206-N13. Harizanov was partially supported by the NSF grant DMS-1202328 and by GWU Columbian College Facilitating Fund. Melnikov was partially supported by Tier 2 grant MOE2011-T2-1-071 (ARC 17/11, M45110030).

[†]The authors are grateful to the anonymous referee, S. Goncharov, J. Knight, R. Miller, A. Montalbán, and A. Morozov for valuable suggestions.

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1 Introduction and preliminaries

In the last few decades there has been increasing interest in computable model theory. Computable model theory uses the tools of computability theory to explore algorithmic content (effectiveness) of notions, theorems, and constructions in various areas of ordinary mathematics. In algebra this investigation based on intuitive notion of effectiveness dates back to van der Waerden who in his 1930 book *Modern Algebra* defined an *explicitly* given field as one the elements of which are uniquely represented by distinguishable symbols with which we can perform the field operations algorithmically. In his pioneering paper [329] on non-factorability of polynomials from 1930, van der Waerden essentially proved that an explicit field $(F, +, \cdot)$ does not necessarily have an algorithm for splitting polynomials in $F[x]$ into their irreducible factors.

Hilbert proposed in the early 1920s that the formalization of classical mathematical theories be based on consistent axiomatic systems, which are complete. Gödel's incompleteness theorem from 1931 showed that Hilbert's proposal was unattainable for a consistent system with an algorithmic set of axioms, capable of expressing arithmetic. Gödel's theorem is an astonishing early result of computable model theory. He showed that "there are in fact relatively simple problems in the theory of ordinary whole numbers which cannot be *decided* from the axioms."

In 1936, Turing invented Turing machine, which marked the beginning of computability theory. The work of Church, Gödel, Kleene, Markov, Post, Turing and others in the next decade established the rigorous mathematical foundations for the computability theory. In the 1950s, a famous problem, involving the interplay of algebra and computability, the *word problem*, was resolved. It was shown independently by Novikov [279] and Boone [28] that there exists a finitely presented group G such that the word problem for G is *undecidable*. Adyan [1] further investigated undecidability of various group-theoretic problems. In 1956, Fröhlich and Shepherdson [107] used the precise notion of a computable function to obtain a collection of results and examples about explicit rings and fields. For example, Fröhlich and Shepherdson proved that "there are two explicit fields that are isomorphic but not explicitly isomorphic." Several years later, Rabin [288] and Mal'cev [228, 227] studied more extensively computable groups and other *computable* (also called *recursive* or *constructive*) algebraic structures, including general structures. Another spectacular negative solution to a famous problem, which involves the interplay of number theory and computability, Hilbert's Tenth Problem, was completed by Matiyasevich [233] in 1970. Building on work of Davis, Putnam, and J. Robinson (see [232]), he established that there is no effective procedure to decide whether a given Diophantine equation has a solution in integers.

In the 1970s, Metakides and Nerode [241, 240] and other researchers in the United States (see [157, 26, 259, 294, 293, 297, 292, 247, 313, 225]) initiated a systematic study of computability in mathematical structures and constructions by using modern computability-theoretic tools, such as the priority method and various coding techniques. At the same time and independently, computable model theory was developed in the Siberian school of constructive mathematics (see [285, 280, 197, 128, 127, 126, 87, 124, 125, 284, 88] and also [90, 93]). While in classical mathematics we can replace some constructions by effective ones, for others such replacement is impossible in principle. For example, from the point of view of computable model theory, isomorphic structures may have very different properties.

Several different notions of effectiveness of structures have been investigated. The generalization and formalization of van der Waerden's intuitive notion of an explicitly given field led to the notion of a computable structure, which is one of the main notions in computable model theory. A structure is *computable* if its domain is computable and its relations and functions are uniformly computable. Further generalization led to a countable structure of a certain Turing degree $\mathbf{d}$. (Computable structures are of degree $\mathbf{0}$.) Henkin's construction of a model for a complete decidable theory is effective and produces a structure $\mathcal{A}$ with a computable domain such that the elementary diagram of $\mathcal{A}$ is decidable. Such a structure is called *decidable*. Thus, in the case of a computable structure, our starting point was semantic, while in the case of a decidable structure, the starting point was syntactic. It is easy to see that not every computable structure is decidable since for computable structures only the atomic (open) diagram has to be decidable. We can also assign Turing degrees or some other computability-theoretic degrees to isomorphisms, as well as to various relations on structures. We can also investigate structures, their theories, fragments of diagrams, relations, and isomorphisms within arithmetic and hyperarithmetic hierarchies.

Computability-theoretic notation in this paper is standard and as in [317]. We review some basic notions and notation. For $X \subseteq \omega$, let $\varphi_0^X, \varphi_1^X, \varphi_2^X, \ldots$ be a fixed effective enumeration of all unary X-computable functions. If X is computable, we omit the superscript X. For $e \in \omega$, let $W_e^X = dom(\varphi_e^X)$. Hence $W_0, W_1, W_2, \ldots$ is an effective enumeration of all computably enumerable (c.e.) sets. By $X \leq_T Y$ ($X \equiv_T Y$, respectively) we denote that X is Turing reducible to Y (X is Turing equivalent to Y, respectively). By $X <_T Y$ we denote that $X \leq_T Y$ but $Y \not\leq_T X$. We write $\mathbf{x} = \deg(X)$ for the Turing degree of X. Thus, $\mathbf{0} = \deg(\emptyset)$. Let $n \geq 1$. Then $\mathbf{x}^{(n)} = \deg(X^{(n)})$, where $X^{(n)}$ is the n-th Turing jump of X. A set is Σ_n^0 if it is c.e. relative to $\mathbf{0}^{(n-1)}$. A set is Π_n^0 if its negation is Σ_n^0, and a set is Δ_n^0 if it is both Σ_n^0 and Π_n^0. Let $\Delta_0^0 =_{def} \Delta_1^0$. A set X is *arithmetic* if $X \leq \emptyset^{(k)}$ for some $k \geq 0$. A set $X \leq_T \emptyset'$ and its Turing degree $\mathbf{x}$ are called *low* if $\mathbf{x}' \leq \mathbf{0}'$, and *low*$_n$ if $\mathbf{x}^{(n)} \leq \mathbf{0}^{(n)}$. The *low basis theorem* of Jockusch and Soare [181], establishes that every infinite binary tree $\mathcal{T}$ has an infinite path f with $f' \leq_T \mathcal{T}'$. In particular, every infinite computable binary tree has a low path.

An ordinal is *computable* if it is finite or is the order type of a computable well order on ω. The computable ordinals form a countable initial segment of the ordinals. *Kleene's* $\mathcal{O}$ is the set of notations for computable ordinals, with the corresponding partial order $<_{\mathcal{O}}$ (see [298, 301]). The ordinal 0 gets notation 1. If a is a notation for α, then 2^a is a notation for $\alpha+1$. Then $a <_{\mathcal{O}} 2^a$, and also, if $b <_{\mathcal{O}} a$, then $b <_{\mathcal{O}} 2^a$. Suppose α is a limit ordinal. If φ_e is a total function, giving notations for an increasing sequence of ordinals with limit α, then $3 \cdot 5^e$ is a notation for α. For all n, we have $\varphi_e(n) <_{\mathcal{O}} 3 \cdot 5^e$, and if $b <_{\mathcal{O}} \varphi_e(n)$, then $b <_{\mathcal{O}} 3 \cdot 5^e$. Let $|a|$ denote the ordinal with notation a. If $a \in \mathcal{O}$, then the restriction of $<_{\mathcal{O}}$ to the set $pred(a) = \{b \in \mathcal{O} : b <_{\mathcal{O}} a\}$ is a well order of type $|a|$. For $a \in \mathcal{O}$, $pred(a)$ is c.e., uniformly in a. The set $\mathcal{O}$ is Π^1_1-complete.

The least noncomputable ordinal is denoted by ω_1^{CK}, where CK stands for Church-Kleene. To extend the arithmetic hierarchy, we define the representative sets in the hyperarithmetic hierarchy, H_a for $a \in \mathcal{O}$. The definition is recursive, and is based on iterating Turing jump: $H_1 = \emptyset$, $H_{2^a} = (H_a)'$, and $H_{3 \cdot 5^e} = \{2^x \cdot 3^n : x \in H_{\varphi_e(n)}\}$. Let β be an infinite computable ordinal. Then a set is Σ^0_β if it is c.e. relative to some H_a such that β is represented by notation a. A set is Π^0_β if its negation is Σ^0_β, and a set is Δ^0_β if it is both Σ^0_β and Π^0_β. A set is *hyperarithmetic* if it is Δ^0_α for some computable α. Hence, a set X is hyperarithmetic if $(\exists a \in \mathcal{O})[X \leq_T H_a]$. The hyperarithmetic sets coincide with Δ^1_1 sets.

Ershov classified Δ^0_2 sets as follows. Let α be a computable ordinal. A set $C \subseteq \omega$ is α-c.e. if there are a computable function $f : \omega^2 \to \{0,1\}$ and a computable function $o : \omega \times \omega \to \alpha + 1$ with the following properties:

$$\begin{aligned}
(\forall x)[f(x,0) &= 0 \ \wedge \ \lim_{s\to\infty} f(x,s) = C(x)], \\
(\forall x)(\forall s)[o(x,0) &= \alpha \ \wedge \ o(x,s+1) \leq o(x,s)], \text{ and} \\
(\forall x)(\forall s)[f(x,s+1) &\neq f(x,s) \ \Rightarrow \ o(x,s+1) < o(x,s)].
\end{aligned}$$

In particular, 1-c.e. sets are c.e. sets, and 2-c.e. sets are d.c.e. sets.

Several important notions of computability on effective structures have syntactic characterizations, which involve computable infinitary formulas introduced by Ash. Roughly speaking, these are infinitary formulas involving infinite conjunctions and disjunctions over c.e. sets. More precisely, let α be a computable ordinal. Ash defined computable Σ_α and Π_α formulas of $L_{\omega_1\omega}$ recursively and simultaneously and together with their Gödel numbers. The computable Σ_0 and Π_0 formulas are the finitary quantifier-free formulas. The computable $\Sigma_{\alpha+1}$ formulas are of the form

$$\bigvee_{n \in W_e} \exists \overline{y_n} \psi_n(\overline{x}, \overline{y_n}),$$

where for $n \in W_e$, ψ_n is a Π_α formula indexed by its Gödel number, and $\exists\overline{y_n}$ is a finite block of existential quantifiers. That is, $\Sigma_{\alpha+1}$ formulas are c.e. disjunctions of $\exists\Pi_\alpha$ formulas. Similarly, $\Pi_{\alpha+1}$ formulas are c.e. conjunctions of $\forall\Sigma_\alpha$ formulas. It can be shown that a computable Σ_1 formula is of the form

$$\bigvee_{n \in \omega} \exists \overline{y_n} \theta_n(\overline{x}, \overline{y_n}),$$

where $(\theta_n(\overline{x}, \overline{y_n}))_{n\in\omega}$ is a computable sequence of quantifier-free formulas. If α is a limit ordinal, then Σ_α (Π_α, respectively) formulas are of the form $\bigvee_{n\in W_e} \psi_n$ ($\bigwedge_{n\in W_e} \psi_n$, respectively), such that there is a sequence $(\alpha_n)_{n\in W_e}$ of ordinals having limit α, given by the ordinal notation for α, and every ψ_n is a Σ_{α_n} (Π_{α_n}, respectively) formula. For a more precise definition of computable Σ_α and Π_α formulas see [14]. The important property of these formulas, due to Ash, is the following. For a structure $\mathcal{A}$, if $\theta(\overline{x})$ is a computable Σ_α formula, then the set $\{\overline{a} : \mathcal{A} \models \theta(\overline{a})\}$ is Σ^0_α relative to $\mathcal{A}$. An analogous property holds for computable Π_α formulas. If $\mathcal{A}$ and $\mathcal{B}$ are hyperarithmetic structures satisfying the same computable infinitary sentences, then $\mathcal{A} \cong \mathcal{B}$ (see [139]).

The following is a *compactness theorem* due to Kreisel and Barwise.

Theorem 1. *Let Γ be a Π^1_1 set of computable infinitary sentences. If every Δ^1_1 subset of Γ has a model, then Γ has a model.*

As a corollary we obtain that if Γ is a Π^1_1 set of computable infinitary sentences, and if every Δ^1_1 subset of Γ has a computable model, then Γ has a computable model (see [14]).

Complexity of a countable structure $\mathcal{A}$ can be measured by its Scott rank. There are several different definitions of Scott rank and we will use one in [14] (also see [41]). First we define a family of equivalence relations on finite tuples $\overline{a}$ and $\overline{b}$ of elements in $\mathcal{A}$, of the same length.

1. We say that $\overline{a} \equiv^0 \overline{b}$ if $\overline{a}$ and $\overline{b}$ satisfy the same quantifier-free formulas.

2. For $\alpha > 0$, we say that $\overline{a} \equiv^\alpha \overline{b}$ if for all $\beta < \alpha$, for every $\overline{c}$, there exists $\overline{d}$, and for every $\overline{d}$, there exists $\overline{c}$, such that $\overline{a}, \overline{c} \equiv^\beta \overline{b}, \overline{d}$.

The *Scott rank of a tuple* $\overline{a}$ in $\mathcal{A}$ is the least β such that for all $\overline{b}$, the relation $\overline{a} \equiv^\beta \overline{b}$ implies $(\mathcal{A}, \overline{a}) \cong (\mathcal{A}, \overline{b})$. The *Scott rank of* $\mathcal{A}$, $SR(\mathcal{A})$, is the least ordinal α greater than the ranks of all tuples in $\mathcal{A}$. For example, if $\mathcal{L}$ is a linear order of type ω, then $SR(\mathcal{L}) = 2$. For a hyperarithmetic structure, the Scott rank is at most $\omega_1^{CK}+1$. It can be shown (see [14, 41]) that for a computable structure $\mathcal{A}$, we have:

(i) $SR(\mathcal{A}) < \omega_1^{CK}$ if there is a computable ordinal β such that the orbits of all tuples are defined by computable Π_β formulas;

(ii) $SR(\mathcal{A}) = \omega_1^{CK}$ if the orbits of all tuples are defined by computable infinitary formulas, but there is no bound on the complexity of these formulas;

(iii) $SR(\mathcal{A}) = \omega_1^{CK}+1$ if there is some tuple the orbit of which is not defined by any computable infinitary formula.

There are structures in natural classes, for example, abelian p-groups, where p is a prime number, with arbitrarily large computable ranks, and of rank $\omega_1^{CK} + 1$, but none of rank ω_1^{CK} (see [25]). Makkai [226] was the first to prove the existence of an arithmetic structure of Scott rank ω_1^{CK}, and in [210], J. Millar and Knight showed that such structure can be made computable. Through the recent work of Calvert, Knight, and J. Millar [42], Calvert, Goncharov, and

Knight [39], and Freer [104], we started to better understand the structures of Scott rank ω_1^{CK}. Computable structures of Scott rank ω_1^{CK} were obtained in familiar classes such as trees, undirected graphs, fields of any fixed characteristic, and linear orders [42, 39]. Sacks asked whether for known examples of computable structures of Scott rank ω_1^{CK}, the computable infinitary theories are $\aleph_0$-categorical. In [40], Calvert, Goncharov, J. Millar, and Knight gave an affirmative answer for known examples. In [242], J. Millar and Sacks introduced an innovative technique that produced a countable structure $\mathcal{A}$ of Scott rank ω_1^{CK} such that $\omega_1^{\mathcal{A}} = \omega_1^{CK}$ and the $L_{\omega_1^{CK},\omega}$-theory of $\mathcal{A}$ is *not* $\aleph_0$ -categorical. It is not known whether such a structure can be computable.

In this paper, we will not consider structures that are computable with bounds on the resources that algorithms can use, such as time and memory constraints. For a survey of polynomial time structures see the paper [48] by Cenzer and Remmel. Another approach that turned out to be very interesting, which is beyond the scope of this paper, is to consider functions representable by various types of finite automata. For instance, a function presented by a finite string automaton can be computed in linear time using a constant amount of memory. A seminal paper in this field is [202] by Khoussainov and Nerode. The most interesting property of automatic structures is that they have decidable model checking problems. We can use this property to prove the decidability of the first-order theories of many structures, e.g., Presburger arithmetic. There is also a class of tree automatic structures (see [300, 200]), which is richer than the class of automatic structures. Tree automatic structures have nice algorithmic properties, in particular, decidable model checking problem. Many interesting problems in this area remain open.

2 Degrees and jump degrees of structures and their isomorphism types

We will assume that all structures are at most countable and their languages are computable. Clearly, finite structures are computable. Let $\mathbf{d}$ be a Turing degree. An infinite structure $\mathcal{M}$ is $\mathbf{d}$-*computable* if its universe can be identified with the set of natural numbers ω in such a way that the relations and operations of $\mathcal{M}$ are uniformly $\mathbf{d}$-computable. For example, we may consider structures computable in the halting problem, such as Σ_1^0 and Π_1^0 structures. See Higman [165], Feiner [94], Metakides and Nerode [241], Ershov and Goncharov [93], and Cenzer, Harizanov, and Remmel [45] for more on Σ_1^0 structures, and Remmel [294], Khoussainov, Slaman, and Semukhin [206], and Cenzer, Harizanov, and Remmel [45] for more on Π_1^0 structures.

If an algebraic structure is not computable, then it is natural to ask how close it is to a computable one. This property can be captured by the collection of all Turing degrees relative to which a given structure has a computable isomorphic copy. Thus, we have the following definition.

Definition 1. The *degree spectrum* of a structure $\mathcal{A}$ is

$$DgSp(\mathcal{A}) = \{\deg(D(\mathcal{B})) : \mathcal{B} \cong \mathcal{A}\},$$

where $D(\mathcal{B})$ is the atomic diagram of $\mathcal{B}$.

Knight proved the following fundamental result about the degree spectrum of a structure.

Theorem 2. *([209]) The degree spectrum of any structure is either a singleton or is upward closed.*

A structure $\mathcal{A}$ is *automorphically trivial* if there is a finite subset C of its domain such that every permutation of the domain, which fixes C pointwise, is an automorphism of $\mathcal{A}$. Automorphically trivial structures include all finite structures, of course, and also some infinite structures, such as the complete graph on countably many vertices. If the structure is automorphically nontrivial, the degree spectrum is upward closed [209]. The degree spectrum of an automorphically trivial structure always contains exactly one Turing degree, and if the language is finite, that degree must be $\mathbf{0}$ (see [155]). Jockusch and Richter introduced the following notion.

Definition 2. ([297]) If the degree spectrum of a structure $\mathcal{A}$ has a least element, then this element is called the *degree of the isomorphism type* of $\mathcal{A}$.

Richter [297, 296] initiated the systematic study of such degrees. She proved that if $\mathcal{A}$ is a structure without a computable copy, which satisfies the effective extendability condition, then the isomorphism type of $\mathcal{A}$ has no degree. A structure $\mathcal{A}$ satisfies the *effective extendability condition* if for every finite structure $\mathcal{M}$ isomorphic to a substructure of $\mathcal{A}$, and every embedding f of $\mathcal{M}$ into $\mathcal{A}$, there is an algorithm that determines whether a given finite structure $\mathcal{F}$ extending $\mathcal{M}$ can be embedded into $\mathcal{A}$ by an embedding extending f. Richter showed that every linear order, and every tree, as a partially ordered set, satisfy the effective extendability condition. More recently, A. Khisamiev [194] proved that every abelian p-group satisfies the effective extendability condition. Hence the isomorphism type of a countable linear order, a tree, or an abelian p-group, which is not isomorphic to a computable one, does not have a degree of its isomorphism type. Richter also showed that for any Turing degree $\mathbf{d}$, there is a torsion abelian group the isomorphism type of which has the degree $\mathbf{d}$, as well as that there is such a group the isomorphism type of which does not have a degree. Results of Richter motivated the study of jump degrees of structures. The following definition was also introduced by Jockusch and Richter.

Definition 3. ([297]) Let $\mathcal{A}$ be a structure, and α a computable ordinal. We say that a Turing degree $\mathbf{d}$ is the α^{th} *jump degree* of $\mathcal{A}$ if it is the least degree in

$$\{\mathbf{d}^{(\alpha)} : \mathbf{d} \in DgSp(\mathcal{A})\}.$$

The degree $\mathbf{d}$ is said to be *proper* α^{th} *jump degree* of $\mathcal{A}$ if for every computable ordinal $\beta < \alpha$, the structure $\mathcal{A}$ has no β^{th} jump degree.

Given a class of structures, we may ask for which computable ordinals α there exist representatives of this class having (proper) α^{th} jump degrees.

The following theorem summarizes the results for linear orders due to Knight [209], Ash, Jockusch, and Knight [13], and Downey and Knight [80].

Theorem 3. *([209, 13, 80]) If a linear order has first jump degree, it must be $\mathbf{0}'$. In contrast, for each computable ordinal $\alpha \geq 2$ and every Turing degree $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, there exists a linear order having proper α^{th} jump degree $\mathbf{d}$.*

Ordinal jump degrees of Boolean algebras are well-understood as well, but the results differ from the ones for linear orders. Jockusch and Soare established the following result.

Theorem 4. *([179]) For $n \in \omega$, if a Boolean algebra has n^{th} jump degree, then it is $\mathbf{0}^{(n)}$. In contrast, for each $\mathbf{d} \geq \mathbf{0}^{(\omega)}$, there exists a Boolean algebra with proper ω^{th} jump degree $\mathbf{d}$.*

Oates investigated jump degrees of torsion abelian groups.

Theorem 5. *([281]) For every computable α, there is a torsion abelian group having proper α^{th} jump degree.*

The proof relies on algebraic properties of countable abelian p-groups, which are well-undestood.

The situation becomes more complex in the case of countable, torsion-free, abelian groups, where there is no suitable algebraic classification theory. Nevertheless, there has been a significant progress in this area. If $\mathcal{G} = (G, +)$ is a torsion-free abelian group, a set of nonzero elements $\{g_i : i \in I\} \subset G$ is *linearly independent* if $\alpha_1 g_{i_1} + \cdots + \alpha_k g_{i_k} = 0$ has no solution for $\{i_1, ..., i_k\} \subseteq I$, $\alpha_i \in \mathbb{Z}$ for each i, and $\alpha_i \neq 0$ for some i. A *basis* for $\mathcal{G}$ is a maximal linearly independent set, and the *rank* of $\mathcal{G}$ is the cardinality of a basis. Calvert, Harizanov, and Schlapentokh obtained the results about Turing degrees of isomorphism types for various familiar algebraic classes, including torsion-free abelian groups of finite rank.

Theorem 6. *([37]) There are algebraic fields and torsion-free abelian groups of any finite rank > 1, the isomorphism types of which have arbitrary Turing degrees. There are structures in each of these classes the isomorphism types of which do not have Turing degrees.*

For rank 1, torsion-free, abelian groups the result was previously obtained by Knight, Downey, and Jockusch (see [74]). Such groups are isomorphic to subgroups of $(\mathbb{Q}, +)$, and there is a known classification for these groups due to Baer.

Melnikov [238] showed that not every infinite-rank, torsion-free, abelian group has first jump degree. Results about the existence of proper jump degrees for torsion-free abelian groups were resolved by Downey and Jockusch for the first jump, and by Melnikov for the second and the third jump.

Theorem 7. *([74, 238] For $n \in \{1,2\}$ and every degree $\mathbf{d} \geq \mathbf{0}^{(n)}$, there is a torsion-free group having proper n^{th} jump degree $\mathbf{d}$. For every degree $\mathbf{d} > \mathbf{0}'''$, there is a torsion-free group having proper 3^{rd} jump degree $\mathbf{d}$.*

The case of higher ordinals remained unresolved until the recent work of Andersen, Kach, Melnikov, and Solomon who obtained the following general result.

Theorem 8. *([3]) For every computable $\alpha > 3$, every $\mathbf{d} > \mathbf{0}^{(\alpha)}$ can be realized as a proper α^{th} jump degree of a torsion-free abelian group.*

It is not known whether the result can be strengthened to $\mathbf{d} = \mathbf{0}^{(\alpha)}$ for $\alpha > 2$. The groups from Theorem 7 are of the form $\bigoplus_{i\in\omega} \mathcal{H}_i$, where $\mathcal{H}_i \leqq (\mathbb{Q}, +)$. Such groups, introduced by Baer in 1937, are called *completely decomposable* and have nice algebraic properties. In the case of only one summand, Coles, Downey, and Slaman [58] established the following theorem, as a consequence of their pure computability-theoretic result that for every set $C \subseteq \omega$, there is a Turing degree that is the least degree of the jumps of all sets X for which C is c.e. in X.

Theorem 9. *([58]) Every torsion-free abelian group of rank 1 has first jump degree.*

Theorem 9 can be extended to torsion-free abelian groups of any finite rank, as was observed in [37, 238]. It is not known which ordinals are realized as proper jump degrees of groups of the form $\bigoplus_{i\in\omega} \mathcal{H}_i$, where $\mathcal{H}_i \leqq (\mathbb{Q}, +)$.

For certain classes of countable structures, we can use computable functors to translate results from one class of countable structures to another. A functor $\Phi : \mathcal{K} \to \mathcal{K}_1$ is *computable* if, given an enumeration of an open diagram of $\mathcal{A} \in \mathcal{K}$, we can enumerate the open diagram of $\Phi(\mathcal{A}) \in \mathcal{K}_1$, in a uniform fashion. Computable functors are also called effective transformations. Hirschfeldt, Khoussainov, Shore, and Slinko used injective effective transformations to transfer various computability-theoretic results from graphs to structures in other familiar algebraic classes.

Theorem 10. *([173]) For every automorphically nontrivial structure $\mathcal{A}$, there is a symmetric irreflexive graph, a partial order, a lattice, a ring, an integral domain of arbitrary characteristic, a commutative semigroup, or a 2-step nilpotent group the degree spectrum of which coincides with $DgSp(\mathcal{A})$.*

As a consequence we obtain that these classes have structures with proper α^{th} jump degrees for all computable ordinals α. Frolov, Kalimullin, and R. Miller [109] investigated degree spectra of algebraic fields.

Theorem 11. *([109]) Every algebraic field has first jump degree.*

Not much is known about groups that are far from abelian. There are centerless groups that have arbitrary Turing degrees for their isomorphism classes, as well as no degrees [68]. Recently, Calvert, Harizanov, and Shlapentokh [38] started to investigate effective content of geometric objects, such as ringed spaces and schemes. In particular, they showed that ringed spaces corresponding to unions of varieties, ringed spaces corresponding to unions of subvarieties of certain fixed varieties, and schemes over a fixed field can have arbitrary Turing degrees for their isomorphism classes, as well as no degrees.

Lempp asked if there is a nontrivial sufficient condition on a structure, which will guarantee that its degree spectrum contains **0**. Slaman [315] and Wehner [332] independently obtained the following result, with different proofs.

Theorem 12. *([315, 332]) There exists a structure the degree spectrum of which is the set of all noncomputable Turing degrees.*

Wehner [332] constructed a family of sets that yields a structure with isomorphic copies in exactly the noncomputable Turing degrees. While Wehner's structure is elementarily equivalent to a computable structure, Slaman's is not. We will say that a structure such as one in Theorem 12 has *Slaman-Wehner degree spectrum*. More recently, Hirschfeldt [168] proved that there is a structure with Slaman-Wehner degree spectrum, which is a prime model of a complete decidable theory. This also gives another proof of Theorem 12. Hirschfeldt's structure is elementarily equivalent to a decidable structure. Hirschfeldt's degree spectrum result follows from his theorem in [168] that if $\mathcal{T}$ is a computable tree with no dead ends and with all infinite paths computable, and D is a noncomputable set, then there is a D-computable listing of the isolated paths in $\mathcal{T}$. Previously, Goncharov and Nurtazin [142] and T. Millar [247] established that this result does not hold if D is computable.

Downey asked if there exists a structure in a natural algebraic class of structures, such as a linear order or an abelian group, which has Slaman-Wehner spectrum. We can also ask which sets of degrees can be realized as degree spectra of structures. Since co-null collections of degrees are of a particular interest, we have the following definition due to Kalimullin.

Definition 4. ([188]) An automorphically nontrivial structure $\mathcal{M}$ is called *almost computable* if the measure of $DgSp(\mathcal{M})$ is equal to 1 under the standard Lebesgue measure on the Cantor space.

For example, every structure with Slaman-Wehner spectrum is almost computable. More examples have been obtained recently. Kalimullin [187, 186, 185] investigated the relativization of Slaman-Wehner theorem to nonzero degrees. He showed that such a relativization holds for every *low* Turing degree, as well as every c.e. degree, but not for every Δ^0_3 Turing degree. Using the enumeration result of Wehner, also relativized, Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon [133] showed that for every computable successor ordinal α, there is a structure with copies in just the degrees of sets X such that $\Delta^0_\alpha(X)$ is not Δ^0_α. As a consequence, they obtained the following result.

Theorem 13. *([133]) For each finite n, there is a structure with the degree spectrum consisting of exactly all non-*low_n *Turing degrees.*

Consequently, there are almost computable structures without arithmetic isomorphic copies. Csima and Kalimullin provided another interesting example of a possible degree spectrum.

Theorem 14. *([66]) The set of hyperimmune degrees is the degree spectrum of a structure.*

We could ask the following analogue of Lemmp's question for almost computable structures. If a structure is almost computable, must it contain a hyperarithmetic or a Π^1_1 degree? Greenberg, Montalbán, and Slaman [144] and independently Kalimullin and Nies (unpublished) obtained the following positive result.

Theorem 15. *([144]) If $\mathcal{M}$ is an almost computable structure, then there is some copy of $\mathcal{M}$ that is computable from Kleene's $\mathcal{O}$.*

This bound cannot be improved to be hyperarithmetic. Recently, Greenberg, Montalbán, and Slaman [143] constructed a linear order the degree spectrum of which is the set of all non-hyperarithmetic degrees. There are other examples of almost computable structures in various natural algebraic classes and we will discuss some of them.

Although the degree spectra of linear orders have been intensively studied, the following question remains open. Is there a linear order the degree spectrum of which is the set of all nonzero degrees? Jockusch and Soare [180] established that for every nonzero c.e. Turing degree $\mathbf{d}$, there is a linear order $\mathcal{L}$ of Turing degree $\mathbf{d}$ such that $\mathcal{L}$ does not have a computable copy. Downey, Seetapun, and Knight extended this result to an arbitrary nonzero Turing degree (see [74]). R. Miller [252] constructed a linear order with the spectrum containing all nonzero Δ^0_2 degrees but not $\mathbf{0}$. Recently, Frolov, Harizanov, Kalimullin, Kudinov, and R. Miller obtained the following examples.

Theorem 16. *([108]) Let $n \geq 2$. For every Turing degree $\mathbf{c}$, there is a linear order with spectrum $\{\mathbf{d} : \mathbf{d}^{(n)} > \mathbf{c}\}$. In particular, there is a linear order the spectrum of which contains exactly the non-*low_n *degrees.*

For a survey of related results on linear orders see [108].

Slaman-Wehner's degree spectrum is not possible when restricted to the class of countable Boolean algebras. Knight and Stob [212] obtained the following result about low_4 Boolean algebras, extending a result of Downey and Jockusch [77] for *low* Boolean algebras, and of Thurber [328] for low_2 Boolean algebras.

Theorem 17. *([212]) Every* low_4 *Boolean algebra has a computable isomorphic copy.*

One of the main open questions in this area is the following. Is every low_n, $n \geq 5$, Boolean algebra isomorphic to a computable one? The affirmative answer to this question is known as the *low_n Boolean algebra conjecture*. There is some evidence that if every low_5 Boolean algebra has a computable copy, then the proof of that statement should be different from the proof for low_4 Boolean algebras. This follows from work of Harris and Montalbán in [160] where they showed that there are over 1000 invariants that have to be considered for the

low_5 case, as well as from work of Harris and Montalbán on the complexity of isomorphisms in [161].

Similarly to linear orders, the following question is open. Is there an abelian group having Slaman-Wehner degree spectrum? Recently, Khoussainov, Kalimullin, and Melnikov proved the following result about abelian p-groups.

Theorem 18. *([189]) There exists an abelian p-group, which has an $\mathbf{d}$-computable copy relative to every noncomputable Δ_2^0 Turing degree $\mathbf{d}$, but has no computable copy.*

In addition, Khoussainov, Kalimullin, and Melnikov [189] proved that there exists a noncomputable torsion abelian group the degree spectrum of which contains all hyperimmune degrees. They also showed that this result cannot be generalized to co-countable collections of degrees, when restricted to direct sums of cyclic groups. These results can be re-formulated in terms of effective monotonic approximations that we will later introduce. It is also known that there exists a torsion-free abelian group having exactly *nonlow* isomorphic copies [238]. Other structures studied in this context come from [173]. There are also some related results on equivalence structures (see [45, 189]).

In many cases, the existence of a computable copy of a structure is related to the ability to enumerate a certain invariant of the structure.

Examples

(i) Given a set S, define the algebraic extension $\mathcal{F}_S$ of the prime field $\mathbb{Q}$ to be $\mathbb{Q}(\{\sqrt{p_x} : x \in S\})$. The field $\mathcal{F}_S$ has an X-computable copy if and only if S is c.e. in X.

(ii) Given a set S, define a subgroup $\mathcal{G}(S)$ of $(\mathbb{Q}, +)$ by having a generator $\frac{1}{p_x}$ for $\mathcal{G}(S)$ if and only if $x \in S$. Then $\mathcal{G}(S)$ has an X-computable copy if and only if S is c.e. in X.

It is well known that under an appropriate choice of S neither $\mathcal{F}_S$ nor $\mathcal{G}(S)$ has a Turing degree for its isomorphism type (see, for example, [37]). Nevertheless, in the examples above, we may define the *enumeration degree* of $\mathcal{F}_S$ or $\mathcal{G}(S)$ to be the degree of the set S under the enumeration reducibility $\leq_e$. There is also a direct way to define an enumeration degree spectrum of a structure, as A. Soskova and Soskov did in [321, 320]. More generally, we may view a degree spectrum as a mass problem. The following general definition is due to Medvedev.

Definition 5. ([237]) A *mass problem* is a collection of total functions from ω to ω.

Stukachev defined various reducibilities between mass problems of structures, such as Muchnik reducibility. As usual, we identify the atomic diagram $\mathcal{D}(\mathcal{B})$ of a countable structure $\mathcal{B}$ with its characteristic function $\chi_{\mathcal{D}(\mathcal{B})} \in 2^\omega$, under Gödel coding of formulas.

Definition 6. ([326])

(i) The *mass problem* of a countable structure $\mathcal{A}$ is the set

$$\{\chi_{\mathcal{D}(\mathcal{B})} : \mathcal{B} \cong \mathcal{A}\}.$$

(ii) Given countable structures $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ is *Muchnik reducible* to $\mathcal{B}$, in symbols $\mathcal{A} \leq_w \mathcal{B}$, if $DgSp(\mathcal{A}) \subseteq DgSp(\mathcal{B})$.

Thus, $\mathcal{A}$ is Muchnik equivalent to $\mathcal{B}$, written as $\mathcal{A} =_w \mathcal{B}$, if $\mathcal{A} \leq_w \mathcal{B}$ and $\mathcal{B} \leq_w \mathcal{A}$. Selman's theorem [310] states that if a structure has an enumeration degree as defined above, then $\leq_w$ coincides with the enumeration reducibility $\leq_e$. Thus, the notion of enumeration degree is a special case of Definition 6. For other reducibilities on mass problems of structures see Stukachev [326, 325].

Whenever a reducibility is defined, we look for a suitable definition of the *jump*. Various authors recently and independently introduced the notion of the jump of an abstract structure: Baleva [22] and Soskov and A. Soskova [320] using Moschovakis extensions; Morozov [260] and Puzarenko [287] in the context of admissible sets; Montalbán [257] using predicates for computable infinitary Σ_1 formulas; Stukachev [324] using hereditarily finite extensions. It is remarkable that these different approaches turned out to be equivalent. We give the definition due to Montalbán.

Definition 7. ([257]) Given a language L, let $\{\theta_i : i \in \omega\}$ be a computable enumeration of all computable infinitary Σ_1 formulas in L. Given a structure $\mathcal{A}$ for L, let $\mathcal{A}'$ be the structure obtained by adding to $\mathcal{A}$ infinitely many relations P_i, for $i \in \omega$, where

$$\mathcal{A} \models P_i(\overline{x}) \Leftrightarrow \theta_i(\overline{x}),$$

and the arity of P_i is the same as the length of $\overline{x}$ in $\theta_i(\overline{x})$.

Several results on degree spectra of structures can be re-formulated in terms of the jumps of structures. For instance, the result of Downey and Jockusch in [77] that every *low* Boolean algebra is isomorphic to a computable one follows from the following result. If $\mathcal{B}$ is a Boolean algebra, and $\mathbf{0}'$ computes a copy of $\mathcal{B}'$, then $\mathcal{B}$ has a computable copy. A better understanding of the jump operator on structures may help us establish or refute the low_n Boolean algebra conjecture.

A. Soskova and Soskov, and also Montalbán showed that the spectrum of a structure behaves well with respect to the jump operator of the structure. More precisely, they established the following *jump inversion theorem*.

Theorem 19. *([320, 257]) For every structure $\mathcal{A}$, we have*

$$DgSp(\mathcal{A}') = \{\mathbf{d}' : \mathbf{d} \in DgSp(\mathcal{A})\}.$$

Other authors also independently proved the jump inversion theorem. See Stukachev [324] for more on the jump inversion results. Recently, Puzarenko [286] and Montalbán [256] showed independently and simultaneously that the jump operator has a fixed point.

Theorem 20. *([286, 256]) There is a structure $\mathcal{A}$ such that $\mathcal{A} =_w \mathcal{A}'$.*

Montalbán proved this theorem under the assumption that "$0^\#$ exists", and Puzarenko obtained another proof that does not use this assumption.

Andrews and J. Miller [9] have recently defined the *spectrum of a theory T* to be the set of Turing degrees of models of T. The idea behind this notion is to better understand the relationship between the model-theoretic properties of a theory and the computability-theoretic complexity of its models. Theory spectra may coincide with degree spectra of structures, e.g., the cones above arbitrary Turing degrees are theory spectra, as well as the set of all noncomputable degrees. On the other hand, there are examples of theory spectra that are not degree spectra for any structure, and *vice versa*. We say that a real x is *Martin-Löf random* or 1-*random* iff for every computable collection of c.e. open sets $\{U_n : n \in \omega\}$, with $\mu(U_n) \leq 2^{-n}$, $n \in \omega$, we have $x \notin \cap_{n\in\omega} U_n$, where μ is the standard Lebesgue measure on the Cantor space. A Turing degree is called *1-random* if it contains a set that is 1-random. For more on randomness see [277, 76].

Theorem 21. *([9]) The following sets of Turing degrees can be theory spectra:*

(a) the degrees of complete extensions of Peano arithmetic,

(b) 1*-random degrees,*

(c) the union of the cones above two incomparable Turing degrees.

However, as it follows from [321] and [9], these sets are not the degree spectra of any structures. On the other hand, by [144], there is a structure the degree spectrum of which consists of exactly the non-hyperarithmetic degrees.

Theorem 22. *([9]) The collection of non-hyperarithmetic degrees is not the spectrum of a theory.*

Further interesting examples can be found in [9], and for the case of atomic theories in [7].

The notion of the degree spectrum of a structure turned out to be useful in order to find a new approach to resolve one of the most famous conjectures in mathematical logic: Vaught's conjecture. Recall that *Vaught's conjecture* states that the number of countable models of a first-order theory is either countable or continuum. In [255], Montalbán analyzed computability-theoretic properties of a possible counterexample to Vaught's conjecture in terms of degree spectra of its models. The analysis is done under the assumption of projective determinacy (PD). The result of [255] is stated not only for finitary first-order theories, but for $L_{\omega_1\omega}$ sentences. When the continuum hypothesis (CH) does not hold, we say that an $L_{\omega_1\omega}$-theory T is a *counterexample* to Vaught's conjecture if it has uncountably many countable models but not continuum many. Montalbán [255] also gives another definition, which is equivalent to the given definition under $\neg$CH, and also makes sense when CH holds. He defines a class K of structures to satisfy the property *hyperarithmetic-is-computable on a cone* if there exists Y such that for all X with $X \geq_T Y$, every X-hyperarithmetic structure in K has an X-computable copy.

Theorem 23. *(ZFC+PD) ([255]) Let T be an $L_{\omega_1\omega}$-sentence with uncountably many countable models. The following are equivalent:*

(i) T is a counterexample to Vaught's conjecture;

(ii) The class of models of T satisfies the property hyperarithmetic-is-computable on a cone*;*

(iii) There exists an oracle relative to which

$$\{DgSp(\mathcal{A}) : \mathcal{A} \models T\} = \{\{X \in 2^\omega : \omega_1^X \geq \alpha\} : \alpha \in \omega_1\}.$$

3 Theories, types, models, and diagrams

We will assume that our theories are consistent, countable, and have infinite models. We will denote the *elementary* (*complete*) *diagram* of $\mathcal{A}$ by $D^c(\mathcal{A})$. It is easy to see that the theory of a structure $\mathcal{A}$ is computable in $D^c(\mathcal{A})$, and that $D^c(\mathcal{A})$ is computable in $(D(\mathcal{A}))^{(\omega)}$. The atomic diagram of a model of a theory may be of much lower Turing degree than the theory itself. Henkin's construction of models is effective and establishes that a decidable theory has a decidable model. The *low basis theorem* can be used to obtain for a theory S, a model $\mathcal{A}$ with

$$(D^c(\mathcal{A}))' \leq_T S'.$$

Harizanov, Knight, and Morozov [154] showed that for every automorphically nontrivial structure $\mathcal{A}$, and every set $X \geq_T D^c(\mathcal{A})$, there exists $\mathcal{B} \cong \mathcal{A}$ such that

$$D^c(\mathcal{B}) \equiv_T D(\mathcal{B}) \equiv_T X.$$

For every automorphically trivial structure $\mathcal{A}$, we have $D^c(\mathcal{A}) \equiv_T D(\mathcal{A})$.

A structure $\mathcal{A}$ is called *n-decidable* for $n \geq 1$ if the Σ_n-diagram of $\mathcal{A}$ is decidable. We will denote Σ_n-diagram $\mathcal{A}$ by $D_n(\mathcal{A})$. For sets X and Y, we say that Y is *c.e. in and above* (*c.e.a. in*) X if Y is c.e. relative to X, and $X \leq_T Y$. For any structure $\mathcal{A}$, $D_{n+1}(\mathcal{A})$ is c.e.a. in $D_n(\mathcal{A})$, uniformly in n, where $D_0(\mathcal{A}) = D(\mathcal{A})$. Chisholm and Moses [54] established that there is a linear order that is n-decidable for every $n \in \omega$, but has no decidable copy. Goncharov [127] earlier obtained a similar result for Boolean algebras. There are familiar structures $\mathcal{A}$ such that for all $\mathcal{B} \cong \mathcal{A}$, we have $D^c(\mathcal{B}) \equiv_T D(\mathcal{B})$. In particular, this is true for algebraically closed fields, and for other structures for which we have effective elimination of quantifiers. In [154], Harizanov, Knight, and Morozov gave syntactic conditions on $\mathcal{A}$ under which for all $\mathcal{B} \cong \mathcal{A}$, we have $D^c(\mathcal{B}) \equiv_T D_n(\mathcal{B})$ for $n \in \omega$.

In the early 1960s, Vaught [330] developed the theory of prime, saturated, and homogeneous models using types. A countable structure $\mathcal{A}$ is *homogeneous* if for every two finite sequences $\overline{a}$ and $\overline{b}$ of the same length n, if $\overline{a}$ and $\overline{b}$ realize the same n-type in $\mathcal{A}$, then there is an automorphism of $\mathcal{A}$ taking $\overline{a}$ to $\overline{b}$. Every countable complete theory has a countable homogeneous model. Prime models and countable saturated models are examples of homogeneous models. The study of the computable content of these models was initiated in the 1970s. The set of all computable types of a complete decidable theory is a Π^0_2 set. Every principal type of such a theory is computable, and the set of all its principal types is Π^0_1.

A model $\mathcal{A}$ of a theory T is *prime* if for all models $\mathcal{B}$ of T, $\mathcal{A}$ elementarily embeds into $\mathcal{B}$. For example, the algebraic numbers form a prime model of the theory of algebraically closed fields of characteristic 0. All prime models of a given theory are isomorphic. It is well known that every complete atomic theory has a prime model. It is not difficult to show that if a complete decidable theory T has a decidable prime model, then the set of all principal types of T is uniformly computable. Goncharov and Nurtazin [142] and independently Harrington [157] established the converse.

Theorem 24. *([142, 157]) For a complete decidable theory T, the following are equivalent.*

1. *There is a uniform procedure that maps a formula consistent with T into a computable principal type of T, which contains this formula.*
2. *The theory T has a decidable prime model.*
3. *The theory T has a prime model and the set of all principal types of T is uniformly computable.*

For a set X and its Turing degree $\mathbf{x} = \deg(X)$, we say that a structure $\mathcal{A}$ is *decidable in* X or $\mathbf{x}$-*decidable* if $D^c(\mathcal{A}) \leq_T X$. Drobotun [87] and T. Millar [247] independently showed that a complete, atomic, decidable theory has a $\mathbf{0}'$-*decidable* prime model. More recently, Csima [61] strengthened this result by showing that every complete, atomic, decidable theory T has a prime model $\mathcal{A}$ such that $D^c(\mathcal{A})$ is *low*. Although Csima's result has the same flavor as the *low basis theorem*, it does not follow from it. Epstein extended Csima's result by establishing the following.

Theorem 25. *([89]) Let T be a complete, atomic, decidable theory with a prime model $\mathcal{A}$ such that $D^c(\mathcal{A})$ has a c.e. degree $\mathbf{c} > \mathbf{0}$. Then there is a prime model $\mathcal{B}$ of T such that $D^c(\mathcal{B})$ has a* low *c.e. degree $\mathbf{a}$, where $\mathbf{a} < \mathbf{c}$.*

On the other hand, there are theories with prime models the elementary diagrams of which have minimal degrees, but the theories have no decidable prime models.

Goncharov [121] proved that there is a complete, decidable, ω-stable theory in a finite language having no computable homogeneous model. A theory T is ω-*stable* if for every $\mathcal{M} \models T$ and every countable $X \subseteq M$, there are only countably many types of T over X. (Uncountably categorical theories, which will be investigated in the next section, are notable examples of ω-stable theories.) Goncharov's theory has infinitely many axioms. Peretyat'kin [283] constructed a complete, atomic, finitely axiomatizable (hence decidable) theory without a computable prime model. T. Millar [244] came up with a weaker notion of a decidable model, the notion of an *almost decidable model*, and showed that if a complete decidable theory has fewer than continuum many complete types, then the theory has an almost decidable prime model. Since not every decidable complete theory with only countably many complete types has a decidable model [121], T. Millar's result cannot be extended to decidable prime models. Hirschfeld obtained an interesting result about the degree spectrum of a prime model, already mentioned in the previous section.

Theorem 26. *([168]) There is a prime model of a complete decidable theory with Slaman-Wehner degree spectrum.*

We can also consider theories of algebraic structures from natural classes, such as groups or linear orders. Even if their theories are not necessarily decidable, they can have computable models. N. Khisamiev obtained the following negative result.

Theorem 27. *([195]) There is a complete theory of abelian groups with both a computable model and a prime model, but no computable prime model.*

Interestingly, the proof of this result has influenced other investigations in computable model theory, outside group theory. Khisamiev's proof uses the concept of a limitwise monotonic function, which he introduced in [196] to study which abelian p-groups have computable isomorphic copies.

Definition 8. ([196]) A total function $F : \omega \to \omega$ is *limitwise monotonic* if there is a computable function $f : \omega^2 \to \omega$ such that for all $i, s \in \omega$, we have $f(i,s) \leq f(i,s+1)$, the limit $\lim_{s\to\infty} f(i,s)$ exists, and $F(i) = \lim_{s\to\infty} f(i,s)$.

See [189] for more on limitwise monotonic functions. Using limitwise monotonic functions, Hirschfeldt obtained a negative solution to a long-standing problem posed by Rosenstein [299].

Theorem 28. *[170] There is a complete theory of linear orders having a computable model and a prime model, but no computable prime model.*

A set X and its Turing degree are called *prime bounding* if every complete, atomic, decidable theory has a prime model $\mathcal{A}$ such that $D^c(\mathcal{A}) \leq_T X$. Thus, $\mathbf{0}'$ is prime bounding. Csima, Hirschfeldt, Knight, and Soare obtained the following equivalence.

Theorem 29. *([65]) Let $X \leq_T \emptyset'$. Then X is prime bounding if and only if X is not* low$_2$.

This theorem gives an interesting characterization of *low*$_2$ sets in terms of prime models of certain theories, thus providing a link between computable model theory and degree theory. To prove that a *low*$_2$ set X is not prime bounding, we use a $\emptyset'$-computable listing of the array of sets $\{Y : Y \leq_T X\}$ to find a complete, atomic, decidable theory T, which diagonalizes against all potential prime models of T the elementary diagrams of which are computable in X. To prove that any set X that is not *low*$_2$ is indeed prime bounding, we fix a function

$f \leq_T X$ that dominates every total $\emptyset'$-computable function. Given a complete, atomic, decidable theory T, we use f to build a prime model of T. In addition to the two properties in Theorem 29, Csima, Hirschfeldt, Knight, and Soare [65] consider a number of other properties equivalent to these two, some of which are related to limitwise monotonic functions.

Recall that a countable *saturated* model is a model realizing every type of its language augmented by any finite tuple of constants for its elements. The earliest effective notion related to saturated models was the notion of a recursively saturated model introduced and first studied by Barwise and Schlipf in [26]. A *recursively saturated* model is a model (of a computable language) realizing every *computable* set of formulas consistent with its theory, in the language expanded by any finite set of constants. Note that every saturated model is recursively saturated. It is well known that a complete theory has a countable saturated model if and only if the theory has only countably many n-types for every $n \geq 1$. On the other hand, *every* complete theory in a computable language with infinite models has a countable recursively saturated model. In fact, in the case of a computable language, early proofs of several classical results in model theory can be simplified using recursively saturated models (see [49]). The simplification is done by replacing "large" models by recursively saturated models in the proofs [26]. The "large" models exist only under certain set-theoretic restrictions [49]. Being a computable language is often not a severe restriction since many important languages are computable or even finite. These remarkable results provide an application of computability theory to classical model theory. However, a recursively saturated model does not have to be decidable or even computable, so we will turn our attention to decidable saturated models.

Decidable saturated models of complete decidable theories are fairly well-understood. There is a complete description of decidable saturated models in terms of types, due to Morley [259] and T. Millar [247] independently.

Theorem 30. *[259, 247] Let T be a complete decidable theory. The set of all types of T is uniformly computable if and only if T has a decidable saturated model.*

Thus, a complete theory with a decidable saturated model also has a decidable prime model. Morozov obtained a general positive result for Boolean algebras.

Theorem 31. *([271]) Every countable saturated Boolean algebra has a decidable isomorphic copy.*

If the types are not uniformly computable, then the existence of a decidable saturated model is not guaranteed, as shown independently by Goncharov and Nurtazin [142], Morley [259] and T. Millar [247], who constructed counterexamples.

Theorem 32. *[142, 259, 247] There is a complete decidable theory with all types computable, which does not have a decidable saturated model.*

Any saturated model of a complete decidable theory with all types computable has a $\mathbf{0}'$-decidable isomorphic copy [142, 259, 247]. This result leads to the investigation of the effective content of saturated models using degree-theoretic concepts and machinery. The following definition was introduced by Harris and is similar to the one for prime models. A Turing degree $\mathbf{d}$ is *saturated bounding* if every complete decidable theory with types all computable has a $\mathbf{d}$-decidable saturated model. Macintyre and Marker [229] showed that the degrees of complete extensions of Peano arithmetic are saturated bounding. There is a recent negative result due to Harris.

Theorem 33. *([159]) For every $n \in \omega$, no* low_n *c.e. degree is saturated bounding.*

For a structure $\mathcal{A}$, the *type spectrum* of $\mathcal{A}$ is the set of all types realized in $\mathcal{A}$. Since a countable homogeneous structure is uniquely determined, up to isomorphism, by the set of types it realizes, Morley posed the following natural question for a complete decidable theory T. If the type spectrum of a countable homogeneous model $\mathcal{A}$ of T consists only of computable types and is computable, does $\mathcal{A}$ have a decidable isomorphic copy? Independently, Goncharov [125], Peretyat'kin [284], and T. Millar [246] answered Morley's question negatively.

Theorem 34. *([125, 284, 246]) There exists a complete decidable theory T having a homogeneous model $\mathcal{M}$ without a decidable copy, such that the type spectrum of $\mathcal{M}$ consists only of computable types and is computable.*

In fact, Goncharov [125] and Peretyat'kin [284] provided a criterion for a homogeneous model to be decidable. Their criterion can be stated in terms of the effective extension property. A computable set of computable types of a theory has the *effective extension property* if there is a partial computable function f which, given a type Γ_n of arity k and a formula θ_i of arity $k+1$ (identified with their indices), outputs the index for a type containing Γ_n and θ_i, if there exists such a type.

It is well known that every countable model has a countable homogeneous elementary extension. Ershov conjectured that every decidable model can be elementary embedded into a decidable homogeneous elementary extension. Peretyat'kin refuted Ershov's conjecture in a strong way.

Theorem 35. *([285]) There exists a decidable model, which does not have a* computable *homogeneous elementary extension.*

Goncharov and Drobotun [130] constructed a computable linear order that does not have a computable homogeneous elementary extension.

Regarding more recent investigation of degree-theoretic content of homogeneous models, similarly to prime bounding and saturated bounding degrees, we have the following definition. A Turing degree $\mathbf{d}$ is *homogeneous bounding* if every complete decidable theory has a $\mathbf{d}$-decidable homogeneous model. Csima, Harizanov, Hirschfeldt, and Soare obtained the following result about homogeneous bounding degrees.

Theorem 36. *([63]) There is a complete decidable theory T such that every countable homogeneous model of T has the degree of a complete extension of Peano arithmetic.*

This theorem implies that every homogeneous bounding degree is the degree of a complete extension of Peano arithmetic, but it is in fact stronger, since we build a *single* theory T such that the use of the degrees of complete extensions of Peano arithmetic is necessary to compute even the atomic diagram of a homogeneous model of T. Together with the converse of Theorem 36 due to Macintyre and Marker [229], we have the following consequence.

Corollary 1. *A Turing degree* $\mathbf{d}$ *is homogeneous bounding if and only* $\mathbf{d}$ *is the degree of a complete extension of Peano arithmetic.*

Lange introduced the following definition of a $\mathbf{0}$-homogeneous bounding degree.

Definition 9. ([220])

1. A countable structure $\mathcal{A}$ has a $\mathbf{d}$-*basis* if the types realized in $\mathcal{A}$ are all computable and the Turing degree $\mathbf{d}$ can list Δ^0_0-indices for all types realized in $\mathcal{A}$.
2. A Turing degree $\mathbf{c}$ is $\mathbf{0}$-*basis homogeneous bounding* if for every automorphically nontrivial homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis, there exists $\mathcal{B}$ such that $\mathcal{B} \cong \mathcal{A}$ and $\mathcal{B}$ is $\mathbf{c}$-decidable.

Now we can restate Theorem 34 as follows: There exists a homogeneous model $\mathcal{A}$ having a $\mathbf{0}$-basis but no decidable isomorphic copy.

Theorem 37. *([220]) Let T be a complete decidable theory and let $\mathcal{A}$ be a homogeneous model of T with a $\mathbf{0}'$-basis. Then $\mathcal{A}$ has an isomorphic copy decidable in a* low *degree.*

This theorem implies Csima's result that every complete, atomic, decidable theory T has a prime model decidable in a *low* degree (see [61]).

Theorem 38. *([220]) Let T be a complete decidable theory with all types computable. Let $\mathcal{A}$ be a homogeneous model of T with a $\mathbf{0}$-basis. Then $\mathcal{A}$ has an isomorphic copy $\mathcal{B}$ decidable in any nonzero degree.*

Lange also gave a characterization of $\mathbf{0}$-basis homogeneous bounding degrees.

Theorem 39. *([220, 219]) A degree $\mathbf{d} \leq \mathbf{0}'$ is $\mathbf{0}$-basis homogeneous bounding if and only if $\mathbf{d}$ is* nonlow_2.

4 Small theories and their models

We now consider the question of the existence of effective (computable, decidable, etc.) models for *small theories*, that is, theories with at most countably many countable models.

Definition 10. Let κ be a cardinal. A theory is called *κ-categorical* if it has exactly one model of cardinality κ, up to isomorphism.

The following result is well known as *Morley's categoricity theorem* (see [49]).

Theorem 40 (Morley)**.** *If a theory T is κ-categorical for some uncountable cardinal κ, then T is λ-categorical for all uncountable λ.*

Hence, theories categorical in an uncountable cardinal are also called *uncountably categorical*. The theories that are $\aleph_0$-categorical are also called *countably categorical*. A theory that is both countably and uncountably categorical is simply called *totally categorical*. For the case of an uncountably categorical but not countably categorical theory, Baldwin and Lachlan [21] established that its countable models can be listed in a chain of proper elementary embeddings:

$$\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \mathcal{A}_2 \preceq \cdots \preceq \mathcal{A}_\omega,$$

where $\mathcal{A}_0$ is a prime model, and $\mathcal{A}_\omega$ is a saturated model of the theory. Thus, an uncountably categorical theory has either only one countable model or countably many countable models, up to isomorphism.

Definition 11. A theory is called *Ehrenfeucht* if it has finitely many but more than one countable models, up to isomorphism.

By Vaught's theorem, if a theory has two nonisomorphic models, then it has at least three nonisomorphic models. An example of a theory with exactly three countable models was given by Ehrenfeucht. His result can be easily generalized to obtain a theory with exactly n countable models, for any finite $n \geq 3$.

An important question in computable model theory is when a small theory has a computable model. For the case of countably categorical theories, Lerman and Schmerl [225] gave sufficient conditions, which were later extended by Knight as follows.

Theorem 41. *([208]) Let T be a countably categorical theory. If $T \cap \Sigma_{n+2}$ is Σ^0_{n+1} uniformly in n, then T has a computable model.*

The natural question posed by Knight is whether there exist countably categorical theories of high complexity, which satisfy the conditions of the previous theorem. First examples were given by Goncharov and Khoussainov in [137], and then generalized by Fokina as follows.

Theorem 42. *([97]) There exists a countably categorical theory of arbitrary arithmetic complexity, which has a computable model.*

The proof is based on the method of Marker's extensions from [137]. (This method was later applied to investigate various other properties of computable structures, such as in [95, 102].)

The case of a countably categorical theory with a nonarithmetic complexity was resolved by Khoussainov and Montalbán [201]. The unique model of their theory, up to isomorphism, is a modification of the random graph.

Theorem 43. *([201]) There exists a countably categorical theory S with a computable model such that $S \equiv_T \mathbf{0}^{(\omega)}$.*

Another proof of Theorem 43 can be found in [4].

Recall that a consistent decidable theory always has a decidable model. For small theories we can say more. Obviously, if a theory is countably categorical and decidable, then its only (up to isomorphism) countable model always has a decidable copy. For the case of uncountably categorical but not countably categorical theories, Harrington [157] and N. Khisamiev [197] showed that such a theory T is decidable if and only if all countable models of T have decidable isomorphic copies. If T is uncountably categorical but not decidable, then it is possible that some of its models can be isomorphic to computable models, while the others cannot be isomorphic to computable ones.

The following definition of a spectrum of computable models was introduced by Khoussainov, Nies, and Shore.

Definition 12. ([203]) Let T be an uncountably categorical theory with Baldwin-Lachlan elementary chain of countable models:

$$\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \mathcal{A}_2 \preceq \cdots \preceq \mathcal{A}_\omega.$$

The *spectrum of computable models* of the theory T is the set:

$$SCM(T) = \{i \leq \omega : \mathcal{A}_i \text{ has a computable isomorphic copy}\}.$$

A number of researchers investigated which sets can be realized as spectra of computable models of uncountably categorical theories. The first example of a nontrivial spectrum of computable models for uncountably categorical theories was given by Goncharov in [124], where he produced a theory with only the prime model $\mathcal{A}_0$ being isomorphic to a computable one. Goncharov's example was followed by a series of results about various spectra by Kudaibergenov [215], Khoussainov, Nies, and Shore [203], Nies [278], Herwig, Lempp, and Ziegler [164], Hirschfeldt, Khoussainov, and Semukhin [171], and Andrews [5, 6]. All these spectra of computable models are finite or co-finite. On the other hand, the upper bound Nies gave in [278] is $\Sigma^0_{\omega+3}$. The above mentioned uncountably categorical theories are $\mathbf{0}''$-decidable; in particular, all their countable models are isomorphic to $\mathbf{0}''$-decidable ones. Two natural questions arise:

1. What could be the complexity of an uncountably categorical theory with a computable model?

2. Is there a bound on the complexity of all countable models, up to isomorphism, of an uncountably categorical theory with a computable model?

Concerning the first question, the examples of arbitrary arithmetic complexity were given in [97, 137]. Again, the authors used Marker's extensions to build the structures. Andrews [4] resolved the nonarithmetic case by adapting famous Hrushovski's examples from [177] to computable model-theoretic setting.

Theorem 44. *([4]) There exist uncountably categorical theories of arbitrary arithmetic complexity, as well as of nonarithmetic complexity, which have computable models.*

Andrews used the same method to obtain the spectra of computable models in [5, 6]. The original Hrushovski's construction [177] is a powerful model-theoretic tool for building strongly minimal theories. Its modification by Andrews allows us to carry out the construction effectively, and with much greater control, thus providing a remarkable application of model-theoretic methods to solve computability-theoretic problems.

The second question was raised in the mid-1990s by Lempp. He asked whether it was possible to construct an uncountably categorical theory T with a computable prime model such that none of the countable nonprime models is even arithmetic. The answer to this question is negative for a subclass of uncountably categorical theories (see [136]). As usual, *acl* stands for the algebraic closure operation.

Definition 13. (i) A complete theory T is *strongly minimal* if any definable subset of any model $\mathcal{M}$ of T is finite or co-finite. A structure $\mathcal{M}$ is *strongly minimal* if it has a strongly minimal theory.

(ii) A strongly minimal model $\mathcal{M}$ is *trivial* if for all subsets $A \subseteq M$,

$$acl(A) = \bigcup_{a \in A} acl(\{a\}).$$

Goncharov, Harizanov, Lempp, Laskowski, and McCoy established the following result for trivial, strongly minimal models.

Theorem 45. *([136]) Let $\mathcal{M}$ be a computable, trivial, strongly minimal model. Then $Th(\mathcal{M})$ forms a $\mathbf{0}''$-computable set of sentences, and thus all countable models of $Th(\mathcal{M})$ are isomorphic to $\mathbf{0}''$-decidable ones.*

In particular, all countable models of $Th(\mathcal{M})$ are isomorphic to $\mathbf{0}''$-computable models. The proof of Theorem 45 shows an interesting interplay between algorithmic and model-theoretic properties of structures. Namely, the authors proved that for any trivial, strongly minimal theory T in language L, the elementary diagram of any model $\mathcal{M}$ of T is a *model complete* L-theory. This implies that T is $\forall\exists$-axiomatizable, which in turn implies $\mathbf{0}''$-decidability. Furthermore, it was established in [136] that for any strongly minimal, trivial, not totally categorical theory T, the spectrum of computable models is Σ^0_5.

As Khoussainov, Laskowski, Lempp, and Solomon showed in [199], the result in Theorem 45 is best possible in the following sense.

Theorem 46. *([199]) There exists a trivial, strongly minimal (and hence uncountably categorical) theory, which has a computable prime model and each of the other countable models computes $\mathbf{0}''$.*

In [73], Dolich, Laskowski, and Raichev generalized the results of [136] to any uncountably categorical, trivial theory of Morley rank 1. A new, more constructive proof of the same results can be found in [222].

In the case of Ehrenfeucht theories, the question which models can be computable or decidable also has a long history. In the mid-70s, Nerode asked whether all models of a decidable Ehrenfeucht theory must be decidable, by analogy with the results in [157, 197]. Morley [259] gave an example of a theory with six models, of which only the prime model was decidable. A good overview of further related results can be found in [113].

Sudoplatov [327] gave a model-theoretic characterization of Ehrenfeucht models, that is, models of Ehrenfeucht theories. In particular, he introduced the notion of a *limit model*, and a special kind of a pre-order on the set of almost prime models. Recall that a model is *almost prime* if it becomes prime after an enrichment by finitely many constants. Analogously to the case of uncountably categorical theories, Gavryushkin introduced in [114] a notion of the spectrum of computable models for Ehrenfeucht theories. He characterized these spectra in Sudoplatov's terms of pre-orders on almost prime models and the number of limit models over almost prime models. Moreover, Gavryushkin constructed examples of computable Ehrenfeucht models of arbitrarily high arithmetic and nonarithmetic complexity.

Theorem 47. *([114]) For every $n \geq 3$, there exists an Ehrenfeucht theory T of arbitrary arithmetic complexity such that it has n countable models, up to isomorphism, and it has a computable model among them. There also exists such a theory, which is Turing equivalent to the true first-order arithmetic.*

For further examples of Ehrenfeucht theories with various spectra of computable models see [113].

5 Effective categoricity

We are interested in the complexity of isomorphisms between a computable structure and its computable and noncomputable copies. The main notion in this area of investigation is that of computable categoricity. A computable structure $\mathcal{M}$ is *computably categorical* if for every computable structure $\mathcal{A}$ isomorphic to $\mathcal{M}$, there exists a computable isomorphism from $\mathcal{M}$ onto $\mathcal{A}$. This concept has been part of computable model theory since 1956 when Fröhlich and Shepherdson [107] produced examples of computable fields, extensions of the rationals, of both finite and infinite transcendence degrees, which were not computably categorical. These examples refute the natural conjecture that a computable field is computably categorical exactly when it has finite transcendence degree over its prime subfield (which is either $\mathbb{Q}$ or the p-element $\mathbb{F}_p$, depending on characteristic). Later, Ershov [92] showed that an algebraically closed field is computably categorical if and only if it has finite transcendence degree over its prime subfield. This also follows from work of Nurtazin [280] and can be found in Metakides and Nerode [240]. In [227], Mal'cev considered the notion of a recursively (computably) stable structure. A computable structure $\mathcal{M}$ is *computably stable* if every isomorphism from $\mathcal{M}$ to another computable structure is computable. In the same paper Mal'cev investigated the notion of *autostability* of structures, which is equivalent to that of computably categoricity. Since then computable categoricity has been studied extensively. It has been extended to arbitrary levels of hyperarithmetic hierarchy, and more precisely to Turing degrees $\mathbf{d}$. Computable categoricity of a computable structure $\mathcal{M}$ can also be relativized to all (including noncomputable) structures $\mathcal{A}$ isomorphic to $\mathcal{M}$ (see [14]).

Definition 14. A computable structure $\mathcal{M}$ is $\mathbf{d}$-*computably categorical* if for every computable structure $\mathcal{A}$ isomorphic to $\mathcal{M}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{M}$ onto $\mathcal{A}$.

In the case when $\mathbf{d} = \mathbf{0}^{(n-1)}$, $n \geq 1$, we also say that $\mathcal{M}$ is Δ^0_n-*categorical*. Thus, computably categorical is the same as $\mathbf{0}$-computably categorical or Δ^0_1-categorical. We can similarly define Δ^0_α-categorical structures for any computable ordinal α.

Computably categorical structures tend to be quite rare. For a structure in a typical algebraic class, being computably categorical is usually equivalent to having a finite basis or a finite generating set (for example, in the case of a vector space), or to being highly homogeneous (for example, in the case of a random graph). For instance, Goncharov and Dzgoev [129], and Remmel [290] independently proved that a computable linear order is computably categorical if and only if it has only finitely many successor pairs (also called adjacencies). They also established that a computable Boolean algebra is computably categorical if and only if it has finitely many atoms (see also LaRoche [221]). As usual, by $\mathbb{Z}(p^n)$ we denote the cyclic group of order p^n, and by $\mathbb{Z}(p^\infty)$ the quasicyclic (Prüfer) abelian p-group. The length of an abelian p-group G, $\lambda(G)$, is the least ordinal α such that $p^{\alpha+1}G = p^\alpha G$. The divisible part of G is $Div(G) = p^{\lambda(G)}G$ and is a direct summand of G. The group G is said to be reduced if $Div(G) = \{0\}$. Goncharov [120] and Smith [316] independently characterized computably categorical abelian p-groups as those that can be written in one of the following forms: $(\mathbb{Z}(p^\infty))^l \oplus F$ for $l \in \omega \cup \{\infty\}$ and F is a finite group, or $(\mathbb{Z}(p^\infty))^n \oplus H \oplus (\mathbb{Z}(p^k))^\infty$, where $n, k \in \omega$ and H is a finite group. Goncharov, Lempp, and Solomon [140] proved that a computable, ordered, abelian group is computably categorical if and only if it has finite rank. Similarly, they showed that a computable, ordered, Archimedean group is computably categorical if and only if it has finite rank. Lempp, McCoy, R. Miller, and Solomon [223] characterized computably categorical trees of finite height. R. Miller [251] previously established that no computable tree of infinite height is computably categorical.

An *equivalence structure* is a structure with a single equivalence relation. Calvert, Cenzer, Harizanov, and Morozov [33] established that a computable equivalence structure $\mathcal{A}$ is computably categorical if and only if either $\mathcal{A}$ has finitely many finite equivalence classes, or $\mathcal{A}$ has finitely many infinite classes, upper bound on the size of finite classes, and exactly one finite k with infinitely many classes of size k. An *injection structure* $\mathcal{A} = (A, f)$ consists of a nonempty set A and an $1-1$ function $f : A \to A$. Given $a \in A$, the *orbit* $O_f(a)$ of a under f is $\{b \in A : (\exists n \in \mathbb{N})[f^n(a) = b \vee f^n(b) = a]\}$. An injection structure

(A, f) may have two types of infinite orbits: Z-orbits, which are isomorphic to $(\mathbb{Z}, S)$, and ω-orbits, which are isomorphic to (ω, S). Cenzer, Harizanov, and Remmel [46] characterized computably categorical injection structures as those that have finitely many infinite orbits.

R. Miller and Schoutens [254] solved a long-standing problem by constructing a computable field that has *infinite* transcendence degree over the rationals, yet is computably categorical. Their idea uses a computable set of rational polynomials (more specifically, the Fermat polynomials) to "tag" elements of a transcendence basis. Hence their field has an infinite intrinsically computable transcendence basis (that is, computable in every isomorphic computable copy of the field), with each single element effectively distinguishable from the others.

Very little is known about Δ_n^0-categoricity, for $n \geq 2$, of structures from natural classes of algebraic structures. Obtaining their classification is usually a difficult task. The reason is either the absence of invariants (such as for linear orders, abelian and nilpotent groups), or the lack of a suitable computability-theoretic notion which would capture the property of being Δ_n^0-categorical (see discussion of Δ_2^0-categoricity for equivalence structures below). There is a complete description of higher levels categoricity (in fact, stability) for well-orders due to Ash [11]. Harris [158] has recently announced a description of Δ_n^0-categorical Boolean algebras, for any $n < \omega$. McCoy [234] characterized, under certain restrictions, Δ_2^0-categorical linear orders and Boolean algebras. Barker [23] proved that for every computable ordinal α, there are $\Delta_{2\alpha+2}^0$-categorical but not $\Delta_{2\alpha+1}^0$-categorical abelian p-groups. Lempp, McCoy, R. Miller, and Solomon [223] proved that for every $n \geq 1$, there is a computable tree of finite height, which is Δ_{n+1}^0-categorical but not Δ_n^0-categorical.

The following problems remain open. Describe Δ_2^0-categorical linear orders. Describe Δ_2^0-categorical equivalence relations. Describe Δ_2^0-categorical abelian p-groups. Resolving these problems may require new algebraic invariants or new computability-theoretic notions.

In the next theorem we present several recent results on the upper bounds for categoricity. Recall that a set X is *semi-low* if $\{e : W_e \cap X \neq \emptyset\}$ is Δ_2^0.

Theorem 48. *(i) (follows from [44, 236]) Every computable, free, nonabelian group is Δ_4^0-categorical, and the result cannot be improved to Δ_3^0.*

(ii) ([84]) Every computable, free, abelian group is Δ_2^0-categorical, and the result cannot be improved to computable categoricity.

(iii) ([84]) Every computable abelian group of the form $\bigoplus_{i\in\omega} H_i$, where $H_i \leq (\mathbb{Q}, +)$ for $i \in \omega$, is Δ_3^0-categorical. A computable group of this form is Δ_2^0-categorical if and only if it is isomorphic to a free module over a localization of $\mathbb{Z}$ by a set of primes with a semi-low *complement.*

(iv) ([33]) Every computable equivalence relation is Δ_3^0-categorical, and the result cannot be improved to Δ_2^0.

We may compare these results with those stated in Theorems 90 and 91. More generally, the study of higher categoricity is often equivalent to the study of algebraic properties of a family of relations specific for a given class (such as independence relations, back-and-forth relations, etc.). The result in Theorem 48 (iii) has been recently extended to arbitrary direct sums of rational subgroups [83], for which the sharp upper bound is Δ^0_5.

We can relativize the notion of Δ^0_α-categoricity by studying the complexity of isomorphisms from a computable structure to any countable isomorphic structure.

Definition 15. A computable structure $\mathcal{M}$ is *relatively* Δ^0_α*-categorical* if for every $\mathcal{A}$ isomorphic to $\mathcal{M}$, there is an isomorphism from $\mathcal{M}$ to $\mathcal{A}$, which is Δ^0_α relative to the atomic diagram of $\mathcal{A}$.

Clearly, a relatively Δ^0_α-categorical structure is Δ^0_α-categorical. For linear orders [129, 290], Boolean algebras [129, 290], trees of finite height [223], abelian p-groups [120, 316, 32], equivalence structures [33], and injection structures [46], computable categoricity implies relative computable categoricity. R. Miller and Shlapentokh [253] proved that a computable algebraic field F with a splitting algorithm is computably categorical iff it is decidable which pairs of elements of F belong to the same orbit under automorphisms. They also showed that this criterion is equivalent to relative computable categoricity of F.

A remarkable feature of relative Δ^0_α-categoricity is that it admits a syntactic characterization. This characterization involves the existence of certain effective Scott families. Scott families come from *Scott isomorphism theorem*, which says that for a countable structure $\mathcal{A}$, there is an $L_{\omega_1\omega}$-sentence the countable models of which are exactly the isomorphic copies of $\mathcal{A}$. For proof of Scott isomorphism theorem see [14]. A *Scott family* for a structure $\mathcal{A}$ is a countable family Φ of $L_{\omega_1\omega}$-formulas with finitely many fixed parameters from A such that:

(i) Each finite tuple in $\mathcal{A}$ satisfies some $\psi \in \Phi$;

(ii) If $\overline{a}$, $\overline{b}$ are tuples in $\mathcal{A}$, of the same length, satisfying the same formula in Φ, then there is an automorphism of $\mathcal{A}$, which maps $\overline{a}$ to $\overline{b}$.

If we strengthen condition (ii) to require that the formulas in Φ define each tuple in $\mathcal{A}$, then Φ is called a *defining family* for $\mathcal{A}$. A *formally* Σ^0_α *Scott family* is a Σ^0_α Scott family consisting of computable Σ_α formulas. In particular, it follows that a formally c.e. Scott family is a c.e. Scott family consisting of finitary existential formulas. The following equivalence was established by Goncharov [128] for $\alpha = 1$, and by Ash, Knight, Manasse, and Slaman [19] and independently by Chisholm [50] for any computable ordinal α.

Theorem 49. *([19, 50]) The following are equivalent for a computable structure* $\mathcal{A}$.

1. *The structure* $\mathcal{A}$ *is relatively* Δ^0_α*-categorical.*
2. *The structure* $\mathcal{A}$ *has a formally* Σ^0_α *Scott family* Φ *with finitely many fixed parameters.*
3. *The structure* $\mathcal{A}$ *has a c.e. Scott family consisting of computable* Σ_α *formulas with finitely many fixed parameters.*

Infinitary language is essential for Scott families. Cholak, Shore, and Solomon [55] proved the existence of a computably stable rigid graph that does not have a Scott family of finitary formulas.

In [234], McCoy characterized relatively Δ^0_2-categorical linear orders and Boolean algebras. In [235], McCoy gave a complete description of relatively Δ^0_3-categorical Boolean algebras, and proved that there are $2^{\aleph_0}$ relatively Δ^0_3-categorical linear orders. More recently, Calvert, Cenzer, Harizanov, and Morozov investigated relative Δ^0_2-categoricity for equivalence structures [33] and abelian p-groups [32], and Cenzer, Harizanov, and Remmel [46] investigated relative Δ^0_2-categoricity for injection structures. In the following theorem we state some of these characterizations of relative Δ^0_2-categoricity. As usual, by ω^* we denote the reverse order of ω, and by η the order type of rationals. For a group G, the *period* of G is $max\{order(g) : g \in G\}$ if this quantity is finite, and ∞ otherwise.

Theorem 50. *(i) ([234]) A computable linear order is relatively Δ^0_2-categorical if and only if it is a sum of finitely many intervals, each of type $m, \omega, \omega^*, \mathbb{Z}$, or $n \cdot \eta$, so that each interval of type $n \cdot \eta$ has a supremum and infimum.*

(ii) ([234]) A computable Boolean algebra is relatively Δ^0_2-categorical if and only if it can be expressed as a finite direct sum $c_1 \vee \cdots \vee c_n$, where each c_i is either atomless, an atom, or a 1-atom.

(iii) ([33]) A computable equivalence structure is relatively Δ^0_2-categorical if and only if it either has finitely many infinite equivalence classes, or there is an upper bound on the size of its finite equivalence classes.

(iv) ([46]) A computable injection structure is relatively Δ^0_2-categorical if and only if it has finitely many orbits of type ω, or finitely many orbits of type Z.

(v) ([32]) A computable abelian p-group G is relatively Δ^0_2-categorical if and only if G is reduced and $\lambda(G) \leq \omega$, or G is isomorphic to $\bigoplus_{\alpha} \mathbb{Z}(p^\infty) \oplus H$, where $\alpha \leq \omega$ and H has finite period.

Every Δ^0_2-categorical injection structure is relatively Δ^0_2-categorical (see [46]). Every computable injection structure is relatively Δ^0_3-categorical. Every computable equivalence structure is relatively Δ^0_3-categorical. There is no such bound for a computable abelian p-group G. For example, it follows from the index set results in [36] that if $\lambda(G) = \omega \cdot n$ and $m \leq 2n - 1$, or if $\lambda(G) > \omega \cdot n$ and $m \leq 2n - 2$, then G is not Δ^0_m-categorical.

Goncharov [126] was the first to show that computable categoricity of a computable structure does not imply its relative computable categoricity. The main idea of his proof was to code a special kind of family of sets into a computable structure. Such families were constructed independently by Badaev [20] and Selivanov [311]. The result of Goncharov was lifted to higher levels in the hyperarithmetic hierarchy by Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon for successor ordinals [133], and by Chisholm, Fokina, Goncharov, Harizanov, Knight, and Quinn for limit ordinals [51].

Theorem 51. *([133, 51]) For every computable ordinal α, there is a Δ^0_α-categorical but not relatively Δ^0_α-categorical structure.*

It is not known whether every (computable) Δ^1_1-categorical structure must be relatively Δ^1_1-categorical (see [134]). Kach and Turetsky [183] showed that there exists a Δ^0_2-categorical equivalence structure, which is not relatively Δ^0_2-categorical. Hirschfeldt, Kramer, R. Miller, and Shlapentokh [174] characterized relative computable categoricity for computable algebraic fields and used their characterization to construct a field with the following property.

Theorem 52. *([174]) There is a computably categorical algebraic field, which is not relatively computably categorical.*

The notions of computable categoricity and relative computable categoricity coincide if we add more effectiveness requirements on the structure. Goncharov [128] proved that in the case of 2-decidable structures, computable categoricity and relative computable categoricity coincide. Kudinov showed that the assumption of 2-decidability cannot be weakened, by giving in [216] an example of 1-decidable and computably categorical structure, which is not relatively computably categorical. Ash [10] established that for every computable ordinal α, under certain decidability conditions on $\mathcal{A}$, if $\mathcal{A}$ is Δ^0_α-categorical, then $\mathcal{A}$ is relatively Δ^0_α-categorical.

T. Millar [243] proved that if a structure $\mathcal{A}$ is 1-decidable, then any expansion of $\mathcal{A}$ by finitely many constants remains computably categorical. Cholak, Goncharov, Khoussainov, and Shore showed that the assumption of 1-decidability is important.

Theorem 53. *([52]) There is a computable structure, which is computably categorical, but ceases to be after naming any element of the structure.*

Clearly, the structure in this theorem is not relatively computably categorical. Khoussainov and Shore [205] proved that there is a computably categorical structure $\mathcal{A}$ without a formally c.e. Scott family such that the expansion of $\mathcal{A}$ by any finite number of constants is computably categorical.

Downey, Kach, Lempp, and Turetsky have recently obtained the following result.

Theorem 54. *([79]) Any* 1*-decidable computably categorical structure is relatively Δ^0_2-categorical.*

Based on this theorem, we could conjecture that every computable structure that is computably categorical should be relatively Δ^0_3-categorical. However, this is not the case, as recently announced by Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky.

Theorem 55. *([78]) For every computable ordinal α, there is a computably categorical structure that is not relatively Δ^0_α-categorical.*

Thus, a natural question arises whether there is a computably categorical structure that is not relatively hyperarithmetically categorical. In [78], the uniformity of the constructed structures together with an overspill argument allowed the authors to establish that the problem of computable categoricity is Π^1_1-complete, which was a long-standing open question.

Definition 16. The **d**-*computable dimension* of a computable structure $\mathcal{M}$ is the number of computable isomorphic copies of $\mathcal{M}$, up to **d**-computable isomorphism.

Hence, a computably categorical structure has computable dimension 1. Many natural structures have computable dimension 1 or ω. For example, it was shown in [240] that it is impossible for a computable algebraic field to have finite computable dimension greater than 1. Goncharov was the first to produce examples of computable structures of finite computable dimension greater than 1.

Theorem 56. *([123, 119]) For every finite $n \geq 2$, there is a computable structure of computable dimension n.*

After Goncharov's examples, structures of finite computable dimension $n \geq 2$ were found in several familiar classes, such as 2-step nilpotent groups [141] and other classes [173].

For a computable structure $\mathcal{A}$, some Turing degree, which is not necessarily $\mathbf{0}^{(n)}$, may compute an isomorphism between any two computable copies of the structure. The following notion of the categoricity spectrum, introduced by Fokina, Kalimullin, and R. Miller, aims to capture the set of all Turing degrees capable of computing isomorphisms between arbitrary computable copies of $\mathcal{A}$.

Definition 17. ([102]) Let $\mathcal{A}$ be a computable structure.

(i) The *categoricity spectrum* of $\mathcal{A}$ is

$$\mathrm{CatSpec}(\mathcal{A}) = \{\mathbf{x} : \mathcal{A} \text{ is } \mathbf{x}\text{-computably categorical}\}.$$

(ii) A Turing degree **d** is the *degree of categoricity* of $\mathcal{A}$, if it exists, if **d** is the least degree in $\mathrm{CatSpec}(\mathcal{A})$.

(iii) A Turing degree **d** is *categorically definable* if it is the degree of categoricity of some computable structure.

This terminology intends to parallel the notions of the degree spectrum of a structure $\mathcal{A}$, and the degree of the isomorphism class of $\mathcal{A}$. Since there are only countably many computable structures, most Turing degrees are not categorically definable. Fokina, Kalimullin, and R. Miller investigated which Turing degrees are categorically definable. Their main result in [102] gives a partial answer for the case of arithmetic degrees, and was later extended by Csima, Franklin, and Shore to hyperarithmetic degrees.

Theorem 57. *([62]) (i) For every computable ordinal α, $\mathbf{0}^{(\alpha)}$ is the degree of categoricity of a computable structure.*

(ii) For a computable successor ordinal α, every degree **d** *that is c.e.a. in $\mathbf{0}^{(\alpha)}$ is a degree of categoricity.*

Negative results were also obtained in [102, 62]. Namely, if **d** is a non-hyperarithmetic degree, then **d** cannot be the degree of categoricity of a computable structure. Furthermore, Anderson and Csima showed that not all hyperarithmetic degrees are degrees of categoricity.

Theorem 58. *([2]) (i) There exists a Σ^0_2 degree that is not categorically definable.*

(ii) Every degree of a set that is 2*-generic relative to some perfect tree is not a degree of categoricity.*

(iii) Every noncomputable hyperimmune-free degree is not a degree of categoricity.

Thus, it is natural to ask whether all Δ^0_2 degrees are categorically definable.

Not every computable structure has a degree of categoricity. The first negative example was built by R. Miller.

Theorem 59. *([250]) There exists a computable field with a splitting algorithm, which is not computably categorical, and such that its categoricity spectrum must contain degrees $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ with $\boldsymbol{d}_0 \wedge \boldsymbol{d}_1 = \mathbf{0}$.*

Subsequently, R. Miller built another computable field the categoricity spectrum of which has no least degree and does not contain $\mathbf{0}'$. R. Miller used the algebraicity of the field to present the isomorphisms between it and a computable isomorphic copy as infinite paths through a finite-branching computable tree. If the field has a splitting algorithm, then the branching of this tree is computable, and we can apply the *low basis theorem*. If the field does not have a splitting algorithm, then we relativize to the degree of the branching and apply the relativized *low basis theorem*.

Further interesting examples of structures without the degree of categoricity were built by Fokina, Frolov, and Kalimullin [101]. The main property of their structures is that they are *rigid*, that is, have no nontrivial automorphisms, which was not the case for the examples in [250]. If a rigid structure $\mathcal{M}$ is $\mathbf{d}$-categorical, then it is also $\mathbf{d}$-*stable*, i.e., every isomorphism from $\mathcal{M}$ onto a computable copy is $\mathbf{d}$-computable. (The converse is not true, for example, a computable copy of a two-dimensional vector space over $\mathbb{Q}$ is computably stable but not rigid.) Constructions from [101] give for every nonzero c.e. degree $\mathbf{d}$, a rigid $\mathbf{d}$-computably categorical structure with no degree of categoricity. The authors construct similar rigid structures for all degrees $\mathbf{d}$ that are c.e.a. in $\mathbf{0}^{(n)}$, for any $n \in \omega$. When we pass to d.c.e. structures, we lose the property of rigidity. It is natural to ask whether there is a computable structure the categoricity spectrum of which is the set of all noncomputable Turing degrees. It is also interesting to find out whether the union of two cones of Turing degrees can be a categoricity spectrum.

In recent papers [116, 117, 118], Goncharov investigated categoricity restricted to decidable structures.

Definition 18. A decidable structure $\mathcal{A}$ is called *decidably categorical* if every two decidable copies of $\mathcal{A}$ are computably isomorphic.

Nurtazin gave the following characterization of decidably categorical structures. Recall that for a complete theory T, a formula $\theta(\overline{x})$ is called *complete* if for every formula $\psi(\overline{x})$, either $T \vdash \theta(\overline{x}) \Rightarrow \psi(\overline{x})$ or $T \vdash \theta(\overline{x}) \Rightarrow \neg\psi(\overline{x})$.

Theorem 60. *([280]) Let $\mathcal{A}$ be a decidable structure. Then $\mathcal{A}$ is decidably categorical if and only if there is a finite tuple $\bar{c}$ of elements in A such that $(\mathcal{A}, \bar{c})$ is a prime model of the theory $Th(\mathcal{A}, \bar{c})$ and the set of complete formulas of this theory is computable.*

Moreover, Nurtazin proved that if there is no such $\bar{c}$, then there are infinitely many decidable copies of $\mathcal{A}$, no two of which are computably isomorphic.

Similarly to the case of computable categoricity, we define *decidable categoricity spectrum* of $\mathcal{A}$ to be the collection of degrees that can compute at least one isomorphism between *decidable* copies of $\mathcal{M}$. In [116], Goncharov studied decidable categoricity of almost prime models. It is not difficult to see that the collection of atomic formulas in a decidable almost prime model $\mathcal{M}$ is co-c.e. Therefore, a c.e. degree is always contained in the decidable categoricity spectrum of $\mathcal{M}$. Goncharov established the following result.

Theorem 61. *([116]) Every c.e. degree* $\mathbf{d}$ *is the degree of decidable categoricity of some decidable almost prime model.*

Goncharov also investigated decidable categoricity of Ehrenfeucht models.

Theorem 62. *([117]) There exists a decidable Ehrenfeucht theory T such that T has a decidable prime model that is decidably categorical, and T has a decidable almost prime model that is not decidably categorical.*

Effective categoricity of computable structures has also been recently investigated within Ershov's difference hierarchy: for graphs by Khoussainov, Stephan, and Yang [207], and for the equivalence structures by Cenzer, LaForte, and Remmel [47].

6 Automorphisms of effective structures

In algebra, automorphism groups of structures often reflect the algebraic properties of structures (for example, as in Galois theory). In computable model theory, the study of *effective* automorphisms help us better understand computability-theoretic properties of countable structures. The set of all automorphisms of a computable structure forms a group under composition, and we may ask questions about the isomorphism types of this group and its natural subgroups. Thus, the theory of automorphisms of effective structures provides another link between computable algebra and classical group theory. We may also study the Turing degrees of members of the automorphism group. This line of investigation is related to the study of effective categoricity of structures. Finally, we may restrict ourselves to computable structures from familiar classes (such as Boolean algebras, linear orders, etc.) and study groups of effective automorphisms for these structures. As usual, we assume that all infinite computable structures have ω as their domains. The next definition captures one of the main notions of this investigation.

Definition 19. For an infinite computable structure $\mathcal{M}$ (with domain ω) and a Turing degree $\mathbf{d}$, we define $Aut_{\mathbf{d}}(\mathcal{M})$ to be the set of all permutations of ω, which are computable in $\mathbf{d}$ and induce automorphisms of $\mathcal{M}$.

We write $Aut_c(\mathcal{M})$ for $Aut_{\mathbf{0}}(\mathcal{M})$ (the subscript c stands for *computable*). For every Turing degree $\mathbf{d}$, the set $Aut_{\mathbf{d}}(M)$ forms a group under composition. In contrast, the set $Aut_p(\omega)$ of all primitive recursive permutations of ω is not a group under composition, as shown by Kuznetsov [218]. One of the central objectives here is to study classical and effective properties of the group $Aut_{\mathbf{d}}(\mathcal{M})$ for various $\mathcal{M}$ and $\mathbf{d}$. We can start with a structure in the empty language, that is, ω with equality, and consider its automorphism group $Aut_{\mathbf{d}}(\omega)$ as a structure. Recall that the degree of the isomorphism type of a structure, if it exists, is the least Turing degree in its Turing degree spectrum. Morozov established the following result.

Theorem 63. *([268]) For every Turing degree $\mathbf{d}$, the degree of the isomorphism type of the group $Aut_{\mathbf{d}}(\omega)$ is $\mathbf{d}''$.*

Morozov showed that the embedding $\mathcal{F} : \mathbf{d} \to Aut_{\mathbf{d}}(\omega)$ can be used to substitute Turing reducibility with the group-theoretic embedding.

Theorem 64. *[262] For every pair $\mathbf{c}$,$\mathbf{d}$ of Turing degrees, we have*

$$(Aut_{\mathbf{d}}(\omega) \leqq Aut_{\mathbf{c}}(\omega)) \Leftrightarrow (\mathbf{d} \leq \mathbf{c}),$$

where $\leqq$ stands for the usual group-theoretic embedding.

It follows from this theorem that $\mathbf{c} = \mathbf{d}$ if and only if $Aut_{\mathbf{d}}(\omega) \cong Aut_{\mathbf{c}}(\omega)$. In contrast, there exists a Turing degree $\mathbf{a}$ such that $Aut_{\mathbf{a}}(\omega)$ and $Aut_{\mathbf{b}}(\omega)$ are elementary equivalent for all $\mathbf{b} \geq \mathbf{a}$ (see [265]). Intuitively, the last statement says that this first-order theory cannot recognize the difference between very "large" Turing degrees. Kent investigated group-theoretic properties of $Aut_{\mathbf{d}}(\omega)$.

Theorem 65. *([192]) For every Turing degree $\mathbf{d}$, the unique normal series for $Aut_{\mathbf{d}}(\omega)$ has the form*

$$\{1\} \lhd E \lhd F \lhd Aut_{\mathbf{d}}(\omega),$$

where F is the subgroup of permutations that change only finitely many numbers, E is the subgroup of even permutations of F, and 1 is the identity permutation.

Notice that a finitely generated subgroup of $Aut_c(\omega)$ has to be a Π^0_1 group. Higman asked if every Π^0_1 finitely generated group can be isomorphically embedded into $Aut_c(\omega)$. The following result of Morozov answers Higman's question negatively.

Theorem 66. *([261]) There exists a 2-generated Π^0_1 group G such that $G \not\leqq Aut_c(\omega)$.*

Morozov syntactically characterized subgroups of $Aut_c(\omega)$, which are isomorphic to the whole $Aut_c(\omega)$.

Theorem 67. *([266]) There exists a first-order sentence in the language of groups such that for every* $G \leqq Aut_c(\omega)$,

$$(G \models \phi) \Leftrightarrow (G \cong Aut_c(\omega)).$$

More specifically, Morozov [266] proved that the class of all groups of the form $Aut_c(\mathcal{M})$, where $\mathcal{M}$ is a computable structure, is definable in the monadic second-order language within $Aut_c(\omega)$. He also showed that the theories of the following three classes of groups are all distinct and differ from the theory of all groups: (i) groups that can be embedded into $Aut_c(\omega)$, (ii) groups that are $Aut_c(\mathcal{M})$ for computable $\mathcal{M}$, and (iii) computable groups. The first class cannot be axiomatized by a hyperarithmetic set of axioms, the other two cannot be axiomatized by any arithmetic set of axioms. Furthermore, Morozov [266] proved that there exists a single sentence, consistent with the theory of groups, which is not true in any group $Aut_c(\mathcal{M})$ where $\mathcal{M}$ is a computable structure.

Now, for various computable structures $\mathcal{M}$, we compare $Aut_{\mathbf{d}}(\mathcal{M})$ and $Aut(\mathcal{M})$. For $\mathbf{d} = \mathbf{0}$, Dzgoev [88], and independently Manaster and Remmel [231] established the following result.

Theorem 68. *([88, 231]) There exists a computable structure* $\mathcal{M}$ *such that* $Aut(\mathcal{M})$ *has* 2^ω *elements, while* $Aut_c(\mathcal{M})$ *has only one element.*

The previous theorem can be strengthened in several ways. Kudaibergenov [214] showed that we can make such $\mathcal{M}$ decidable and homogeneous. Morozov [264] proved that there exists a computable structure $\mathcal{M}$ with $card(Aut(\mathcal{M})) = 2^\omega$ such that every hyperarithmetic structure isomorphic to $\mathcal{M}$ has no nontrivial hyperarithmetic automorphisms. For a criterion for the existence of two isomorphic but not hyperarithmetically isomorphic tuples in a hyperarithmetic structure, and examples of well-known structures with this phenomenon see [135].

For a computable structure $\mathcal{M}$, the group $Aut_c(\mathcal{M})$ does not have to be isomorphic to a computable one. Morozov [269] gave the following characterization of $Aut_c(\mathcal{M})$ having a computable copy.

Theorem 69. *[269] For a computable structure* $\mathcal{M}$*, the group* $Aut_c(\mathcal{M})$ *is isomorphic to a computable one if and only if there exists a finite tuple* $\overline{p}$ *such that* $Aut(\mathcal{M}, \overline{p}) = \{1\}$*, and the set* $\{(\overline{m}, \overline{n}) : \overline{m} \cong_c \overline{n}\}$ *is c.e., where*

$$\overline{m} \cong_c \overline{n} \Leftrightarrow (\exists f \in Aut_c(\mathcal{M}))[f : \overline{m} \to \overline{n}].$$

This theorem has some interesting corollaries.

Corollary 2. *([269]) A finitely generated group* G *is isomorphic to* $Aut_c(\mathcal{M})$ *for some computable structure* $\mathcal{M}$ *if and only if* G *has a decidable word problem.*

For groups that are not finitely generated the situation is rather complex. Even if a group is abelian, not much can be said. It is not difficult to show that $\bigoplus_{p \in S} \mathbb{Z}_p$, where S is a set of primes, is isomorphic to $Aut_c(\mathcal{M})$ for some computable structure $\mathcal{M}$ if and only if S is Σ^0_3 (see Morozov and Buzykaeva [272]). The general case of arbitrary abelian groups is unresolved. Theorem 69 also implies that for every infinite computable Boolean algebra $\mathcal{B}$, the group $Aut_c(\mathcal{B})$ is not computable, and the same is true for every decidable infinite model of an $\aleph_0$-categorical theory with a computable set of atomic formulas.

We can show that the group $Aut_c(\mathcal{M})$ for a computable structure $\mathcal{M}$ is $\mathbf{0}''$-computable (folklore). This upper bound is sharp, as shown in the following theorem due to Morozov.

Theorem 70. *([263]) For every Turing degree $\mathbf{d} \leq \mathbf{0}''$, there exists a computable structure $\mathcal{M}$ such that $deg\,(D(Aut_c(\mathcal{M}))) = \mathbf{d}$.*

We may ask whether for various computable $\mathcal{M}$, the group $Aut_c(\mathcal{M})$ has a degree of its isomorphism type. As we have seen earlier, this was the case when $\mathcal{M}$ is ω with equality. Nonetheless, Morozov [263] constructed a computable structure $\mathcal{M}$ such that $Aut_c(\mathcal{M})$ has no degree of its isomorphism type. We may also ask which Turing degrees contain only groups isomorphic to $Aut_c(\mathcal{M})$ for some computable $\mathcal{M}$. Morozov [269, 263] proved that this collection of degrees is the singleton $\{\mathbf{0}\}$.

Recently Harizanov, Morozov, and R. Miller [156] introduced another notion in the study of $Aut(\mathcal{M})$.

Definition 20. ([156]) The *automorphism (Turing) degree spectrum* of a computable structure $\mathcal{M}$, in symbols $AutSp(\mathcal{M})$, is the set

$$\{deg(f) : f \in Aut(\mathcal{M}) - \{1_{\mathcal{M}}\}\},$$

where $1_{\mathcal{M}}$ is the identity automorphism of $\mathcal{M}$.

Harizanov, Morozov, and R. Miller [156] showed that various collections of Turing degrees, including many upper cones, can be realized as automorphism degree spectra. Let $\mathcal{M}$ be a computable structure. If $AutSp(\mathcal{M})$ is the upper cone of degrees $\geq \boldsymbol{d}$, then $\boldsymbol{d}$ is hyperarithmetic. Harizanov, Morozov, and R. Miller [156] showed that for any computable ordinal α, and any Turing degree $\mathbf{d}$ with $\mathbf{0}^{(\alpha)} \leq \boldsymbol{d} \leq \mathbf{0}^{(\alpha+1)}$, the upper cone of degrees $\geq \mathbf{d}$ forms an automorphism spectrum. They also showed that there exists a computable structure $\mathcal{A}$ the automorphism spectrum of which is the union of the upper cones above each degree of an infinite antichain of Σ^0_n degrees for $n \geq 1$. The spectrum $AutSp(\mathcal{M})$ is at most countable if and only if it contains only hyperarithmetic degrees. Since for every $f, g \in Aut(\mathcal{M})$ the composition fg is also an automorphism, the automorphism degree spectrum cannot contain exactly two incomparable degrees, as Harizanov, Morozov, and R. Miller showed.

Theorem 71. *([156])*

1. *Let $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ be incomparable Turing degrees. Then no computable structure $\mathcal{M}$ has $AutSp(\mathcal{M}) = \{\boldsymbol{d}_0, \boldsymbol{d}_1\}$ or $AutSp(\mathcal{M}) = \{\mathbf{0}, \boldsymbol{d}_0, \boldsymbol{d}_1\}$.*

2. *There exist pairwise incomparable Δ^0_2 Turing degrees $\boldsymbol{d}_0$, $\boldsymbol{d}_1$, $\boldsymbol{d}_2$, and computable structures $\mathcal{A}$ and $\mathcal{B}$ such that $AutSp(\mathcal{A}) = \{\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2\}$ and $AutSp(\mathcal{B}) = \{\mathbf{0}, \boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2\}$.*

It was shown in [156] that there exists a computable structure $\mathcal{A}$ such that for every c.e. degree $\mathbf{d}$, some computable copy of $\mathcal{A}$ has the automorphism degree spectrum $\{\mathbf{d}\}$. If $\mathbf{0}^{(\alpha)} \leq \boldsymbol{d} \leq \mathbf{0}^{(\alpha+1)}$ for some computable ordinal α, then there exists a computable structure with automorphism degree spectrum $\{\boldsymbol{d}\}$. A total function $f : \omega \to \omega$ is said to be a Π^0_1-*function singleton* if there exists a computable tree $\mathcal{T} \subseteq \omega^{<\omega}$ through which f is a unique infinite path. It was proved in [156] that a Turing degree $\mathbf{d}$ contains a Π^0_1-function singleton if and only if $\{\mathbf{d}\}$ is the automorphism spectrum of some computable structure.

For a computable structure $\mathcal{M}$ from some well-known algebraic class of structures, the typical question we might ask is: Given $Aut(\mathcal{M})$, what can we say about the isomorphism type of $\mathcal{M}$? Obtaining a satisfactory answer to this question is usually a difficult task. The effective analogue of the question – when $\mathcal{M}$ is computable and $Aut(\mathcal{M})$ is replaced by $Aut_c(\mathcal{M})$ – is not any easier. In the case of computable Boolean algebras, Morozov [270] obtained a positive partial result. By $\mathcal{B} \cong_c \mathcal{A}$ we denote that $\mathcal{B}$ and $\mathcal{A}$ are computably isomorphic.

Theorem 72. *([270]) Let $\mathcal{A}$ be an atomic decidable Boolean algebra. For every computable Boolean algebra $\mathcal{B}$, we have*

$$(Aut_c(\mathcal{B}) \cong Aut_c(\mathcal{A})) \Rightarrow (\mathcal{B} \cong_c \mathcal{A}).$$

In contrast, Remmel [289] showed that for every computable Boolean algebra $\mathcal{B}$, there exists $\mathcal{C} \cong \mathcal{B}$ such that every $f \in Aut_c(\mathcal{C})$ moves only finitely many atoms of $\mathcal{C}$. It is also proven in [270] that there exist two decidable Boolean algebras, $\mathcal{B}_0$ and $\mathcal{B}_1$, such that $\mathcal{B}_0 \ncong \mathcal{B}_1$ and $Aut_c(\mathcal{B}_0) \cong Aut_c(\mathcal{B}_1)$. Morozov [270] also showed that there exists a computable Boolean algebra $\mathcal{B}$, and a Boolean algebra $\mathcal{C}$ having no computable copy, such that $Aut(\mathcal{B}) \cong Aut(\mathcal{C})$.

In [56], Chubb, Harizanov, Morozov, Pingrey, and Ufferman investigated the relationship between algebraic structures and their inverse semigroups of partial automorphisms. An *inverse semigroup* is a semigroup where for each element f there is a unique g so that $gfg = g$ and $fgf = f$. For a structure $\mathcal{M}$, the authors considered the semigroup $I_{fin}(\mathcal{M})$ of all finite automorphisms, and, in the case of a computable structure $\mathcal{M}$, the semigroup of all partial computable automorphisms, $I_{pc}(\mathcal{M})$. As usual, $\equiv$ stands for elementary equivalence of structures. In [56], it was shown that structures from certain classes can be recovered, up to isomorphism or elementary equivalence, from these semigroups. For example, for all nontrivial countable equivalence structures $\mathcal{A}_0$ and $\mathcal{A}_1$, we have:

$$\begin{aligned}
&\text{(i) } (I_{fin}(\mathcal{A}_0) \cong I_{fin}(\mathcal{A}_1)) \Leftrightarrow (\mathcal{A}_0 \cong \mathcal{A}_1);\\
&\text{(ii) } (I_{fin}(\mathcal{A}_0) \equiv I_{fin}(\mathcal{A}_1)) \Leftrightarrow (\mathcal{A}_0 \equiv \mathcal{A}_1).
\end{aligned}$$

We call an equivalence structure (A, E) *nontrivial* if E differs from the diagonal relation $\{(a, a) : a \in A\}$ and from the set $A \times A$. It was shown in [56] that for a nontrivial computable equivalence structure $\mathcal{E}_0$, there is a first-order sentence σ in the language of inverse semigroups such that for any nontrivial computable equivalence structure $\mathcal{E}_1$, we have

$$(I_{pc}(\mathcal{E}_1) \models \sigma) \Rightarrow (\mathcal{E}_1 \cong_c \mathcal{E}_0).$$

The authors of [56] also considered partial orders, relatively complemented distributive lattices, and Boolean algebras. It would be interesting to investigate for other natural algebraic structures how structures themselves can be recovered, up to isomorphism or elementary equivalence, from various inverse semigroups of their partial automorphisms.

There are also interesting results about computable automorphisms of computable linear orders. Schwartz obtained the following characterization of computable linear orders containing dense intervals.

Theorem 73. *([302]) A computable linear order $\mathcal{A}$ contains a dense interval if and only if $card(Aut_c(\mathcal{L})) > 1$ for every computable $\mathcal{L}$ such that $\mathcal{L} \cong \mathcal{A}$.*

In order to state the next result by Morozov and Truss [273], we will first introduce some notation. For a computable structure $\mathcal{M}$ and a *Turing ideal* I, let $Aut_I(\mathcal{M})$ be the collection of all automorphisms of $\mathcal{M}$ computable from members of I. Let $\mathcal{Q} = (\mathbb{Q}, \leq)$.

Theorem 74. *([273]) For Turing ideals I and J we have:*

$$\begin{aligned}(Aut_I(\mathcal{Q}) &\leqq Aut_J(\mathcal{Q})) \Leftrightarrow (I \subseteq J),\\ (Aut_I(\mathcal{Q})) &\cong Aut_J(\mathcal{Q}) \Leftrightarrow (I = J).\end{aligned}$$

The proof uses techniques from the theory of ordered abelian groups (see [115]). It is interesting to compare Theorem 74 with Theorem 64. The next result of Morozov and Truss can be compared with Theorem 67.

Theorem 75. *([274]) There is a first-order sentence τ such that, up to isomorphism, the group $Aut_c(\mathcal{Q})$ is the only model of τ among all subgroups of $Aut_c(\omega)$.*

Lempp, McCoy, Morozov, and Solomon studied the algebraic properties of $Aut_c(\mathcal{Q})$ and compared them with those of $Aut(\mathcal{Q})$. They obtained the following result distinguishing $Aut_c(\mathcal{Q})$ from $Aut(\mathcal{Q})$.

Theorem 76. *([224]) The following three properties, known to be true for $Aut(\mathcal{Q})$, fail for $Aut_c(\mathcal{Q})$:*

(a) the group is divisible;

(b) every element is a commutator of itself with some other element;

(c) two elements are conjugate if and only if they have isomorphic orbital structures.

Not much is known about effective automorphisms of computable modules, including vector spaces and abelian groups. Many algebraic difficulties arise in the study of their automorphism groups. The following result about modules, due to Morozov, is similar to Theorem 73.

Theorem 77. *([267]) For every computable division ring $\mathcal{R}$, there exists a computable copy of the module $\mathcal{M} = \bigoplus_{i\in\omega} \mathcal{R}$ such that $Aut_c(\mathcal{M})$ contains only multiplications by scalars from $\mathcal{R}$.*

Further related results can be found in [217].

7 Degree spectra of relations

One of the important questions in computable model theory is how a specific property of a computable structure may change if the structure is isomorphically transformed so that it remains computable. A computable property of a computable structure $\mathcal{A}$, which Ash and Nerode [18] considered, is given by an additional computable relation R on the domain of $\mathcal{A}$. (That is, R is not named in the language of $\mathcal{A}$.) Ash and Nerode investigated syntactic conditions on $\mathcal{A}$ and R under which for every isomorphism f from $\mathcal{A}$ onto a computable structure $\mathcal{B}$, $f(R)$ is c.e. Such relations are called *intrinsically c.e.* on $\mathcal{A}$. In general, we have the following definition. Let $\mathcal{P}$ be a certain complexity class.

Definition 21. ([18]) An additional relation R on the domain of a computable structure $\mathcal{A}$ is called *intrinsically* $\mathcal{P}$ on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to a computable structure belongs to $\mathcal{P}$.

For example, the successor relation, and being an even number are not intrinsically computable relations on $(\omega, <)$. Clearly, if $\mathcal{A}$ is a computably stable structure, then every computable relation on its domain is intrinsically computable.

If R is definable in $\mathcal{A}$ by a computable Σ_1 formula with finitely many parameters, then R is intrinsically c.e. Ash and Nerode [18] proved that, under a certain extra decidability condition on $\mathcal{A}$ and R, the relation R is intrinsically c.e. on $\mathcal{A}$ iff R is definable by a computable Σ_1 formula with finitely many parameters. The Ash-Nerode condition for an m-ary relation R says that there is an algorithm, which determines for every existential formula $\psi(x_0, \dots, x_{m-1}, \overline{y})$ and every $\overline{c} \in A^{lh(\overline{y})}$, whether the following implication holds for every $\overline{a} \in A^m$:

$$(\mathcal{A} \vDash \psi(\overline{a}, \overline{c})) \Rightarrow R(\overline{a}).$$

Barker [24] extended this result by showing that for every computable ordinal α, under certain additional decidability conditions on $\mathcal{A}$, the relation R is intrinsically Σ^0_α on $\mathcal{A}$ iff R is definable by a computable Σ_α formula with finitely many parameters. For the relative notions, the effectiveness conditions are not needed. Let $\mathcal{P}$ be a certain complexity class, which can be relativized, such as the class of all Σ^0_α sets.

Definition 22. An additional relation R on the domain of a computable structure $\mathcal{A}$ is called *relatively intrinsically* $\mathcal{P}$ on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to any structure $\mathcal{B}$ is $\mathcal{P}$ relative to the atomic diagram of $\mathcal{B}$.

The following equivalence is due to Ash, Knight, Manasse, and Slaman [19], and independently Chisholm [50].

Theorem 78. *([19, 50]) Let $\mathcal{A}$ be a computable structure. A relation R on $\mathcal{A}$ is relatively intrinsically Σ^0_α iff R is definable by a computable Σ_α formula with finitely many parameters.*

Goncharov [126] and Manasse [230] gave examples of intrinsically c.e. relations on computable structures, which are not relatively intrinsically c.e. This result was lifted to higher levels in the hyperarithmetic hierarchy by Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon for successor ordinals [133], and by Chisholm, Fokina, Goncharov, Harizanov, Knight, and Quinn for limit ordinals [51].

Theorem 79. *([133, 51]) For every computable ordinal α, there is a computable structure $\mathcal{A}$ with an intrinsically Σ^0_α relation R such that R is not definable by a computable Σ_α formula with finitely many parameters.*

In addition to considering the complexity of relations on computable structures within hyperarithmetic hierarchy, we can also consider their degrees, such as Turing degrees or strong degrees. Harizanov introduced the following notion.

Definition 23. ([151]) The *Turing degree spectrum* of R on $\mathcal{A}$, in symbols $DgSp_{\mathcal{A}}(R)$, is the set of all Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ onto computable structures.

If for some isomorphism f from $\mathcal{A}$ to a computable structure, we have $X = f(R)$ and $\mathbf{x} = deg(X)$, then we say that $\mathbf{x}$ is realized in $DgSp_{\mathcal{A}}(R)$ *via* X, or *via* f. Uncountable degree spectra of relations were studied by Harizanov [150, 147], and Ash, Cholak, and Knight [12]. In particular, they showed independently that if every Turing degree $\leq \mathbf{0}''$ can be realized in $DgSp_{\mathcal{A}}(R)$ *via* an isomorphism of the same Turing degree as its image of R, then $DgSp_{\mathcal{A}}(R)$ contains every Turing degree. In [53], the authors investigated the spectra of relations on computable structures under strong reducibilities such as *weak truth-table* (wtt) reducibility and *truth-table* (tt) reducibility.

In [149], Harizanov studied when every c.e. degree can be obtained in $DgSp_{\mathcal{A}}(R)$ *via* an isomorphism of the same degree as its image of R. Ash, Cholak, and Knight [12] lifted her result to arbitrary α-c.e. degrees, where α is a computable ordinal, in Ershov's difference hierarchy. For example, the degree spectrum of the successor relation on $(\omega, \leq)$ contains all c.e. degrees, and the same holds for the set of all even numbers. The degree spectrum of the set of algebraic elements in an algebraically closed field of infinite transcendence degree contains all c.e. Turing degrees.

One of the general results by Harizanov about $DgSp_{\mathcal{A}}(R)$ containing all c.e. degrees is the following theorem, which requires extra effectiveness condition – it is enough that the existential diagram of $(\mathcal{A}, R)$ is computable.

Theorem 80. *([149]) Let $\mathcal{A}$ be a computable structure, and let R be a relation that is intrinsically c.e. on $\mathcal{A}$, while $\neg R$ is not. Then, under a certain extra decidability condition, for any c.e. degree $\mathbf{d}$, we have $\mathbf{d} \in DgSp_{\mathcal{A}}(R)$.*

Ash and Knight [15] generalized the previous theorem. Their generalization involves degrees that are coarser than Turing degrees. In the following definition we will use the symbol Δ^0_α to denote a complete Δ^0_α set.

Definition 24. ([15]) (i) $A \leq_{\Delta^0_\alpha} B$ iff $A \leq_T B \oplus \Delta^0_\alpha$
(ii) $A \equiv_{\Delta^0_\alpha} B$ iff ($A \leq_{\Delta^0_\alpha} B$ and $B \leq_{\Delta^0_\alpha} A$)
(iii) The equivalence classes under $\equiv_{\Delta^0_\alpha}$ are called *α-degrees*.

Note that $\leq_{\Delta^0_1}$ is the same as $\leq_T$.

Theorem 81. *([15]) Let $\mathcal{A}$ be a computable structure, and let R be a relation that is not intrinsically Δ^0_α on $\mathcal{A}$. Then, under certain extra effectiveness conditions, for any Σ^0_α set C, there is an isomorphism f from $\mathcal{A}$ onto a computable copy with $f(R) \equiv_{\Delta^0_\alpha} C$.*

Ash and Knight also showed that it is not possible to substitute Turing degrees for α-degrees. In [16], they produced examples of structures $\mathcal{A}$ and relations R, satisfying a great deal of effectiveness, in which certain Σ^0_α Turing degrees, in particular, minimal degrees, are impossible for the image of R. Hirschfeldt and White [175] constructed a family of relations on computable structures, the degrees of which coincide with the levels of the hyperarithmetic hierarchy. Their examples are built up from back-and-forth trees, which explicitly code the alternations of quantifiers.

Using Goncharov's result from the theory of numberings [122], we can show that there is a computable non-intrinsically c.e. relation R on a computable structure $\mathcal{A}$ such that $DgSp_{\mathcal{A}}(R) = \{\mathbf{0}, \mathbf{d}\}$, where $\mathbf{d} \leq \mathbf{0}''$ but $\mathbf{d} \not\leq \mathbf{0}'$ (see [149]). Harizanov [148] showed that there is a two-element degree spectrum $DgSp_{\mathcal{A}}(R) = \{\mathbf{0}, \mathbf{d}\}$, such that $\mathbf{0} < \mathbf{d} \leq \mathbf{0}'$ where $\mathbf{d}$ cannot be realized *via* a c.e. set. Goncharov and Khoussainov [138], and Khoussainov and Shore [205] proved that there is a two-element degree spectrum $DgSp_{\mathcal{A}}(R) = \{\mathbf{0}, \mathbf{c}\}$ such that $\mathbf{c}$ is a nonzero degree realized *via* a c.e. set. Khoussainov and Shore broadly generalized this result.

Theorem 82. *([205]) Let $(P, \preceq)$ be a computable partially ordered set. There are a computable structure $\mathcal{A}$ and a computable unary relation R on its domain such that $(DgSp_{\mathcal{A}}(R), \leq) \cong (P, \preceq)$ and every degree in $DgSp_{\mathcal{A}}(R)$ is realized* via *a c.e. set.*

For some familiar relations on computable structures, their Turing degree spectra exhibit the dichotomy: either singletons or infinite. Harizanov [149] established that if for a non-intrinsically c.e. relation R on $\mathcal{A}$, the Ash-Nerode decidability condition holds, then $DgSp_{\mathcal{A}}(R)$ must be infinite. Hirschfeldt [169] gave a sufficient condition for a relation to have infinite degree spectrum. Applying this condition to linear orders and using the proof of a result of Moses [275], Hirschfeldt established that a computable relation on a computable linear order is either intrinsically computable or has an infinite Turing degree spectrum. Downey, Goncharov, and Hirschfeldt proved the same dichotomy for relations on Boolean algebras.

Theorem 83. *([75]) A computable relation on a computable Boolean algebra is either intrinsically computable or has infinite Turing degree spectrum.*

A similar question can be asked for computable relations on other classes of structures such as computable abelian groups. Another interesting question from [75] is whether the degree spectrum of an intrinsically Δ_2^0 relation on a computable linear order is always a singleton or infinite.

Degree spectra have also been investigated for specific important relations on natural classes of structures. One such relation is the successor relation S on a computable linear order $\mathcal{L}$. There are two known examples of singleton degree spectra of the successor relation. If $\mathcal{L}$ has only finitely many successor pairs, then the order is computably categorical, hence the successor relation is intrinsically computable. Downey and Moses [85] constructed a linear order $\mathcal{L}$ having an intrinsically complete successor relation, that is, $DgSp_{\mathcal{L}}(S) = \{\mathbf{0}'\}$. It was a long-standing open question to investigate upward closure in c.e. degrees of the degree spectrum of the successor relation in computable linear orders. Harizanov, Chubb, and Frolov [57] showed that if $\mathcal{A}$ is a computable linear order with domain A where for all $x \in A$ there is a successor pair (a, b) in $\mathcal{A}$ with $x < a$, then the degree spectrum of the successor relation of $\mathcal{A}$ is closed upward in the c.e. Turing degrees. As a consequence, they established that for every c.e. Turing degree $\mathbf{b}$, the upper cone of c.e. Turing degrees determined by $\mathbf{b}$ is the degree spectrum of the successor relation of some computable linear order. Downey, Lempp, and Wu [82] established the positive result in full generality by developing a new method of constructing Δ_3^0 isomorphisms. Their proof uses a result from [57].

Theorem 84. *([82]) If a computable linear order has infinitely many successor pairs, then the degree spectrum of the successor relation is closed upward in the c.e. Turing degrees.*

In [323], Soskov established that a Δ_1^1 relation on computable $\mathcal{A}$, which is invariant under automorphisms of $\mathcal{A}$, is definable in $\mathcal{A}$ by a computable infinitary formula with no parameters. This led to the following characterization of intrinsically Δ_1^1 relations.

Theorem 85. *([323]) For a computable structure $\mathcal{A}$, and a relation R on $\mathcal{A}$, the following are equivalent:*

(i) R is intrinsically Δ_1^1 on $\mathcal{A}$;

(ii) R is relatively intrinsically Δ_1^1 on $\mathcal{A}$;

(iii) R is definable in $\mathcal{A}$ by a computable infinitary formula with finitely many parameters.

In the following theorem characterizing intrinsically Π^1_1 relations, Soskov [322] established the equivalence $(ii) \Leftrightarrow (iii)$, while $(i) \Leftrightarrow (ii)$ was established in [134].

Theorem 86. *([322, 134]) For a computable structure $\mathcal{A}$ and relation R on $\mathcal{A}$, the following are equivalent:*

(i) R is intrinsically Π^1_1 on $\mathcal{A}$;

(ii) R is relatively intrinsically Π^1_1 on $\mathcal{A}$;

(iii) R is definable in $\mathcal{A}$ by a Π^1_1 disjunction of computable infinitary formulas with finitely many parameters.

Goncharov, Harizanov, Knight, and Shore [134] considered a general family of examples of intrinsically Π^1_1 relations arising in computable structures of Scott rank $\omega_1^{CK}+1$. A *Harrison order* is a computable linear order of type $\omega_1^{CK}(1+\eta)$. Harrison [162] showed that such an order exists. The initial segment of this order of type ω_1^{CK} is intrinsically Π^1_1 since it is defined by the disjunction of computable infinitary formulas saying that the interval to the left of x has order type α, for computable ordinals α. A *Harrison Boolean algebra* is a computable Boolean algebra of type $I(\omega_1^{CK}(1+\eta))$, where for an order $\mathcal{L}$, the interval algebra $I(\mathcal{L})$ is the algebra generated, under finite union, by the intervals $[a, b)$, $(-\infty, b)$, $[a, \infty)$, with endpoints in $\mathcal{L}$. The set of superatomic elements of this Boolean algebra is intrinsically Π^1_1. A *Harrison group* is a countable abelian p-group G such that $\lambda(G) = \omega_1^{CK}$, every element in its Ulm sequence $(u_\alpha(G))_{\alpha<\omega_1^{CK}}$ is ∞, and the divisible part has infinite dimension. Recall that the Ulm subgroups G_α are defined by $G_\alpha = p^{\omega\alpha}G$, and $u_\alpha(G) =_{def} \dim_{\mathbb{Z}_p} P_\alpha(G)/P_{\alpha+1}(G)$, where $P_\alpha(G) = G_\alpha \cap \{x \in G : px = 0\}$. The set of elements of a Harrison group, which have computable ordinal heights, is intrinsically Π^1_1. It is the complement of the divisible part. By a *path* through Kleene's $\mathcal{O}$ we mean a subset of $\mathcal{O}$ that is linearly ordered under $<_\mathcal{O}$ and includes a notation for every computable ordinal.

Theorem 87. *([134]) The following sets are equal:*

1. *the set of Turing degrees of Π^1_1 paths through $\mathcal{O}$;*
2. *the set of Turing degrees of left-most paths of computable trees $\mathcal{T} \subseteq \omega^{<\omega}$ such that $\mathcal{T}$ has a path, but no hyperarithmetic path;*
3. *the set of Turing degrees of maximal well-ordered initial segments of Harrison orders;*
4. *the set of Turing degrees of superatomic parts of Harrison Boolean algebras;*
5. *the set of Turing degrees of divisible parts of Harrison groups.*

For certain types of structures, there is a close connection between the notions of degree spectra of structures and of relations. Harizanov and R. Miller [155] defined a computable structure $\mathcal{U}$ to be *spectrally universal* for a theory T if for every automorphically nontrivial countable model $\mathcal{A}$ of T, there is an embedding $f : \mathcal{A} \to \mathcal{U}$ such that $\mathcal{A}$ as a structure, has the same degree spectrum as $f(A)$, as a relation on the domain of $\mathcal{U}$. Spectrally universal structures investigated in [155] are the countable dense linear order and the random graph. Both are Fraïssé limits. This led Csima, Harizanov, R. Miller, and Montalbán to develop the theory of computable Fraïssé limits in [64]. They gave a sufficient condition for certain Fraïssé limits to be spectrally universal, which they used to show that the countable atomless Boolean algebra is spectrally universal.

For syntactic characterizations of relations having Post-type properties on structures, or their degree-theoretic complexity see [167, 166, 17, 146, 132, 145, 131].

8 Families of relations on a structure

Many important algebraic properties can be investigated by considering natural families of relations on a structure. For example, for a vector space V we can consider the family of its bases:

$$\mathcal{B}(V) = \{X \subseteq V : X \text{ is a basis of } V\}.$$

For an orderable field F we can consider the set of all linear orders on its domain, which are invariant under the field operations:

$$O(F) = \{R \subseteq F \times F : R \text{ is an order on } F\}.$$

Such a family of relations does not necessarily have a computable member even when the structure is computable. Mal'cev [227] showed that there exists a computable vector space without a computable basis. Metakides and Nerode [240] and Ershov [92] showed that there exists a computable orderable field that cannot be computably ordered. We could ask for a sufficient condition on a family of relations on a computable structure to have a computable member. More generally, we may ask what the collection of Turing degrees of its members is.

Definition 25. ([70]) Given a family of relations $\mathcal{R}$ on a computable structure $\mathcal{M}$, define

$$DgSp(\mathcal{R}) = \{deg(R) : R \in \mathcal{R}\}.$$

In the next definition we are computing all relations simultaneously (uniformly).

Definition 26. Let $\mathcal{A}$ be a computable structure, and let $\mathcal{R} = (R_i)_{i\in I}$ be a family of relations on $\mathcal{A}$, where $l(i)$ is the arity of R_i. Define

$$DgSp(\mathcal{R}; \mathcal{A}) = deg\{\overline{a} \subseteq A^{l(i)} : \mathcal{A} \models R_i(\overline{a}), i \in I\}.$$

In many interesting examples, the index set I and the arities of relations are computable. The previous two definitions are dependent on a given presentation of a structure. We could let the definitions range over all computable copies of $\mathcal{A}$. However, this approach is not common.

Let us consider the problem of computing a generating set (or a basis) of a given computable structure. The definition of a basis depends on the class of structures. The study of the problem of computing a basis in several classical algebraic examples provides a natural link between Definition 26 and Definition 25. More specifically, to build a basis stage-by-stage (Definition 25), one usually needs a corresponding notion of independence (Definition 26). Consider the following example.

*Example***.** Let V be a countable vector space of infinite dimension. Define the following sets of relations on V.

1. For every $i \in \omega$, and any $x_0, \dots, x_i \in V$, we set $P_i(x_0, \dots, x_i) = 1$ if and only if $x_0, \dots, x_i$ are linearly independent.

2. Let $\mathbb{B}$ be the collection of maximal linearly independent sets (bases) in V.

If $\mathcal{P} = (P_i)_{i \in \omega}$ is uniformly computable, then we say that V has an algorithm for linear independence.

Theorem 88. *(folklore; see [227, 241]) Every computable vector space over a computable field has a* $\mathbf{0}'$*-computable basis, and this bound is sharp.*

Let us now consider another natural example from algebra.

Example. Let F be a countable algebraically closed field of infinite transcendence degree. Define the following sets of relations on F.

1. For every $i \in \omega$, and any $a_0, \dots, a_i \in F$, we set $R_i(a_1, \dots, a_i) = 1$ if and only if, $a_1, \dots, a_i$ are algebraically independent.

2. Let $\mathbb{A}$ be the collection of maximal algebraically independent subsets of F.

If $\mathcal{R} = (R_i)_{i \in \omega}$ is uniformly computable in F, then we say that F has an algorithm for algebraic independence.

Theorem 89. *(folklore; see [107, 240, 288]) The algebraic closure of* $\mathbb{Q}(x_i : i \in \omega)$ *has a* $\mathbf{0}'$*-maximal algebraically independent set, and this bound is sharp.*

It is clear that independence can be formalized using families of relations as in Definition 26, and the collection of bases should be studied according to Definition 25. It is important to observe that in the context of vector spaces and algebraically closed fields, the existence of a generating set is equivalent to the problem of computable categoricity relative to an oracle. The same can be said about many other natural examples.

A number of researchers investigated complexity of independent sets and other subsets and subspaces of c.e. vector spaces and c.e. algebraically closed fields (see, for example, [241, 184, 293, 313, 239, 71]). In many of their results the operations (vector addition and scalar multiplication, or field operations, respectively) play no direct role. For instance, in the proofs of Theorems 88 and 89 only the phenomenon of independence occurs. In fact, Metakides and Nerode [239] initiated the study of the effective content of abstract independence relations (Steinitz closure systems). For an extended survey of the results about computable Steinitz closure systems, see the paper [86] by Downey and Remmel.

We will now discuss recent results about bases of various structures. Downey and Melnikov [84] studied free modules over localizations of integers.

Theorem 90. *([84]) Let $S \subseteq \omega$ be a c.e. set of primes.*

(i) Every computable free module $\mathcal{F}(S)$ over the localization of $\mathbb{Z}$ by S has a Σ^0_3 (actually, Π^0_2 in S) set of generators.

(ii) Every computable copy of $\mathcal{F}(S)$ has a Σ^0_2 set of generators if and only if the complement of S is semi-low.

The theorem can be equivalently re-formulated in terms of computable categoricity relative to an oracle. The corresponding analogue of linear independence for free modules of this kind is *S-independence,* which is a generalization of the classical notion of p-independence (see [84]). As a consequence of Theorem 90 with $S = \emptyset$, it follows that every free abelian group has a Π^0_1 generating set.

Algebraic structure becomes more complex in the case of free nonabelian groups. Relatively recently, Sela in a series of papers [309, 308, 307, 306, 305, 304, 303] gave a positive solution to the problem of elementary equivalence of free groups of different finite ranks greater than 1, posed by Tarski in the 1940s. (See also Kharlampovich and Myasnikov [193].) Inspired by this result, Carson, Harizanov, Knight, Lange, McCoy, Morozov, Safranski, Quinn, and Wallbaum [44], and McCoy and Wallbaum [236] investigated free groups in the context of computable model theory. Let F_∞ be the free group of rank $\aleph_0$.

Theorem 91. *([44, 236]) Every computable copy of F_∞ has a Π^0_2 basis, and the result cannot be improved to Σ^0_2.*

The proof of the theorem uses deep results in algebra. The corresponding notion of independence is what is called *primitiveness* in every finitely generated subgroup (see [44, 236]).

In general, not every family of unary relations (Definition 25) has a hyperarithmetic "notion of independence" (Definition 26). For example, consider the collection of paths on $\mathcal{T} \subset \omega^{<\omega}$, where $\mathcal{T}$ codes a Σ^1_1-complete set. In contrast, we have seen that natural structures well-understood in algebra tend to have arithmetic bases. Thus, we can ask whether there is a natural structure (such as a ring, a module, or a group) for which finding a generating set is not (hyper)arithmetic. A possible candidate is the pure transcendental ring over the rationals, $\mathbb{Q}[x_i : i \in \omega]$. Does every computable copy of $\mathbb{Q}[x_i : i \in \omega]$ have a (hyper)arithmetic basis? Describing automorphism orbits of generators in $\mathbb{Q}[x_i : i \in \omega]$ is a long-standing open problem in algebra. There has been some progress in this direction; see the recent paper by Shestakov and Umirbaev [312].

We will now discuss some old and recent results on orders on orderable groups and fields. Recall that a left order on a group $\mathcal{G} = (G, \cdot)$ is a linear order $<$ of its elements, which is left-invariant under the group operation:

$$(\forall x, y)(\forall z)[x < y \Rightarrow z \cdot x < z \cdot y].$$

A right order is defined similarly. A bi-order (or simply order) is invariant under both left and right multiplication. The definition of an order for a field is similar. Clearly, every left order on an abelian group is a bi-order. Every left order $<_l$ on $\mathcal{G}$ induces a right order $<_r$ on $\mathcal{G}$ as follows:

$$a <_r b \Leftrightarrow b^{-1} <_l a^{-1}.$$

It is well known that an abelian group is orderable if and only if it is torsion-free. A field is orderable exactly when it is formally real (see [110]). As for fields, in the case of computable orderable groups, the effective analogue of the classical result fails. Downey and Kurtz [81] showed that there exists a computable group isomorphic to $\mathbb{Z}^\omega = \bigoplus_{i\in\omega} \mathbb{Z}$, which does not have a computable order. On the other hand, Dobritsa [72] previously showed that every computable, torsion-free, abelian group is isomorphic to a computable group with a computable order.

For a group $\mathcal{G}$, by $LO(\mathcal{G})$ we denote the set of all left orders on $\mathcal{G}$, and by $BiO(\mathcal{G})$ the set of all bi-orders on $\mathcal{G}$. There is a natural topology on these sets (when nonempty), making the topological spaces compact, even when $\mathcal{G}$ is a semigroup instead of a group, or just a structure with a single binary operation (see [69]). In some cases this space is homeomorphic to the Cantor set. Sikora [314] established that the space $BiO(\mathbb{Z}^n)$ for $n > 1$ is homeomorphic to the Cantor set. Dabkowska [67] established that the space $BiO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set. (Her result can also be obtained from [81].) Solomon [318] obtained the following results about Turing degrees of orders on abelian groups.

Theorem 92. *([318])*

1. *A computable, torsion-free, abelian group of finite rank greater than* 1 *has an order in every Turing degree.*

2. *A computable, torsion-free, abelian group of infinite rank has an order in every Turing degree* $\mathbf{d} \geq \mathbf{0}'$.

3. *Let* $n > 1$. *A computable, torsion-free, properly* n*-step nilpotent group has an order in every Turing degree* $\mathbf{d} \geq \mathbf{0}^{(n)}$.

The positive cone of an order $<$ on a group $\mathcal{G}$ is $P = \{a \in G : e \leq a\}$, where $e \in G$ is the identity element. The negative cone is $P^{-1} = \{a \in G : a \leq e\}$. Clearly, $a \leq b$ iff $a^{-1}b \in P$. Hence, we can effectively pass from binary relations (orders) to unary relations (positive cones) and *vice versa*. We can easily verify that if $P \subseteq G$ is a *subsemigroup* of $\mathcal{G}$ (i.e., $PP \subseteq P$), which satisfies $P \cap P^{-1} = \{e\}$, then P defines a left order on $\mathcal{G}$ if and only if P is *total* (i.e., $P \cup P^{-1} = G$). Moreover, P defines a bi-order on $\mathcal{G}$ if, in addition, P is a *normal* subsemigroup (i.e., $g^{-1}Pg \subseteq P$ for every $g \in G$). Denote by $\mathbb{C}(\mathcal{G})$ the set of all positive cones of orders on $\mathcal{G}$. Clearly, $DgSp(BiO(\mathcal{G})) = \{\deg(C) : C \in \mathbb{C}(\mathcal{G})\}$.

Solomon [319] established that for every orderable computable group $\mathcal{G}$, there is a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection from $\mathbb{C}(\mathcal{G})$ to the set of all infinite paths of $\mathcal{T}$. Hence $\mathbb{C}(\mathcal{G})$ corresponds to a Π^0_1 class, and, by the *low basis theorem*, $BiO(\mathcal{G})$ contains an order of *low* Turing degree. Previously, Metakides and Nerode [240] established the same results for computable orderable fields. Moreover, they showed that the sets of orders of computable orderable fields are in exact correspondence to the collections of Π^0_1 subsets of 2^ω.

Theorem 93. *([240]) For every nonempty Π^0_1 class $\mathbb{P}$, there is a computable orderable field $\mathcal{F}$ and a Turing degree preserving bijection $f : \mathbb{P} \to \mathbb{C}(\mathcal{F})$.*

The proof is based on a result by Craven [60] that for every Boolean topological space $\mathcal{T}$, there is a formally real field $\mathcal{F}$ such that $\mathbb{C}(\mathcal{F})$ is homeomorphic to $\mathcal{T}$. Many corollaries about degree spectra of orders on fields follow from Theorem 93. It is not hard to see that the situation is different for torsion-free abelian groups. Solomon [319], using a result by Jockusch and Soare [181], showed that there is a Π^0_1 class $\mathbb{P}$ such that for any computable, torsion free, abelian group $\mathcal{G}$, we have $\{deg(f) : f \in \mathbb{P}\} \neq DgSp(BiO(\mathcal{G}))$.

More recently, Dabkowska, Dabkowski, Harizanov, and Togha [70] studied topological and computability-theoretic properties of left orders and bi-orders on (not necessarily abelian) groups. They obtained general sufficient conditions for the degree spectra of orders on groups to contain upper cones of Turing degrees. As a corollary they established the following result about the free groups F_n of rank n.

Theorem 94. *([70]) Every computable copy of F_n, where $n > 1$, has an order in every Turing degree.*

Sikora [314] conjectured that $BiO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set. The conjecture still remains open. It was shown in [153] that there is a computable copy of F_∞ with no computable left order, and hence the space $BiO(F_\infty)$ (as well as $L(F_\infty)$) is homeomorphic to the Cantor set.

Kach, Lange, and Solomon [182] constructed computable, torsion-free, abelian groups such that the degree spectra of their orders are not upward closed. The groups are isomorphic to effectively completely decomposable groups. N. Khisamiev and Krykpaeva [198] defined a computable, infinite-rank, torsion-free, abelian group $\mathcal{H}$ to be *effectively completely decomposable* if there is a uniformly computable sequence of rank one groups $\mathcal{H}_i$, $i \in \omega$, such that $\mathcal{H}$ is equal to $\oplus_{i\in\omega}\mathcal{H}_i$.

Theorem 95. *([182]) Let $\mathcal{H}$ be a computable and effectively completely decomposable group. Then there is a computable copy $\mathcal{G}$ of $\mathcal{H}$ such that $DgSp(BiO(\mathcal{G}))$ contains $\mathbf{0}$, but is not upward closed.*

More precisely, Kach, Lange, and Solomon showed that there is a noncomputable, c.e. set C such that $\mathcal{G}$ has exactly two computable orders, and every C-computable order on $\mathcal{G}$ is computable. On the other hand, since $\mathcal{H}$ is effectively completely decomposable, it has a computable basis formed by choosing a nonzero element h_i from every $\mathcal{H}_i$. Hence $DgSp(BiO(\mathcal{H}))$ contains every Turing degree, and $\mathcal{G}$ is not effectively completely decomposable. Kach, Lange, and Solomon [182] conjectured that the conclusion of Theorem 95 holds for all computable, infinite-rank, torsion-free, abelian groups $\mathcal{H}$.

Natural relations in partial orders are their chains and antichains. Complexity of infinite chains and antichains in computable partial orders was studied by Herrmann [163] and Harizanov, Jockusch, and Knight [152]. It follows from an effective version of Ramsey's theorem for pairs, due to Jockusch [178], that a computable partial order of ω has either an infinite Δ^0_2 chain, or an infinite Δ^0_2 antichain, or else both an infinite Π^0_2 chain and an infinite Π^0_2 antichain. On the other hand, Herrmann [163] showed that there is a computable partial order of ω with no infinite Σ^0_2 chain or antichain. Harizanov, Jockusch, and Knight [152] showed that there is a computable partial order with an infinite chain but none that is Σ^1_1 or Π^1_1, and they obtained the analogous result for antichains. They also showed that there is a computably axiomatizable theory T of partial orders such that T has a computable model with arbitrarily long finite chains but no computable model with an infinite chain. They also established the corresponding result for antichains.

9 Classes of structures and equivalence relations

Our goal is to measure the complexity of classes of computable structures and equivalence relations on these classes. More precisely, we want to know how complex the answers to the following types of questions are. Does a computable structure belong to a particular class of structures with fixed algebraic, model-theoretic, or algorithmic properties (e.g., a class of groups, uncountably categorical structures, decidable structures, etc.)? Are two structures from such a class isomorphic, computably isomorphic, bi-embeddable, etc.? We are looking for a criterion that will allow us to say whether such questions have "nice" answers.

There are many papers investigating the complexity of classes of countable structures. There is earlier work in descriptive set theory [248, 249] investigating subsets of the Polish space of structures with universe ω for a given countable relational language. Concerning the possible complexity (in the noneffective Borel hierarchy) of the set of copies of a given structure, D. Miller [249] showed that if this set is $\Delta^0_{\alpha+1}$, then it is d-Σ^0_α. In [248], A. Miller showed that this set cannot be properly Σ^0_2. There are also examples illustrating other possibilities.

The main issue here is to find an optimal definition of the class of structures under investigation. This often requires the use of various internal properties of the structures in the class. After a reasonable definition is found, it is necessary to prove its sharpness. Usually, this is done by proving completeness in some complexity class.

In the case of equivalence relations, the study of Borel reducibility has developed into a rich area of descriptive set theory. The notion of Borel reducibility allows us to compare the complexity of equivalence relations on Polish spaces (see [190, 111]). In particular, natural equivalence relations on classes of countable structures, such as isomorphism and bi-embeddability, have been widely studied; for example, see [106, 176, 105]). An effective version of this study was introduced by Calvert, Cummins, Knight, and S. Miller (Quinn) [34], and Knight, S. Miller (Quinn), and Vanden Boom [211]. The main idea is that the complexity of the isomorphism relation on various classes of countable structures can be measured using the effective transformations. The introduced c-embeddings and tc-embeddings are based on uniform enumeration reducibility and uniform Turing reducibility, respectively. The main advantage of this approach is that allows distinctions among classes with countably many isomorphism types.

In computable model theory, we may state our goal as follows. Let K be a class of structures. We denote by K^c the set of computable structures in K. A *computable characterization* of K should separate computable structures in K from all other structures (those not in K, or noncomputable ones). A *computable classification* for K up to an equivalence relation E (isomorphism, computable isomorphism, etc.) should determine each computable element, up to the equivalence E, in terms of relatively simple invariants. In [139], Goncharov and Knight presented three possible approaches to the study of computable characterizations of classes of structures.

Within the framework of the first approach, we say that K has a *computable characterization* if K^c is the set of computable models of a computable infinitary sentence.

Proposition 1. *(i) The class of linear orders can be characterized by a single first-order sentence.*

(ii) The class of abelian p-groups is characterized by a single computable Π_2 *sentence.*

(iii) The classes of well orders and reduced abelian p-groups cannot be characterized by single computable infinitary sentences.

Furthermore, we say that there is a *computable classification* for K if there is a computable bound on the ranks of elements of K^c. By a *computable rank* $R^c(\mathcal{A})$ of a structure $\mathcal{A}$ we mean the least ordinal α such that for all tuples $\overline{a}$ and $\overline{b}$ in A, of the same length, if for all $\beta < \alpha$, all computable Π_β formulas that true of $\overline{a}$ are also true of $\overline{b}$, then there is an automorphism of $\mathcal{A}$ taking $\overline{a}$ to $\overline{b}$. For example, the computable rank of a vector space over $\mathbb{Q}$ is 1. There is no computable bound on computable ranks of linear orders and abelian p-groups. The computable rank is not the same as the Scott rank. However, for a hyperarithmetic structure, its computable rank is a computable ordinal just in case its Scott rank is computable (see [139]). If $\mathcal{A}$ is hyperarithmetic, then $R^c(\mathcal{A}) \leq \omega_1^{CK}$.

The second approach involves the notion of an index set. A *computable index* for a structure $\mathcal{A}$ is a number e such that $D(\mathcal{A}) = W_e$, where $D(\mathcal{A})$ is the atomic diagram of $\mathcal{A}$. We denote the structure with index e by $\mathcal{A}_e$. For a class K of structures, the *index set* $I(K)$ is the set of computable indices of members of K^c:

$$I(K) = \{e : W_e = D(\mathcal{A}) \wedge \mathcal{A} \in K\}.$$

For an equivalence relation E on a class K, we define

$$I(E, K) = \{(m, n) : m, n \in I(K) \wedge \mathcal{A}_m E \mathcal{A}_n\}.$$

Within this approach, we say that K has a *computable characterization* if $I(K)$ is hyperarithmetic. The class K has a *computable classification* up to E if $I(E, K)$ is hyperarithmetic.

The first and the second approach are known to be equivalent [139]. In fact, we do not know a better way to estimate the complexity of an index set than by giving a description by a computable infinitary formula.

Proposition 2. *([139]) (i) For the following classes K, the index set $I(K)$ is Π^0_2:*

(a) linear orders,

(b) Boolean algebras,

(c) abelian p-groups,

(d) vector spaces over $\mathbb{Q}$.

(ii) (Kleene, Spector) For the following classes K, the index set $I(K)$ is not hyperarithmetic:

(a) well-orders,

(b) superatomic Boolean algebras,

(c) reduced abelian p-groups.

In the next theorem, the calculations of the complexity of index sets for classes of structures with interesting model-theoretic properties are due to White [333], Calvert, Fokina, Goncharov, Knight, Kudinov, Morozov, and Puzarenko [35], Fokina [96], and Pavlovskii [282]. In (v), $\Sigma^0_3 - \Sigma^0_3$ denotes the difference of two Σ^0_3 sets.

Theorem 96. *(i) ([333, 282]) The index set of computable prime models is an m-complete $\Pi^0_{\omega+2}$ set.*

(ii) ([333]) The index set of computable homogeneous models is an m-complete $\Pi^0_{\omega+2}$ set.

(iii) ([282]) The index set of structures with uncountably categorical theories is a Δ^0_ω-hard $\Sigma^0_{\omega+1}$ set.

(iv) ([282]) The index set of structures with countably categorical theories is a Δ^0_ω-hard $\Pi^0_{\omega+2}$ set.

(v) ([96]) The index set of structures with decidable countably categorical theories is an m-complete $\Sigma^0_3 - \Sigma^0_3$ set.

(vi) ([35]) (a) The index set of computable structures with noncomputable Scott ranks is m-complete Σ^1_1.

(b) The index set of structures with the Scott rank ω_1^{CK} *is m-complete* Π^0_2 *relative to Kleene's* $\mathcal{O}$.

(c) The index set of structures with the Scott rank $\omega_1^{CK}+1$ *is m-complete* Σ^0_2 *relative to Kleene's* $\mathcal{O}$.

The index sets for structures with specific algorithmic properties were also studied by White [333], Fokina [95], and Downey, Kach, Lempp, and Turetsky [79].

Theorem 97. *(i) ([95]) The index set of decidable structures is* Σ^0_3*-complete.*

(ii) ([333]) The index set of hyperarithmetically categorical structures is Π^1_1*-complete.*

(iii) ([79]) The index set of relatively computably categorical structures is Σ^0_3*-complete.*

The following result of Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky resolves an important old problem.

Theorem 98. *([78]) The index set of computably categorical structures is* Π^1_1*-complete.*

The structures constructed to establish this result are computable trees of special kind. It would be worthwhile to calculate the complexity of the index sets of other classes of computable structures having interesting algebraic, model-theoretic, or algorithmic properties.

The third approach of Goncharov and Knight [139] to computable characterization of classes of structures involves the notion of *enumeration*. A class of structures has a good characterization if all its structures are represented in the list, up to isomorphism or some other equivalence relation. A good classification of the class would mean listing each equivalence class only once.

Definition 27. (i) An *enumeration* of K^c/E is a sequence $(\mathcal{M}_n)_{n\in\omega}$ representing all E-equivalence classes in K^c.

(ii) A *Friedberg enumeration* of K^c/E is an enumeration in which every E-equivalence class is represented only once.

(iii) An enumeration is Δ^0_α-*computable* if there is a Δ^0_α-computable sequence of computable indices for the structures.

We say that K has a *computable characterization* if there is a hyperarithmetic enumeration of $K^c/\cong$. We say that K has a *computable classification* up to E if there is a hyperarithmetic Friedberg enumeration of K^c/E. It is known that this approach is not equivalent to the previous two approaches, by is only implied by them. Recall that a Harrison order is a computable linear order of type $\omega_1^{CK}(1+\eta)$.

Proposition 3. *([139]) Consider the class K consisting of copies of the Harrison order and of the linear orders of rank at most ω. Then $K^c/\cong$ has a hyperarithmetic Friedberg enumeration, but the index set $I(K)$ is not hyperarithmetic.*

We will now focus on the classification problems up to important equivalence relations. The most interesting cases are isomorphism, bi-embeddability, and isomorphism of bounded algorithmic complexity. Possible ways to compare the complexity of various equivalence relations are:

1. comparison among sets;
2. comparison among equivalence relations.

The former case was discussed above. It corresponds to the second approach from [139]. Within this approach, we usually prove m-completeness among sets in some complexity class. There has been quite a lot of work on the isomorphism problem for various classes of computable structures by Goncharov and Knight [139], Calvert [31, 29, 30], and Calvert, Harizanov, Knight, and S. Miller (Quinn) [36].

Theorem 99. *(i) ([29]) The isomorphism problem for computable vector spaces over $\mathbb{Q}$ is m-complete among Π^0_3 sets.*

(ii) ([29]) The isomorphism problem for torsion-free abelian groups of finite characteristic is m-complete among Σ^0_3 sets.

(iii) ([139]) (a) The isomorphism problem for abelian p-groups is m-complete among Σ^1_1 sets.

(b) The isomorphism problem for trees is m-complete among Σ^1_1 sets.

Recently, Carson, Fokina, Harizanov, Knight, Safranski, Quinn, and Wallbaum initiated the study of the *computable embedding problem*. In [43], they investigated the relation between the isomorphism problem and the embedding problem for some well-known classes of structures. The isomorphism problem and the embedding problem were compared as sets, that is, using the standard m-reducibility. While for some classes of structures the two problems have the same complexity, for other classes the isomorphism problem is more complicated than the embedding problem, or *vice versa*.

Further comparison of complexity of equivalence relations was done using the 2-dimensional versions of reducibilities. This approach can be seen as an analogue of investigation done in descriptive set theory. Recall that in descriptive set theory, two equivalence relations, E and F, on Borel classes of structures, K and L, respectively, can be compared using Borel reducibility. In the computable case, instead of arbitrary invariant Borel classes of countable structures, we consider classes of computable structures with hyperarithmetic index sets. In other words, we consider classes consisting of computable models of computable infinitary sentences. As mentioned above, this corresponds to a "nice" characterization of a class.

A straightforward analogue of the Borel reducibility is the hyperarithmetic reducibility.

Definition 28. For equivalence relations E_1, E_2 on (hyperarithmetic subsets of) ω, we say that E_1 is *h-reducible to* E_2, in symbols $E_1 \leq_h E_2$, if there is a hyperarithmetic function f such that for all x, y,

$$xE_1y \Leftrightarrow f(x)E_2f(y).$$

A stronger reducibility would be a 2-dimensional version of the m-reducibility. This reducibility is traditionally used in the general study of equivalence relations on ω. It was introduced by Ershov in [91] where he studied properties of numberings. Later it was used, for example, in [27, 112, 59, 8] and denoted simply by $\leq$. As sometimes we need to emphasize the difference between m-reducibility and h-reducibility, we will denote the reducibility *via* a computable function by $\leq_m$, specifying when necessary that we consider the 2-dimensional version of m-reducibility among relations. When the results hold for both h-reducibility and m-reducibility we will use the symbol $\leq$.

Definition 29. Let E_1, E_2 be equivalence relations on hyperarithmetic subsets $X, Y \subseteq \omega$, respectively. The relation E_1 is *m-reducible* to E_2, in symbols $E_1 \leq_m E_2$, iff there exists a partial computable function f with $X \subseteq dom(f)$ and $f(X) \subseteq Y$ such that for all $x, y \in X$,

$$xE_1y \Leftrightarrow f(x)E_2f(y).$$

Each notion of reducibility generates the corresponding notion of completeness.

Definition 30. A relation E on a hyperarithmetic subset of ω is an *h-complete* Σ^1_1 equivalence relation, or *m-complete* Σ^1_1 equivalence relation, if E is Σ^1_1 and every Σ^1_1 equivalence relation E_1 on a hyperarithmetic subset of ω is h-reducible to E, or m-reducible to E, respectively.

We use the previous definitions to compare equivalence relations on classes of computable structures. Recall that each such relation E on a class K has the index set $I(E, K)$. We make no distinction between E and $I(E, K)$ in the following sense. If E_1 is an arbitrary equivalence relation on ω, then we say that E_1 h-reduces to E, or m-reduces to E, iff there exists a hyperarithmetic, or computable, respectively, sequence of computable structures $\{\mathcal{A}_x\}_{x\in\omega}$ from K such that for all x, y, we have xE_1y iff A_xEA_y. (This is equivalent to $E_1 \leq_h I(E, K)$ or $E_1 \leq_m I(E, K)$ in the sense of Definitions 28 and 29.) From now on we will write $\leq$ to denote either of $\leq_h$, $\leq_m$. We will use the terms "reduces," "complete," etc. for the corresponding notion of reducibility.

The following result is due to Fokina and S. Friedman.

Proposition 4. *([99]) There is a class K of structures with hyperarithmetic index set such that the bi-embeddability relation on K^c is complete among Σ^1_1 equivalence relations.*

This result corresponds to the analogous result in descriptive set theory due to S. Friedman and Motto Ros [105]. However, the theory of Σ^1_1 equivalence relations on ω under $\leq$-reducibility behaves very differently from the theory of Borel equivalence relations on Polish spaces. In particular, Fokina, S. Friedman, Harizanov, Knight, McCoy, and Montalbán [100] established the following completeness result.

Theorem 100. *([100]) The isomorphism of computable graphs is complete with respect to the chosen effective reducibility in the context of all Σ^1_1 equivalence relations on ω.*

This is false in the context of countable structures and Borel reducibility since Kechris and Louveau [191] showed that there are examples of Borel equivalence relations that are not Borel-reducible to isomorphism of graphs. Moreover, the authors of [100] proved that the isomorphism relation on computable torsion abelian groups is complete among Σ^1_1 equivalence relations on ω, while in the classical case it is known to be incomplete among isomorphism relations on classes of countable structures, as established by H. Friedman and Stanley [106]. In [100], the authors also established that the isomorphism relation on computable, torsion-free, abelian groups is complete among Σ^1_1 equivalence relations on ω, while in the case of countable structures it is not known to be complete for isomorphism relations.

Regarding bounding the complexity of the isomorphism relation, Fokina, S. Friedman, and Nies obtained the following result.

Theorem 101. *([103]) The computable isomorphism relation on computable structures from classes including predecessor trees, Boolean algebras, and metric spaces is a complete Σ^0_3 equivalence relation under the computable reducibility.*

To prove their result, the authors first showed that *one-one equivalence* relation of c.e. sets, as an equivalence relation on indices, is Σ^0_3 complete, and then reduced this equivalence relation to the computable isomorphism on predecessor trees. Using the technique developed by Hirschfeldt and White in [175] and Csima, Franklin, and Shore in [62], the result of Theorem 101 can be lifted to hyperarithmetic levels.

It follows from [105] by S. Friedman and Motto Ros that the following result holds for the bi-embeddability relation on computable structures.

Theorem 102. *([105]) For every Σ^1_1 equivalence relation E on ω, there exists a hyperarithmetic class K of structures, which is closed under isomorphism, and such that E is h-equivalent to the bi-embeddability relation on computable structures from K.*

In fact, the reduction functions have complexity at most $\mathbf{0}'$. In [98], Fokina and S. Friedman showed that the general structure of Σ^1_1 equivalence relations on hyperarithmetic subsets of ω is rich. Theorem 102 states that the structure of bi-embeddability relations on hyperarithmetic classes of computable structures is as complex as the whole structure of Σ^1_1 equivalence relations under

h-reducibility. It would be interesting to answer the following question and possibly get a refinement of Theorem 102. If E is a Σ^1_1 equivalence relation on ω, does there exist a hyperarithmetic class K of structures, which is closed under isomorphism, and such that E is equivalent to the bi-embeddability relation on computable structures from K *via* computable functions?

It is not known whether there exists a hyperarithmetic class of computable structures with Σ^1_1, but not Δ^1_1 isomorphism relation, which is not complete among all isomorphism relations on hyperarithmetic classes of computable structures. An affirmative answer to the following question may help solve this problem. Does there exist a hyperarithmetic class K of computable structures, which contains a unique structure of noncomputable Scott rank (up to isomorphism)? If such a class exists, then the isomorphism relation on the class of computable graphs cannot be reduced to the isomorphism relation on K. Indeed, there exist nonisomorphic graphs of high (that is, ω_1^{CK} or $\omega_1^{CK}+1$) Scott rank. They must be mapped to nonisomorphic structures in K. However, no computable structure of high Scott rank can be mapped to a computable structure of computable Scott rank under a hyperarithmetic reducibility. This question is closely connected with many important open questions in computable model theory concerning computable structures of high Scott rank, such as the question of strong computable approximation (see [139, 35]). It is known that, up to bi-embeddability, this is true in the following sense. In the class of computable linear orders, the equivalence class of linear orders bi-embeddable with the rationals is Σ^1_1-complete, but every computable scattered linear order (that is, one not bi-embeddable with the rationals) has a hyperarithmetic equivalence class. For more information on the bi-embeddability relation in the class of countable linear orders see the paper [258] by Montalbán.

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