Automorphism Groups of Substructure Lattices of Vector Spaces in Computable Algebra

Rumen Dimitrov[1], Valentina Harizanov[2], and Andrei Morozov[3] ⋆

[1] Department of Mathematics, Western Illinois University, Macomb, IL 61455, USA
rd-dimitrov@wiu.edu
[2] Department of Mathematics, George Washington University, Washington, DC 20052, USA harizanv@gwu.edu
[3] Sobolev Institute of Mathmematics, Novosibirsk, 630090, Russia
morozov@math.nsc.ru

Abstract. For a Turing degree $\mathbf{x}$, we investigate the automorphisms of the lattice of $\mathbf{x}$-c.e. vector spaces. We establish the equivalence of the embedding relation for these automorphism groups with the order relation on the corresponding Turing degrees. By a result of Guichard the automorphisms of the lattice of $\mathbf{x}$-c.e. vector spaces are induced by $\mathbf{x}$-computable invertible semilinear transformations, $\mathrm{GSL}_{\mathbf{x}}$. We prove that the Turing degree spectrum of the group $\mathrm{GSL}_{\mathbf{x}}$ is the upper cone of Turing degrees $\geq \mathbf{x}''$.

1 Automorphisms of effective structures

The study of automorphisms on computable or computably enumerable structures connects computability theory and classical group theory. The set of all automorphisms of a computable structure forms a group under composition, and it is natural to ask questions about its complexity as well as the complexity of its subgroups. It is also interesting to connect the embedability of the subgroups with Turing reducibility.

The following notion is the focus of our investigation. Let $\mathbf{d}$ be a Turing degree. For an infinite computable structure $\mathcal{M}$, we define $Aut_{\mathbf{d}}(\mathcal{M})$ to be the set of all automorphisms of $\mathcal{M}$ computable in $\mathbf{d}$. The set $Aut_{\mathbf{d}}(\mathcal{M})$ under composition is a subgroup of $Aut(\mathcal{M})$. When the structure $\mathcal{M}$ is ω with equality, then its automorphism group $Aut(\mathcal{M})$ is usually denoted by $Sym(\omega)$, the symmetric group of ω. Hence we have

$$Sym_{\mathbf{d}}(\omega) = \{f \in Sym(\omega) : \deg(f) \leq \mathbf{d}\},$$

where $\deg(f)$ is the Turing degree of f. Our other computability theoretic notation is also standard and as in [18].

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The *Turing degree spectrum* of a countable structure $\mathcal{A}$ is

$$DgSp(\mathcal{A}) = \{\deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\},$$

where $\deg(\mathcal{B})$ is the Turing degree of the atomic diagram of $\mathcal{B}$. Knight [8] proved that the degree spectrum of any structure is either a singleton or is upward closed. Only the degree spectrum of a so-called automorphically trivial structure is a singleton, and if the language is finite, that degree must be $\mathbf{0}$ (see [7]). Automorphically trivial structures include all finite structures, and also some special infinite structures, such as the complete graph on countably many vertices. Jockusch and Richter (see [15]) defined the *degree of the isomorphism type* of a structure, if it exists, to be the least Turing degree in its Turing degree spectrum. Richter [15, 16] was first to systematically study such degrees. For these and more recent results about these degrees see [4]. In this paper we are especially interested in the following result by Morozov.

Theorem 1. *([10]) The degree of the isomorphism type of the group* $Sym_{\mathbf{d}}(\omega)$ *is* $\mathbf{d}''$.

We will establish a similar result in the context of effective vector spaces.

Let V_∞ be a canonical fully effective $\aleph_0$-dimensional vector space over a computable field F. We can think of the vectors in V_∞ as (the codes of) the finitely non-zero ω-sequences of elements of F. By $\mathcal{L}$ we denote the lattice of all subspaces of V_∞. For a Turing degree $\mathbf{d}$, by $\mathcal{L}_{\mathbf{d}}(V_\infty)$ we denote the following sublattice of $\mathcal{L}$:

$$\mathcal{L}_{\mathbf{d}}(V_\infty) = \{V \in \mathcal{L} : V \text{ is } \mathbf{d}\text{-computably enumerable}\}.$$

Note that in the literature $\mathcal{L}_{\mathbf{0}}(V_\infty)$ is usually denoted by $\mathcal{L}(V_\infty)$. Guichard [6] established that there are countably many automorphisms of $\mathcal{L}_{\mathbf{0}}(V_\infty)$ by showing that each computable automorphism is generated by a $1-1$ and onto computable semilinear transformation of V_∞. Recall that a map $\mu : V_\infty \to V_\infty$ is called a *semilinear transformation* of V_∞ if there is an automorphism σ of F such that

$$\mu(\alpha u + \beta v) = \sigma(\alpha)\mu(u) + \sigma(\beta)\mu(v)$$

for every $u, v \in V_\infty$ and every $\alpha, \beta \in F$.

Notation 2 *By* $GSL_{\mathbf{d}}$ *we denote the group of* $1-1$ *and onto semilinear transformations* $\langle \mu, \sigma \rangle$ *such that* $deg(\mu) \leq \mathbf{d}$ *and* $deg(\sigma) \leq \mathbf{d}$.

Hence Guichard proved that every element of $Aut(\mathcal{L}_{\mathbf{0}}(V_\infty))$ is generated by an element of $GSL_{\mathbf{0}}$. This result can be relativized to an arbitrary Turing degree $\mathbf{d}$. The proof of Theorem 3 below is essentially identical to the proof in [6].

Theorem 3. *([6]) Every* $\Phi \in Aut(\mathcal{L}_{\mathbf{d}}(V_\infty))$ *is generated by some* $\langle \mu, \sigma \rangle \in GSL_{\mathbf{d}}$. *Moreover if* Φ *is also generated by some other* $\langle \mu_1, \sigma_1 \rangle \in GSL_{\mathbf{d}}$, *then there is* $\gamma \in F$ *such that*

$$(\forall v \in V_\infty)\,[\mu(v) = \gamma\mu_1(v)].$$

There are two main results in this paper. The first is Theorem 4 in Section 2, which establishes that for every pair $\mathbf{a}$, $\mathbf{b}$ of Turing degrees, we have $Aut(\mathcal{L}_{\mathbf{a}}(V_\infty)) \hookrightarrow Aut(\mathcal{L}_{\mathbf{b}}(V_\infty))$ if and only if $\mathbf{a} \leq \mathbf{b}$. The second main result is Theorem 7 in Section 3, which establishes that the isomorphism degree type of the group $GSL_{\mathbf{d}}$ is $\mathbf{d}''$.

2 Group embeddings and Turing reducibility

Morozov showed that the correspondence $\mathbf{a} \to Sym_{\mathbf{a}}(\omega)$ can be used to substitute Turing reducibility with group-theoretic embedding. More precisely, Morozov [11] established that

$$(Sym_{\mathbf{a}}(\omega) \hookrightarrow Sym_{\mathbf{b}}(\omega)) \Leftrightarrow (\mathbf{a} \leq \mathbf{b})$$

for every pair $\mathbf{a}$, $\mathbf{b}$ of Turing degrees. It follows from this result that $\mathbf{a} = \mathbf{b}$ if and only if $Sym_{\mathbf{a}}(\omega) \cong Sym_{\mathbf{b}}(\omega)$. Here, we establish an analogous result for the subgroups of the group of automorphisms of the corresponding sublattices of $\mathcal{L}$. In the proof of the next, main theorem we will use the standard notation: $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$.

Theorem 4. *For any pair of Turing degrees* $\mathbf{a}, \mathbf{b}$ *we have*

$$(Aut(\mathcal{L}_{\mathbf{a}}(V_\infty)) \hookrightarrow Aut(\mathcal{L}_{\mathbf{b}}(V_\infty))) \Leftrightarrow \mathbf{a} \leq \mathbf{b}.$$

Proof. Obviously, if $\mathbf{a} \leq \mathbf{b}$, then $Aut(\mathcal{L}_{\mathbf{a}}(V_\infty)) \hookrightarrow Aut(\mathcal{L}_{\mathbf{b}}(V_\infty))$.

Now, assume that $Aut(\mathcal{L}_{\mathbf{a}}(V_\infty)) \hookrightarrow Aut(\mathcal{L}_{\mathbf{b}}(V_\infty))$. Let $\{e_0, e_1, \ldots\}$ be a fixed computable basis of V_∞. For $\langle \mu_1, \sigma_1 \rangle, \langle \mu_2, \sigma_2 \rangle \in GSL_{\mathbf{a}}$, we define $\langle \mu_1, \sigma_1 \rangle \sim \langle \mu_2, \sigma_2 \rangle$ iff:

(1) $\sigma_1 = \sigma_2$, and

(2) there is $\alpha \in F$ such that $\alpha \neq 0$ and $(\forall v \in V_\infty)\,[\mu_1(v) = \alpha\mu_2(v)]$.

Note that $Aut(\mathcal{L}_{\mathbf{a}}(V_\infty)) \cong GSL_{\mathbf{a}}/\sim$. We can define a group embedding $\delta : Sym_{\mathbf{a}}(\omega) \hookrightarrow GSL_{\mathbf{a}}/\sim$ as follows. For any $f \in Sym_{\mathbf{a}}(\omega)$, we let $\delta(f)$ be the $\sim$ -equivalence class of a linear transformation $\langle \widetilde{f}, id \rangle$ such that

$$\widetilde{f}(e_i) = e_{f(i)}.$$

Note that if $\delta(f_1) = \delta(f_2)$, then $\widetilde{f_1} = c\widetilde{f_2}$ for some $c \in F$, and thus

$$(\forall i \in \omega)\,[e_{f_1(i)} = \widetilde{f_1}(e_i) = c\widetilde{f_2}(e_i) = ce_{f_2(i)}].$$

Since the vectors e_i, $i \in \omega$, are independent, we must have

$$(\forall i \in \omega)\,[f_1(i) = f_2(i)].$$

Therefore, there exists a map

$$K : Sym_{\mathbf{a}}(\omega) \hookrightarrow GSL_{\mathbf{b}}/\sim$$

such that if $f \in Sym_{\mathbf{a}}(\omega)$, then $K(f)$ is a **b**-computable linear transformation of V_∞ modulo scalar multiplication.

We claim that if a set A is c.e. in **a**, then A is c.e. in **b**. Fix $A \subseteq \omega$ such that A is c.e. in **a**, and let $h : \omega \to \omega$ be an **a**-computable enumeration of A. Hence $rng(h) = A$. Fix a partition of the natural numbers into uniformly computable infinite sets R_i for $i \in \mathbb{Z}$ with enumerations $R_i = \{c_i^0 < c_i^1 < \cdots\}$. Let the permutations $g_0, g_1, w, b \in Sym_{\mathbf{a}}(\omega)$ be defined as follows:

$w(c_i^j) = c_{i+1}^j$ for each $i \in Z$ and $j \in \omega$,

$g_0 = \prod_{j\in\omega} (c_0^{2j}, c_0^{2j+1})$,

$g_1 = \prod_{j\in\omega} (c_0^{2j+1}, c_0^{2j+2})$, and

$b = \prod_{n,t\in\omega \,\wedge\, h(t)=n} (c_n^t, c_n^{t+1})$.

We will also use the following abbreviation: $w^n = \underbrace{w \cdots w}_{n \text{ times}}$. Then we have

$$n \notin A \Leftrightarrow \Big([g_0, b^{w^n}] = 1 \wedge [g_1, b^{w^n}] = 1 \Big).$$

This is because g_0 and b^{w^n} commute iff n is not enumerated into A at an odd stage t, and, similarly, g_1 and b^{w^n} commute iff n is not enumerated into A at an even stage t. Hence for $\widetilde{g_0} = K(g_0)$, $\widetilde{g_1} = K(g_1)$, $\widetilde{w} = K(w)$, and $\widetilde{b} = K(b)$, we have

$$n \notin A \Leftrightarrow \Big([K(g_0), K(b)^{(K(w))^n}] = 1 \wedge [K(g_1), K(b)^{(K(w))^n}] = 1 \Big)$$
$$\Leftrightarrow \Big([\widetilde{g_0}, \widetilde{b}^{\widetilde{w}^n}]_{/\sim} = 1 \wedge [\widetilde{g_1}, \widetilde{b}^{\widetilde{w}^n}]_{/\sim} = 1 \Big)$$

We will now show that $[\widetilde{g_0}, \widetilde{b}^{\widetilde{w}^n}] \nsim 1$ is c.e. relative to **b**. Let $\tau_n =_{def} [\widetilde{g_0}, \widetilde{b}^{\widetilde{w}^n}]$. Then

$$\tau_n \nsim 1 \Leftrightarrow \tau_n(e_0) \text{ and } e_0 \text{ are linearly independent, or}$$
$$(\exists m \in \omega)\, (\exists \alpha \neq 0)\, [\tau_n(e_0) = \alpha e_0 \wedge \tau_n(e_m) \neq \alpha e_m].$$

Let $A \in \mathbf{a}$. Then A and $\overline{A}$ are both c.e. in **b**, and, therefore, A is computable in **b**. Hence $\mathbf{a} \leq \mathbf{b}$. □

3 Complexity of $GSL_{\mathbf{d}}$

In this section we will determine the Turing degree spectrum of $GSL_{\mathbf{d}}$. For the statement of the main theorem we will use terminology and notation from the following definition.

Definition 1. *A permutation p on a set M is:*

(i) $1_{\inf}2_{\inf}$ on M if it is a product of infinitely many 1-cycles and infinitely many 2-cycles;

(ii) $1_{\inf}2_{fin}$ on M if it is a product of infinitely many 1-cycles and finitely many 2-cycles.

The main theorem about the degree spectrum of $GSL_{\mathbf{d}}$ will be derived from the following embeddability theorem.

Theorem 5. *Let G be an X-computable group, and let $H : Sym_{\mathbf{0}}(\omega) \hookrightarrow G$ be an embedding. Suppose that for every $1_{\inf}2_{\inf}$ permutation $p \in Sym_{\mathbf{0}}(\omega)$, the image $H(p)$ is not a conjugate of the image of any $1_{\inf}2_{fin}$ permutation in $Sym_{\mathbf{0}}(\omega)$.*

Then $\mathbf{0}'' \leq \deg(X)$.

Proof. Let A be a Π^0_2-complete set and let $R(x,t)$ be a computable predicate such that

$$n \in A \Leftrightarrow (\exists^\infty t)\, R(n,t).$$

We will prove that $A \leq_T X$. Fix a partition of the natural numbers into uniformly computable infinite sets $S_{i,j}$ for $i \in \mathbb{Z}$ and $j \in \{1,2\}$ with enumerations $S_{i,j} = \{c^0_{i,j} < c^1_{i,j} < \cdots\}$. The sets $S_{i,1}$ and $S_{i,2}$ will be referred to as the left and the right parts of the i-th column, $S_i = S_{i,1} \cup S_{i,2}$. This reference will be useful when we define certain maps below. We can graphically present this partition as follows:

$$\cdots\;
\underbrace{\begin{array}{|c|c|}\hline \vdots & \vdots \\ c^2_{-1,1} & c^2_{-1,2} \\ c^1_{-1,1} & c^1_{-1,2} \\ c^0_{-1,1} & c^0_{-1,2} \\ \hline S_{-1,1} & S_{-1,2} \\ \hline \end{array}}_{\text{Column } S_{-1}}
\;
\underbrace{\begin{array}{|c|c|}\hline \vdots & \vdots \\ c^2_{0,1} & c^2_{0,2} \\ c^1_{0,1} & c^1_{0,2} \\ c^0_{0,1} & c^0_{0,2} \\ \hline S_{0,1} & S_{0,2} \\ \hline \end{array}}_{\text{Column } S_{0}}
\;
\underbrace{\begin{array}{|c|c|}\hline \vdots & \vdots \\ c^2_{1,1} & c^2_{1,2} \\ c^1_{1,1} & c^1_{1,2} \\ c^0_{1,1} & c^0_{1,2} \\ \hline S_{1,1} & S_{1,2} \\ \hline \end{array}}_{\text{Column } S_{1}}
\;\cdots$$

We will now define the following maps.

(i) $w(c^k_{i+1,j}) =_{def} c^k_{i,j}$ for each $i \in \mathbb{Z}$, $k \in \omega$ and $j = 1, 2$.

Clearly, the map w is such that $w(S_{i+1,1}) = S_{i,1}$ and $w(S_{i+1,2}) = S_{i,2}$. It maps the left (right) part of the $(i+1)$-st column to the left (right) part of the i-th column for each i.

(ii) $p_0 =_{def} \prod\limits_{k \in \omega} (c^k_{0,1}, c^k_{0,2})$

It is easy to see that the map p_0 switches the left and right parts of the 0-th column (i.e., $p_0(S_{0,1}) = S_{0,2}$ and $p_0(S_{0,2}) = S_{0,1}$), and is identity on all other elements of ω.

(iii) $p_n =_{def} p_0^{w^n} = w^{-n} p_0 w^n$

Note that the map p_n switches the left and right parts of the n-th column (i.e., $p_n(S_{n,1}) = S_{n,2}$ and $p_n(S_{n,2}) = S_{n,1}$), and is identity on all other elements of ω.

(iv) $z(k) =_{def} \begin{cases} 0 & \text{if } k = 0, \\ 1 & \text{if } k = 2, \\ k-2 & \text{if } k = 2t \geq 4, \\ k+2 & \text{if } k = 2t+1. \end{cases}$

Note that the map z is a permutation of ω, which contains only one infinite cycle and (0).

(v) $\tau =_{def} (0,1)$
For $k \in \mathbb{Z}$ we have

$$\tau^{z^k} = \begin{cases} (0,2k), & \text{if } k \geq 1, \\ (0,2\,|k|+1), & \text{if } k \leq 0, \end{cases}$$

so

$$(\forall n,m \in \omega)\,(\exists n_1,m_1 \in \mathbb{Z})\,[\left(\tau^{z^{n_1}}\right)^{\tau^{z^{m_1}}} = (n,m)]. \tag{1}$$

Note that property (1) guarantees that any $1_{\inf}2_{\text{fin}}$ permutation on ω can be represented as a finite product of the permutations τ and z.

(vi) We will now construct a permutation b on ω with the following properties:

$b\restriction_{S_{n,1}} = id\restriction_{S_{n,1}}$

$b\restriction_{S_{n,2}}$ is $\begin{cases} 1_{\inf}2_{\inf}, & \text{if } n \in A, \\ 1_{\inf}2_{\text{fin}}, & \text{if } n \notin A. \end{cases}$

We will define b in stages. At each stage s we will have $E^s =_{def} dom(b^s) = rng(b^s)$.

Construction

Stage 0.

Let $b^0 \restriction S_i =_{def} id$ for $i \leq -1$, and $E^0 = \bigcup_{i\leq -1} S_i$.

Stage $s+1 = \langle n,t\rangle$.

Case 1. If $R(n,t)$, then find the least elements $p,q,r \in S_{n,2}$ such that $p,q,r \notin E^s$. Let $b^{s+1} = b^s \cdot (p,q)$ and assume that $b^{s+1}(r) = r$. Thus, we have $E^{s+1} = E^s \cup \{p,q,r\}$ and $b^{s+1} \restriction E^s = b^s$.

Case 2. If $\neg R(n,t)$, then find the least elements $p,q,r \in S_{n,2}$ such that $p,q,r \notin E^s$. Let $b^{s+1} \restriction E^s = b^s$ and $b^{s+1}(p) = p$, $b^{s+1}(q) = q$, $b^{s+1}(r) = r$. Then $E^{s+1} = E^s \cup \{p,q,r\}$.

End of construction.

By construction, $dom(b) = rng(b) = \omega$.

It follows that if $n \in A$, then $(\exists^\infty t)\,R(n,t)$, so Case 1 applies infinitely often for this n, and hence the map b switches infinitely many pairs in the right part of the n-th column. Therefore, $b \restriction S_{n,2}$ is $1_{\inf}2_{\inf}$ and $b \restriction S_{n,1} = id$.

If $n \notin A$, then $(\exists^{<\infty} t)\,R(n,t)$, so Case 1 applies finitely often for this n, and hence the map b switches only finitely many pairs in the right part of the n-th column. Therefore, $b \restriction S_{n,2}$ is $1_{\inf}2_{\text{fin}}$ and $b \restriction S_{n,1} = id$.

In both cases, the map b^{p_n} reverses the action of b on the left and right parts of the n-th column S_n, while for $k \neq n$, we have $b^{p_n} \restriction S_k = b \restriction S_k$.

Then $b \cdot b^{p_n}$ is $\begin{cases} 1_{\inf}2_{\inf} \text{ on } S_n, & \text{if } n \in A, \\ 1_{\inf}2_{\text{fin}} \text{ on } S_n, & \text{if } n \notin A, \\ id \text{ on } S_k, & \text{if } n \neq k. \end{cases}$

Therefore, $b \cdot b^{p_n}$ is $\begin{cases} 1_{\inf}2_{\inf} \text{ on } \omega, & \text{if } n \in A, \\ 1_{\inf}2_{\text{fin}} \text{ on } \omega, & \text{if } n \notin A. \end{cases}$

Finally, note that on ω, every computable $1_{\inf}2_{\inf}$ permutation is the conjugate of a fixed computable $1_{\inf}2_{\inf}$ permutation and some other computable permutation. Therefore, assume that f is a fixed computable $1_{\text{inf}}2_{\text{inf}}$ permutation such that:

$$(\forall z_1 \in Sym_{\mathbf{0}}(\omega))\,(\exists h \in Sym_{\mathbf{0}}(\omega))\,[z_1 = f^h].$$

Hence for every n, we have

$$\begin{aligned} n \in A &\Leftrightarrow b \cdot b^{p_n} \text{ is a } 1_{\text{inf}}2_{\text{inf}} \text{ permutation on } \omega \\ &\Leftrightarrow (\exists h \in Sym_{\mathbf{0}}(\omega))\,[b \cdot b^{p_n} = f^h] \\ &\Leftrightarrow (\exists u \in H(Sym_{\mathbf{0}}(\omega)))\,[H(b) \cdot H(b)^{H(p_n)} = H(f)^u], \text{ and} \end{aligned} \tag{2}$$

$$\begin{aligned} n \notin A &\Leftrightarrow b \cdot b^{p_n} \text{ is a } 1_{\text{inf}}2_{\text{fin}} \text{ permutation on } \omega \\ &\Leftrightarrow b \cdot b^{p_n} = \prod_{(i,j)\in F} \left(\tau^{z^i}\right)^{\tau^{z^j}} \\ &\Leftrightarrow H(b) \cdot H(b)^{H(p_n)} = \prod_{(i,j)\in F} \left(H(\tau)^{H(z)^i}\right)^{H(\tau)^{H(z)^j}}. \end{aligned} \tag{3}$$

The set F in the last line of (3) denotes some finite set of pairwise disjoint cycles and the maps referenced in (2) and (3) are those that we defined in (i)–(vi) above. For the map $H : Sym_{\mathbf{0}}(\omega) \hookrightarrow G$ note that $H(p_n) = H(w)^{-n} \cdot H(p_0) \cdot H(w)^n$.

We claim that the last equivalence in (2) can be strengthened so that we have:

$$n \in A \Leftrightarrow (\exists u \in G)\,[H(z) = H(f)^u] \tag{4}$$

That is, if $n \in A$, then

$$\begin{gathered} (\exists h \in Sym_{\mathbf{0}}(\omega))\,[b \cdot b^{p_n} = f^h] \text{ and, therefore,} \\ (\exists u \in H(Sym_0(\omega)))\,[H(b) \cdot H(b)^{H(p_n)} = H(f)^u], \text{ and} \\ (\exists u \in G)\,[H(b) \cdot H(b)^{H(p_n)} = H(f)^u]. \end{gathered}$$

For the proof of the other direction of (4) suppose that for some fixed $h \in G$ we have

$$H(b) \cdot H(b)^{H(p_n)} = H(f)^h, \text{ but } n \notin A.$$

Then, because of (3), we have the following:

(i) $b \cdot b^{p_n}$ is a $1_{\text{inf}}2_{\text{fin}}$ permutation on ω,

(ii) $H(b) \cdot H(b)^{H(p_n)}$ is the image of the $1_{\text{inf}}2_{\text{fin}}$ permutation $b \cdot b^{p_n}$, while

(iii) $H(f)$ is the image of the $1_{\text{inf}}2_{\text{inf}}$ permutation f.

This contradicts our assumption that the image under H of the $1_{\text{inf}}2_{\text{fin}}$ permutation $b \cdot b^{p_n}$ cannot be the conjugate of the image of the $1_{\text{inf}}2_{\text{inf}}$ permutation f.

Theorem 6. *The degree of the isomorphisms type of the group* $GSL_{\mathbf{0}}$ *is* $\mathbf{0}''$.

Proof. Let $V = \{v_0, v_1\ , \ldots\}$ be a computable basis of V_∞. Define

$$H : Sym_{\mathbf{0}}(\omega) \hookrightarrow GSL_{\mathbf{0}}$$

so that for any $p \in Sym_{\mathbf{0}}(\omega)$ the image $H(p) = \langle L, id\rangle$ is a semilinear map such that

$$L(v_i) = v_{p(i)} \text{ for every } i \in \omega.$$

We claim that under H the image of a $1_{\text{inf}}2_{\text{inf}}$ permutation from $Sym_{\mathbf{0}}(\omega)$ cannot be a conjugate of the image of a $1_{\text{inf}}2_{\text{fin}}$ permutation from $Sym_{\mathbf{0}}(\omega)$. To establish this fact, suppose that $\langle f, id\rangle$, $\langle f_1, id\rangle \in GSL_{\mathbf{0}}$ are the images of some $1_{\text{inf}}2_{\text{inf}}$ and $1_{\text{inf}}2_{\text{fin}}$ computable permutations on ω, respectively. Suppose that $\langle f, id\rangle$ and $\langle f_1, id\rangle$ are conjugates, and let $\langle h, \sigma\rangle \in GSL_{\mathbf{0}}$ be such that $\langle f, id\rangle^{\langle h,\sigma\rangle} = \langle f_1, id\rangle$. Note that the map $h : V_\infty \to V_\infty$ is $1-1$ and onto. The associated field automorphism $\sigma : F \to F$ from $\langle h, \sigma\rangle$ is used to indicate that $h(av + bw) = \sigma(a)\, h(v) + \sigma(b)\, h(w)$. To simplify the notation, we will refer to the semilinear maps $\langle f, id\rangle$, $\langle f_1, id\rangle$, and $\langle h, \sigma\rangle$ simply as f, f_1, and h, respectively.

Note that the definition of the map H allows us to view $f \restriction V$ and $f_1 \restriction V$ as $1_{\text{inf}}2_{\text{inf}}$ and $1_{\text{inf}}2_{\text{fin}}$ permutations on V, respectively. We now claim that f_1 satisfies the following property:

$$(\exists W \subset_{fin} V_\infty)\, (\forall v \in V_\infty)\, [(v - f_1(v)) \in W]. \tag{5}$$

Here, $W \subset_{fin} V_\infty$ stands for W being a finite-dimensional subspace of V_∞. For a set $U \subseteq V_\infty$, by $cl(U)$ we will denote the closure of U, which is the set of all linear combinations of the vectors in U. To prove (5), assume that $B = \{x_1, \ldots, x_k, y_1, \ldots, y_k\} \subseteq V$ is such that $f_1 \restriction V = \prod\limits_{1\le i\le k} (x_i, y_i)$. Note that for every $v \in V_\infty$, there are $v_1 \in cl(V - B)$ and $v_2 \in cl(B)$ such that $v = v_1 + v_2$. Then

$$f_1(v) = f_1(v_1) + f_1(v_2) = v_1 + f_1(v_2),$$

and so

$$v - f_1(v) = v_1 + v_2 - v_1 - f_1(v_2) = v_2 - f_1(v_2) \in cl(B),$$

because $f_1(v_2) \in cl(B)$. Therefore, $W = cl(B)$ is a finite-dimensional subspace of V_∞ for which property (D) holds.

We will now prove that f^h does not satisfy property (5), which will contradict the assumption that $f^h = f_1$. Thus, assume that W is a finite-dimensional subspace of V_∞ such that

$$(\forall x \in V_\infty)\, [\big(x - f^h(x)\big) \in W]. \tag{6}$$

The *support* of a vector x with respect to a basis $Z = \{z_j : j \in J\}$, denoted by $supp_Z(x)$, is the set $\{z_{j_l} : l \in \{0, \ldots, t\}\}$ such that

$$x = \sum_{l=0}^{t} \lambda_l z_{j_l}$$

and $(\forall l \in \{0, ..., t)[\lambda_l \neq 0]$.

Let $W_1 = h(W)$ and note that W_1 is finite-dimensional. Let B_1 be a finite subset of the basis V such that

$$(\forall x \in W_1)\,[supp_V(x) \subseteq B_1].$$

We will now find $u_1 \in V_\infty$ such that $u_1 - f(u_1) \notin W_1$. Since $f \restriction V$ is a $1_{\text{inf}}2_{\text{inf}}$ permutation on V, there are infinitely many pairs $(u, v) \in V \times V$ such that

$$u \neq v, \;\; f(u) = v \text{ and } f(v) = u. \tag{7}$$

Since B_1 is finite, we can also find $u_1, v_1 \in V - B_1$, which have property (7). Then:

(i) $u_1 - f(u_1) = u_1 - v_1 \neq 0$, and

(ii) $u_1 - f(u_1) = (u_1 - v_1) \notin cl(B_1)$ because $B_1 \cup \{u_1, v_1\} \subseteq V$.

Since $W_1 \subseteq cl(B_1)$, we have that $u_1 - f(u_1) \notin W_1$. Therefore,

$$\big(h^{-1}(u_1) - h^{-1}(f(u_1))\big) \notin h^{-1}(W_1), \text{ and so}$$
$$\big(h^{-1}(u_1) - h^{-1}fhh^{-1}(u_1)\big) \notin W.$$

If we let $x_1 = h^{-1}(u_1)$, we obtain

$$x_1 - f^h(x_1) \notin W,$$

which contradicts that f^h satisfies (6).

We constructed an embedding $H : Sym_{\mathbf{0}}(\omega) \hookrightarrow GSL_{\mathbf{0}}$ such that the images of any $1_{\text{inf}}2_{\text{inf}}$ and $1_{\text{inf}}2_{\text{fin}}$ permutations from $Sym_{\mathbf{0}}(\omega)$ cannot be conjugates in $GSL_{\mathbf{0}}$. We use Theorem 5 to conclude that $\mathbf{0}''$ is computable in any copy of $GSL_{\mathbf{0}}$. We can construct a copy of $GSL_{\mathbf{0}}$, which is computable in $\mathbf{0}''$. Therefore, the degree of the isomorphisms type of $GSL_{\mathbf{0}}$ is $\mathbf{0}''$.

Note that the result of the previous theorem can be easily relativized to any Turing degree $\mathbf{d}$.

Theorem 7. *The degree of the isomorphisms type of the group $GSL_{\mathbf{d}}$ is $\mathbf{d}''$.*

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