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Orders on magmas and computability theory

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ABSTRACT

We investigate algebraic and computability-theoretic properties of orderable magmas. A magma is an algebraic structure with a single binary operation. A right order on a magma is a linear ordering of its domain, which is right-invariant with respect to the magma operation. We use tools of computability theory to investigate Turing complexity of orders on computable orderable magmas. A magma is computable if it is finite, or if its domain can be identified with the set of natural numbers and the magma operation is computable. Interesting orderable magmas that are not even associative come from knot theory.

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1. Introduction and Preliminaries

Orderable structures play an important role in several areas of mathematics. They have been studied extensively by algebraists [18, 25, 19, 12] and more recently by topologists and computability theorists [31, 27, 28, 32, 7, 8, 5]. All surface groups, with the exception of the projective plane and the Klein bottle groups, are bi-orderable. All knot groups are left-orderable and many are bi-orderable. Surprisingly many fundamental groups of 3-manifolds are left-orderable or even bi-orderable.

In this paper, we apply the techniques of computability theory to further analyze the spaces of orders on orderable magmas. A *magma* M is a non-empty set with a binary operation $\cdot : M \times M \to M$. A binary relation R on M is a right order

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on $(M, \cdot)$ if R is a strict linear (i.e. total) ordering on the domain M and is right-invariant under the magma operation:

$$(\forall\, a, b, c \in M)[(a, b) \in R \Rightarrow (a \cdot c, b \cdot c) \in R].$$

Similarly, we define a left order on $(M, \cdot)$. A binary relation R on M is a bi-order on $(M, \cdot)$ if R is a linear ordering on the domain M and

$$(\forall\, a, b, c \in M)[(a, b) \in R \Rightarrow ((a \cdot c, b \cdot c) \in R \wedge (c \cdot a, c \cdot b) \in R)].$$

By $RO(M)$ we denote the set of all right orders on $(M, \cdot)$, by $LO(M)$ the set of all left orders, and by $BiO(M)$ the set of all bi-orders. Clearly, $BiO(M) = RO(M) \cap LO(M)$.

For a group G, every left order $<_l$ on G induces the associated right order $<_r$ on G defined by

$$a <_r b \Leftrightarrow b^{-1} <_l a^{-1}.$$

It is easy to see that if G is a left-orderable group, then G must be torsion-free. The converse is not true — there are torsion-free groups that are not left-orderable. However, groups from many important classes, such as torsion-free abelian groups or, more generally, torsion-free nilpotent groups, are bi-orderable. There are left-orderable groups that are not bi-orderable — an example of such a group is the fundamental group of the Klein bottle. Linnell [20] showed that a left-orderable group has either finitely or uncountably many left orders. On the other hand, there is a bi-orderable group with countably infinitely many bi-orders (see [2]).

Orders are often obtained by extending partial orders. A binary relation $\prec$ on a magma $(M, \cdot)$ is a *partial left order* on $(M, \cdot)$ if $\prec$ is a partial ordering on the domain M and

$$(\forall\, a, b, c \in M)[a \prec b \Rightarrow (c \cdot a \text{ and } c \cdot b \text{ are comparable}) \wedge c \cdot a \prec c \cdot b].$$

We define a topology on $LO(M)$ by choosing as a subbasis the collection

$$\mathcal{S} = \{S_{(a,b)}\}_{(a,b) \in (M \times M) - \Delta_M},$$

where $S_{(a,b)} = \{R \in LO(M) \mid (a, b) \in R\}$ and $\Delta_M = \{(a, a) \mid a \in M\}$ (see [28, 7]). The space $LO(M)$ is zero-dimensional, that is, it is a T_1-space with a clopen basis. The space $BiO(M)$ inherits the same topology from $LO(M)$. The authors of [7] showed that $BiO(M)$ is a closed subspace of $LO(M)$. They also proved that if a magma M has cardinality $\mathfrak{m}$, then $LO(M)$ is a compact space that can be embedded into the Cantor cube $\{0, 1\}^{\mathfrak{m}}$. In the case of a group G, they related their result, using Alexander's subbase theorem, to the classical theorem of Conrad giving a criterion for a partial left order to extend to a left order on G.

Clearly, for a computable torsion-free abelian group of rank 1, the space of bi-orders has exactly two elements. Sikora [28] established that for finite $n \geq 2$, the space $BiO(\mathbb{Z}^n)$ is homeomorphic to the Cantor set. Let $\mathbb{Z}^\omega$ be $\bigoplus_{i \in \omega} \mathbb{Z}$, the direct sum of ω copies of $\mathbb{Z}$. Dabkowska [6] proved that the space $BiO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set. Later, Chubb [4] obtained this result as a corollary of a result in [10].

In 1936, Turing introduced a precise notion of an abstract computer, which formalized the intuitive notion of an algorithm. By algorithmically enumerating all Turing machine programs we obtain for any $k \geq 1$ an algorithmic enumeration of all k-ary partial computable functions:

$$\varphi_0^{(k)}, \varphi_1^{(k)}, \ldots, \varphi_n^{(k)}, \ldots.$$

For $k = 1$, we omit the superscript. We can show, using a diagonal argument, that it is not possible to algorithmically enumerate all total functions in this list. The halting set, denoted by $\emptyset'$, is defined as $\emptyset' = \{n : n \in \mathrm{dom}(\varphi_n)\}$. The halting set is also called the *first Turing jump* of the empty set. Turing showed that the halting set is not computable. There are only countably many computable sets.

Turing later introduced the notion of *relative computability.* Relative computability can be formalized by a Turing machine augmented with an additional set called an *oracle* set. This supplies additional information on demand by disclosing its elements and non-elements (thus, such a Turing machine performs finitely many non-algorithmic steps). Let A, B be sets. A set A is *B-computable* if A can be computed by a machine with oracle B. We also say that A is Turing reducible to B, or that A is computable relative to B, and write $A \leq_T B$ or $\deg(A) \leq \deg(B)$. Relative computability provides us with a fine comparison of the sets of natural numbers. For a set X, we have an algorithmic enumeration of all partial X-computable unary functions:

$$\varphi_0^X, \varphi_1^X, \ldots, \varphi_n^X, \ldots$$

We have a similar enumeration for all partial X-computable k-ary functions.

We say that sets A and B have the same *Turing degree* if and only if A is B-computable and B is A-computable. We write $\deg(A) = \deg(B)$, or $A \equiv_T B$ where $\equiv_T$ stands for Turing equivalence of sets. For example, any set and its complement have the same Turing degree. There are $2^{\aleph_0}$ subsets of natural numbers and hence $2^{\aleph_0}$ Turing degrees. The first Turing jump of a set X is defined as $X' = \{n : n \in \mathrm{dom}(\varphi_n^X)\}$. The Turing degree of X is denoted by $\mathbf{x} = \deg(X)$, and the Turing degree of X' by $\mathbf{x}'$. In particular, $\mathbf{0}'$ denotes the Turing degree of the halting set $\emptyset'$. We have $\mathbf{x} < \mathbf{x}'$. We can iterate Turing jump. The nth Turing jump of the empty set is $\emptyset^{(n)}$, and let $\mathbf{0}^{(n)} = \deg(\emptyset^{(n)})$. A set X and its Turing degree $\mathbf{x}$ are called *low* if $X' \leq_T \emptyset'$, hence $\mathbf{x}' = \mathbf{0}'$. There are non-computable low sets. We will denote by $\mathcal{D}$ the set of all Turing degrees. The partially ordered set $\mathcal{D}$ is an upper semilattice. An upper semilattice is a partially ordered set with suprema but not necessarily with infima. For more on computability-theoretic notions see [29, 30].

A countable magma $(M, \cdot)$ is *computable* if its domain M is a computable set and its operation $\cdot$ is computable. Equivalently, the atomic diagram of $(M, \cdot)$ is computable. The atomic diagram of a structure is the set of all atomic sentences and negations of atomic sentences in the language of the structure augmented with new constant symbols for all elements of the domain. For any infinite computable magma we may assume, without loss of generality, by using algorithmic coding by natural numbers, that its domain is the set of all natural numbers. Computability-theoretic properties are not necessarily preserved under isomorphisms — they are preserved under computable isomorphisms. For more on computable structures see [13, 1, 11].

A computable isomorphic copy of $(\mathbb{Z}, +)$ has exactly two orders and they are computable. Moreover, if G is a computable group, then $\prec_r$ and $\prec_l$ have the same Turing degree. Solomon [32] proved that for $n \geq 2$, a computable group isomorphic to $(\mathbb{Z}^n, +)$ has an order in every Turing degree. On the other hand, Downey and Kurtz [10] constructed a computable group isomorphic to $(\mathbb{Z}^\omega, +)$ with no computable order. Dobritsa [9] previously showed that every computable, torsion-free, abelian group is isomorphic to a computable group with a computable order. Recently, Harrison–Trainor [14] proved that there is a computable left-orderable group that is not isomorphic to a computable group with a computable left order. It is not known whether this is true for the case of bi-orderable groups. Solomon also proved that a computable, torsion-free, abelian group G of infinite rank, has an order in every Turing degree $\mathbf{x} \geq \mathbf{0}'$, and that a computable, torsion-free, properly n-step nilpotent group has an order in every Turing degree $\mathbf{x} \geq \mathbf{0}^{(n)}$.

A group G is *free* if there is a set B of elements such that B generates G and there are no nontrivial relations on elements of B. We call B a *basis* for G. All bases for a free group G have the same cardinality, which we call the *rank* of G. We write F_n for the free group of rank n, and F_∞ for the free group of rank $\aleph_0$. The groups F_n, $n \geq 1$, and F_∞ all have computable isomorphic copies. The group F_1 is isomorphic to $(\mathbb{Z}, +)$, so it has two bi-orders. The groups F_n, $n \geq 2$, and F_∞ all have uncountably many bi-orders.

Dabkowska, Dabkowski, Harizanov and Togha proved in [8] that a computable free group G of a finite rank $n > 1$ has a bi-order in every Turing degree. The orders on G are constructed using orders on the quotients of the successive terms of the lower central series of G. Different choices of orders on the quotients of the lower central series yield different bi-orders on G and allow us to code a set of an arbitrary Turing degree into a bi-order on G. Chubb, Dabkowski and Harizanov [5] have recently generalized this result to degrees stronger than Turing degrees and to a large class of computable, finitely presented, residually nilpotent groups. This class includes a variety of important groups such as surface groups, certain nilpotent groups, certain finitely generated one-relator parafree groups, and right-angled Artin groups.

Sikora [28] conjectured that for a free group F_n of finite rank $n > 1$, the space $BiO(F_n)$ is homeomorphic to the Cantor set. It is known that the Cantor set can be embedded into $BiO(F_n)$. McCleary [22] established that for a free group F_n of finite rank $n > 1$, $LO(F_n)$ is homeomorphic to the Cantor set (for another, more recent proof by Navas see [26]). It still remains unknown whether $BiO(F_n)$ for finite $n > 1$ is homeomorphic to the Cantor set.

In Sec. 2, we give algebraic conditions equivalent to the order properties on an arbitrary magma. We then show that the space of orders of a computable magma is an effectively closed subspace of 2^ω, so we can transfer certain computability-theoretic results about effectively closed sets to the orders on magmas. In Sec. 3, we show that there is a computable free group of an infinite rank such that its conjugate quandle does not have a computable right order. As a corollary, we obtain that the space of right orders of this quandle is homeomorphic to the Cantor set.

2. Orders on Magmas and Spaces of Orders on Computable Magmas

Let X be a non-empty set and R, S be binary relations on X. As usual, define the inverse relation of R, $R^{-1} \subseteq X \times X$, as

$$R^{-1} = \{(b, a) \mid (a, b) \in R\},$$

and the composition $S \circ R \subseteq X \times X$ as

$$S \circ R = \{(x, z) : (\exists\, y)[(x, y) \in R \wedge (y, z) \in S]\}.$$

The diagonal of X, Δ_X, is defined by

$$\Delta_X = \{(a, a) : a \in X\}.$$

Let $(M_1, *_1), (M_2, *_2)$ be magmas. The direct product of these magmas, which we write as $(M_1, *_1) \times (M_2, *_2)$, is defined as usual. The underlying set is the Cartesian product $M_1 \times M_2$, and the binary operation $*$ is defined component-wise:

$$(a, b) * (c, d) = (a *_1 c, b *_2 d).$$

Let $(Q, *)$ be a magma and S, T be subsets of Q. The *product of subsets S and T* is defined by

$$ST = \{s * t : s \in S \wedge t \in T\}.$$

Let R be a left order on a magma $(M, *)$. Then the following properties are satisfied:

(1) $R \circ R \subseteq R$,
(2) $R \cap R^{-1} = \emptyset$,
(3) $R \cup R^{-1} = (M \times M) - \Delta_M$,
(4) $\Delta_M R \subseteq R$.

Lemma 2.1. *Properties* (1)–(4) *above hold if and only if R is a left order on* $(M, *)$.

Proof. Property 1 guarantees transitivity of R, Properties 2 and 3 guarantee that R is a linear ordering of the domain. Property 4, where the product is in $(M, *) \times (M, *)$, guarantees that R is left-invariant under the magma operation. □

For a right order R on $(M, *)$, instead of 4, we have

4′ $R\Delta_{M} \subseteq R$.

For a bi-order order R on $(M, *)$, instead of 4, we have

4″ $(\Delta_{M} R \cup R\Delta_{M}) \subseteq R$.

For a group G, a partial left order $\prec$ is determined by and often identified with its *positive partial cone*:

$$P = \{a \in G : e \preceq a\},$$

where $e \in G$ is the identity element. Similarly, the *negative partial cone* is defined as

$$P^{-1} = \{a \in G : a \preceq e\}.$$

We can easily verify that P is a *subsemigroup* of G (i.e. $PP \subseteq P$), which is *pure* (i.e. $P \cap P^{-1} = \{e\}$). Such a subsemigroup $P \subseteq G$ defines a left order on G if and only if P is *total* (i.e. $P \cup P^{-1} = G$). Moreover, P defines a bi-order on G if, in addition, P is a *normal* subsemigroup of G (i.e. $g^{-1}Pg \subseteq P$ for every $g \in G$). For more details see [19].

A subtree $\mathcal{S}$ of the full binary tree is a subset of $2^{<\omega}$, which is closed under initial segments. Such a tree $\mathcal{S}$ is computable if its set of nodes is computable. A subset of the Cantor set 2^ω is called *effectively closed* if it is the collection of infinite paths through a computable subtree of $2^{<\omega}$. There is a computable infinite binary tree without a computable infinite path. Metakides and Nerode [24] investigated orderable fields. A field is orderable exactly when it is formally real. Metakides and Nerode showed that the sets of all orders on computable formally real fields are in exact correspondence to the effectively closed subsets of 2^ω. Solomon [31] established that for every orderable computable group G, there is a computable binary branching tree $\mathcal{T}$ and a Turing degree preserving bijection from $BiO(G)$ to the set of all infinite paths of $\mathcal{T}$. Chubb [4] has a similar result for computable orderable semigroups. Here, we present such a result for any magma in general and investigate how certain computability-theoretic results for effectively closed sets can be transferred to orders on computable magmas.

Theorem 2.2. *Let $(M, \cdot)$ be an infinite right-orderable computable magma. Then there is a computable binary tree $\mathcal{T}_M$ such that the right orders on $(M, \cdot)$ exactly correspond to infinite paths in $\mathcal{T}_M$, and the Turing degree is preserved via a bijection between the right orders on $(M, \cdot)$ and the infinite paths of $\mathcal{T}_M$.*

Proof. Without loss of generality, we can assume that the domain M is the set of all natural numbers. Let $p_0, p_1, p_2, \ldots$ be a computable listing (enumeration) of all elements of $(M \times M) - \Delta_M$. For $p = (a, b)$, let $p^* = (b, a)$. We use characteristic functions of sets of pairs of elements of M to describe the infinite paths in the tree $\mathcal{T}$. We will show that there is a bijection between the right orders on $(M, \cdot)$ and the infinite paths of $\mathcal{T}$, and that the Turing degree is preserved. For a right order R on $(M, \cdot)$, define $f_R \in 2^\omega$ by

$$f_R(i) = \begin{cases} 1 & \text{if } R(a,b) \wedge p_i = (a,b), \\ 0 & \text{if } R(b,a) \wedge p_i = (a,b). \end{cases}$$

It is easy to see that R and f_R have the same Turing degree. For every $f \in 2^\omega$, define

$$R_f = \{p_i : f(i) = 1\} \cup \{p_i^* : f(i) = 0\}.$$

It is easy to see that f and R_f have the same Turing degree. Clearly, $R = R_{f_R}$ and $f = f_{R_f}$.

We will build $\mathcal{T}$ in stages. At every stage $s \geq 0$, we will have a finite tree $\mathcal{T}_s$ that will consist of all nodes of $\mathcal{T}$ of length $\leq s$. We will have $\mathcal{T}_s \subseteq \mathcal{T}_{s+1}$ and $\mathcal{T} = \bigcup_{s \geq 0} \mathcal{T}_s$. For every $\sigma \in \mathcal{T}$, we will define a finite label set S_σ, where $S_\sigma \subseteq M \times M$, which can be viewed as possibly determining a partial order on $(M, \cdot)$. For $f \in 2^\omega$ and $n \in \mathbb{N}$, by $f \upharpoonright n$ we denote $(f(0), \ldots, f(n))$.

Construction

Stage 0. Set $\mathcal{T}_0 = \{\langle\ \rangle\}$, the tree with the empty sequence $\langle\ \rangle$ (root) as its only member. Set $S_{\langle\ \rangle} = \emptyset$.

Stage $s + 1$. Consider each node σ of length s of $\mathcal{T}_s$. If S_σ contains (a, b) and (b, a) for any $a \neq b$, then that node is declared to be terminal, so it will not be further extended. Otherwise, extend σ by adding both $\sigma^\frown 0$ and $\sigma^\frown 1$ to $\mathcal{T}_{s+1}$. Add p_s to $S_{\sigma^\frown 1}$, and add p_s^* to $S_{\sigma^\frown 0}$. Furthermore, for $i = 0, 1$:

- add (s, s) to $S_{\sigma^\frown i}$,
- add every new (a, c) to $S_{\sigma^\frown i}$ such that for some b, both (a, b) and (b, c) are in S_σ,
- add all new (ac, bc) to $S_{\sigma^\frown i}$ for every (a, b) in S_σ and every $c \leq s$.

End of the construction. □

It follows from the construction that $\mathcal{T}$ is computable. We also have the following lemmas.

Lemma 2.3. *Let f be an infinite path in $\mathcal{T}$. Then R_f is a right order on $(M, \cdot)$.*

Proof. For every a, we have that $(a, a) \notin R_f$ by the definition of the sequence $p_0, p_1, p_2, \ldots$. Assume that $a, b \in M$ are such that $a \neq b$. There are $i, j \in \mathbb{N}$ such that $p_i = (a,b)$ and $p_j = p_i^*$. Then $f(i) = 1 - f(j)$ since no initial segment of f terminates at any stage. Hence exactly one of $(a,b) \in R_f$, $(b,a) \in R_f$ holds.

Now assume that $(a,b),(b,c) \in R_f$. Let i,k be such that $p_i = (a,b)$ and $p_k = (b,c)$. Then $f(i) = 1$ and $f(k) = 1$, so $(a,b) \in S_{f\restriction(i+1)}$ and $(b,c) \in S_{f\restriction(k+1)}$. Let $s = \max\{i+1, k+1\}$. Then $(a,c) \in S_{f\restriction(s+1)}$. Let $p_m = (a,c)$. Then $f(m) = 1$ and hence $(a,c) \in R_f$.

Let $(a,b) \in R_f$ and $c \in M$. Let i be such that $p_i = (a,b)$. Then $f(i) = 1$, so $(a,b) \in S_{f\restriction(i+1)}$. Let $s = \max\{i+1, c\}$. Then $(a \cdot c, b \cdot c) \in S_{f\restriction(s+1)}$. Let $p_l = (a \cdot c, b \cdot c)$. Then $f(l) = 1$ since no initial segment of f terminates at stage $l+1$. Hence $(a \cdot c, b \cdot c) \in R_f$. □

Lemma 2.4. *Let R be a right order of $(M,\cdot)$. Then f_R is an infinite path in $\mathcal{T}$.*

Proof. By induction on $n \geq 0$ we prove that for every n,

$$(f_R \restriction n) \in \mathcal{T} \wedge (S_{f_R\restriction n} - \Delta_M) \subseteq R.$$

Let $n = 0$. We have $(f_R \restriction 0) \in \mathcal{T}$ since $f_R \restriction 0 = f_R(0)$ and $(0),(1) \in \mathcal{T}$. Let $p_0 = (a,b)$. Then $S_{f_R\restriction 0} = \{(a,b),(0,0)\}$ if $f_R(0) = 1$, hence $(a,b) \in R$; and $S_{f_R\restriction 0} = \{(b,a),(0,0)\}$ if $f_R(0) = 0$, hence $(b,a) \in R$. Thus, $(S_{f_R\restriction 0} - \Delta_M) \subseteq R$.

Assume that the statement holds for n. Let $\sigma = f_R \restriction n$. Then, since $(S_\sigma - \Delta_M) \subseteq R$ and R is a strict linear ordering of M, there is no pair a,b such that $a \neq b$ and $(a,b),(b,a) \in S_\sigma$. Hence, by construction, both $\sigma^\frown 0$ and $\sigma^\frown 1$ are included in $\mathcal{T}$ at stage $n+1$. Hence $(f_R \restriction (n+1)) \in \mathcal{T}$. Either p_n or p_n^* belongs to R. Without loss of generality, assume $p_n \in R$. Then $f_R(n+1) = 1$ and $p_n \in S_{f_R\restriction(n+1)}$. Since R is closed under transitive closure and multiplication on the right by an element of M, we have $(S_{f_R\restriction(n+1)} - \Delta_M) \subseteq R$. □

Now, we define a function Ψ from the set of all right orders on $(M,\cdot)$ to the set of all infinite paths of $\mathcal{T}$ by $\Psi(R) = f_R$. Clearly, Ψ is a bijection and $\Psi^{-1}(f) = R_f$.

Hence, by the Low Basis Theorem of Jockusch and Soare [15], a computable right-orderable magma has a right order of low Turing degree. Similarly, it can be shown that for every left-orderable (bi-orderable, respectively) computable magma $(M,\cdot)$, there is a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection from $LO(M)$ ($BiO(M)$, respectively) to the set of all infinite paths of $\mathcal{T}$. Hence, a computable left-orderable (bi-orderable, respectively) magma has a left order (bi-order, respectively) of low Turing degree.

In the next three propositions, an order stands for either a right order, a left order, or a bi-order. It is not hard to see that an isolated path in a computable binary tree must be computable. The following proposition follows from Theorem 2.2 and results about the spaces of orders in [7].

Proposition 2.5. *Let $(M,\cdot)$ be a computable orderable magma.*

(i) *If $(M,\cdot)$ has only finitely many orders, then they must be all computable.*

(ii) *If $(M, \cdot)$ has countably infinitely many orders, then it has infinitely many computable orders.*
(iii) *If $(M, \cdot)$ does not have a computable order, then the space of orders on $(M, \cdot)$ is homeomorphic to the Cantor set.*

Using Theorem 2.2 and results of Jockusch and Soare about effectively closed sets from [15], we obtain the following complexity results about orders. For a definition of a hyperimmune-free Turing degree see [29, 30].

Proposition 2.6. *Let $(M, \cdot)$ be an infinite computable orderable magma.*

(i) *Then $(M, \cdot)$ has an order of* low *Turing degree.*
(ii) *Then $(M, \cdot)$ has an order of* hyperimmune-free *Turing degree.*
(iii) *Assume that $(M, \cdot)$ does not have a computable order. Let $(\mathbf{d}_i)_{i \in I}$ be any countable sequence of non-computable (i.e. nonzero) Turing degrees. Then $(M, \cdot)$ has $2^{\aleph_0}$ orders of mutually incomparable Turing degrees such that they are also incomparable with each $\mathbf{d}_i$ for $i \in I$.*
(iv) *Then $(M, \cdot)$ contains two orders of Turing degrees $\mathbf{a}$ and $\mathbf{b}$ such that the infimum of $\mathbf{a}$ and $\mathbf{b}$ is the computable degree, $\mathbf{0}$.*

Using Theorem 2.2 and results of Jockusch and Soare about effectively closed sets from [16], we obtain the following proposition. A set X is *computably enumerable* if there is a computable function g that enumerates it, that is, X is the range of g. A Turing degree is computably enumerable if it contains a computably enumerable set. Computably enumerable degrees form a countable infinite upper semilattice.

Proposition 2.7. *Let $(M, \cdot)$ be a computable orderable magma.*

(i) *Then $(M, \cdot)$ has an order of a computably enumerable Turing degree.*
(ii) *If $(M, \cdot)$ does not have a computable order, then there is a computably enumerable Turing degree $\mathbf{a}$ such that $(M, \cdot)$ has no order of Turing degree $\leq \mathbf{a}$.*

Recall that $(\varphi_n^X)_{n \geq 0}$ is a computable enumeration of all unary partial computable functions with oracle X. A set A is *truth-table reducible* to a set B, in symbols $A \leq_{\mathrm{tt}} B$, if there is a computable function h and an index n such that for every x,

$$A(x) = \varphi_n^{B \restriction h(x)}(x),$$

and for any string $\sigma \in 2^{<\omega}$ of length $h(x)$, we have $x \in \mathrm{dom}(\varphi_n^\sigma)$. Here, as usual in computability theory, the computation $\varphi_n^\sigma(x)$ assumes that only questions about numbers in the domain of σ are posed to the oracle. We say that a set $X \leq_T \emptyset'$ is *super low* if $X' \leq_{\mathrm{tt}} \emptyset'$. Hence, by the *Super Low Basis Theorem* of Jockusch and Soare (see [30]), a computable right-orderable (left-orderable, bi-orderable, respectively) magma contains a *super low* right order (left order, bi-order, respectively).

3. Orders on the Conjugate Quandle of F_∞

Not much is known about orders on magmas where the operation is not necessarily associative. Important examples of such magmas where the operation is self-distributive come from knot theory and are called quandles. A magma $(Q, *)$ is a *quandle* if the following three axioms are satisfied, where the symbol "$\exists\,!$" stands for "there is unique."

(i) $(\forall\, a)[a * a = a]$ (That is, $*$ is idempotent.)
(ii) $(\forall\, b, c)(\exists\,! a)[a * b = c]$ (That is, for every $b \in Q$, the function $*_b : Q \to Q$ defined by $*_b(a) = a * b$ is a bijection.)
(iii) $(\forall\, a, b, c)[(a * b) * c = (a * c) * (b * c)]$ (That is, $*$ is right self-distributive.)

Quandles were first studied in 1982 by Joyce [17] and Matveev [21]. A quandle $(Q, *)$ is called *trivial* if

$$(\forall\, a, b)[a * b = a].$$

It is easy to see that for a trivial quandle, every linear ordering of Q is right-invariant with respect to the quandle operation. Moreover, if Q is a countable infinite trivial quandle, we have that $RO(Q)$ is homeomorphic to 2^ω since $RO(Q)$ has no isolated points.

For a group $(G, \cdot)$, the *conjugate quandle of* G, in symbols $\mathrm{Conj}(G)$, is $(G, *)$ where

$$(\forall\, a, b)[a * b = b^{-1} \cdot a \cdot b].$$

Every bi-order $\prec$ on G is a right order on $\mathrm{Conj}(G)$. That is because if $a \preceq b$ for $a, b \in G$ and if $c \in G$ we have $c^{-1} \cdot a \cdot c \preceq c^{-1} \cdot b \cdot c$, hence $a * c \preceq b * c$. It was remarked in [7] that in general not every right order on $\mathrm{Conj}(G)$ has to be induced by a bi-order on G. For example, in the case of an abelian group G with torsion, G is not bi-orderable, while $\mathrm{Conj}(G)$ is a trivial quandle so any linear order of the domain is a right order on the quandle.

A computable free group G of infinite rank with a computable basis has a computable bi-order, and hence its conjugate quandle, which is also computable, has a computable right order. The elements of a basis will generate G, so we can think of them as letters and of the group elements as reduced words on these letters and their inverses. The identity in G cannot be expressed as a nontrivial word on elements of a basis. However, not every computable free group of infinite rank has a computable basis. In fact, it was shown in [3, 23] that every computable isomorphic copy of F_∞ has a basis that is Π_2 in the arithmetic hierarchy, and the result cannot be improved to Σ_2. A set is both Σ_0 and Π_0 if it is computable, and for $n > 0$, a set is Σ_n, or Π_n, if it can be expressed using n blocks of like quantifiers, beginning with $\exists$, or $\forall$ followed by a computable relation.

Theorem 3.1. *There is an isomorphic computable copy G of F_∞ such that its conjugate quandle* $\mathrm{Conj}(G)$ *has no computable right order.*

Proof. We will build a computable group G isomorphic to F_∞ such that the following requirements are satisfied for all $i \in \mathbb{N}$.

$R_i : \varphi_i^{(2)}$ does not compute a right order on $\mathrm{Conj}(G)$.

At each stage s, we will consider all requirements R_i for $i \leq s$, which have not been satisfied, while determining more and more of the atomic diagram of G. We designate an infinite computable sequence of distinct elements of G:

$$e, a_0, b_0, c_0, \ldots, a_i, b_i, c_i, \ldots,$$

where e is the identity element of G. For every $i \in \mathbb{N}$, we will include in G a finitely generated free factor G_i. We start building a computable free group G_i on generators a_i, b_i, c_i and wait until a stage s at which $\varphi_i^{(2)}$ might compute a right order on $\mathrm{Conj}(F_\infty)$.

We describe the exact strategy for a single requirement R_i. For any $g \in G$, and $n \in \mathbb{N}$, we write $g^n = \underbrace{gg \cdots g}_{n}$. Having enumerated finitely many sentences in the atomic diagram of G, we continue extending the diagram of G and watching $\varphi_i^{(2)}$ on pairs of elements from $S_i = \{a_i, b_i, c_i\}$. For x, y, z, we write $\varphi_{i,s}^{(2)}(x, y) \downarrow= z$ if $i, x, y < s$, and $\varphi_i^{(2)}(x, y)$ halts in fewer than s steps and outputs z. Assume that at some stage s, $\varphi_i^{(2)}$ is defined on $S_i \times S_i$ and is the characteristic function of a strict linear ordering $\prec$ of S_i. Without loss of generality, for example, assume that we have $c_i \prec b_i \prec a_i$; that is, $\varphi_{e,s}^{(2)}(c_i, b_i) = 1$, $\varphi_{e,s}^{(2)}(b_i, c_i) = 0$, $\varphi_{e,s}^{(2)}(c_i, a_i) = 1$, $\varphi_{e,s}^{(2)}(a_i, c_i) = 0$, $\varphi_{e,s}^{(2)}(b_i, a_i) = 1$, $\varphi_{e,s}^{(2)}(a_i, b_i) = 0$, and $\varphi_{e,s}^{(2)}(x, x) = 0$ for $x \in S_i$. At the least such stage s, we make $c_i = (b_i^{-1})^n a_i (b_i)^n$ for some large n. By choosing a sufficiently large n, we will not contradict any quantifier-free statements in the atomic diagram of G to which we have already committed, thus assuring that the group G will be computable. Now, we continue to build a free factor G_i generated just by a_i and b_i. This action will prevent $\varphi_i^{(2)}$ from computing a right order on $\mathrm{Conj}(G)$, hence, the requirement R_i will be satisfied. Assume otherwise, that is, let $\prec$ as above be a right order on $\mathrm{Conj}(G)$. Since $b_i \prec a_i$, we have $b_i * b_i \prec a_i * b_i$, so $b_i \prec a_i * b_i$ since $b_i * b_i = b_i$. Hence $b_i \prec b_i^{-1} a_i b_i$. By continuing we get that $b_i \prec b_i^{-1} a_i b_i * b_i$, i.e. $b_i \prec b_i^{-1} b_i^{-1} a_i b_i b_i, \ldots$. Thus, $c_i \prec (b_i^{-1})^n a_i (b_i)^n$ for any n, which will give us a contradiction. □

Corollary 3.1. *The space* $RO(\mathrm{Conj}(F_\infty))$ *is homeomorphic to the Cantor set.*

Proof. Let G be a computable group isomorphic to F_∞ such that the conjugate quandle $\mathrm{Conj}(G)$ has no computable right orders. By Theorem 2.2, there is a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection from $RO(\mathrm{Conj}(G))$ to the set of all infinite paths of $\mathcal{T}$. Since $\mathcal{T}$ does not have a computable infinite path, it has no isolated infinite paths. Hence the space $RO(\mathrm{Conj}(G))$ is homeomorphic to the Cantor set. □

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